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A REMARK ON A PAPER OF BLASCO

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Abstract. We generalize and give a simple proof of a Theorem of Blasco.

For a topological space X, $C(X)$ denotes the algebra of real valued continuous functions on X under the operation

$$(\alpha f + gh)(x) := \alpha f(x) + h(x)g(x)$$

for each $f, g, h \in C(X)$ and $\alpha \in \mathbf{R}$. An algebra A on X means a subalgebra of $C(X)$ which contains the constant functions. By a $*$-*algebra* A on X we mean an algebra on X and all bounded functions in A separate points from the closed sets.

Let A be an algebra on X and $\pi : A \longrightarrow \mathbf{R}$ be a (algebra) homomorphism and α be a cardinal number. π is said to be α-evaluating if for each subset B of A with cardinality at most α there exits $x_B \in X$ such that

$$\pi(f) = f(x_B)$$

for each $f \in B$. If $\alpha = |\mathbf{N}|$(=the cardinal number of the set of the natural numbers $\mathbf{N}$), the α-evaluating homomorphism is called *countable evaluating*.

Let α be a cardinal number. Recall that a topological space X is called α-Lindelöf if for each open cover $\mathcal{U}$ has a subcover $\mathcal{V}$ such that the cardinality of $\mathcal{V}$ is at most α. A lindelöf space means a $|\mathbf{N}|$-Lindelöf space.

The following theorem is given in [**1**] as one of the main results.

Theorem 1. *Let X be a completely regular Hausdorff space. Then X is Lindelöf if and only if each countable evaluating homomorphism π on a $*$-algebra A on X is point evaluating.*

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This is applied to give a characterization of Lindelöf spaces in terms of algebra homomorphism.

The proof of the sufficiency part of the above theorem is given in $[\mathbf{1}]$ directly, but the necessity part is not direct. We can generalize and give a direct proof of the necessity part of the theorem as in the following remark.

Remark 1. Let X be an α-Lindelöf space (not necessarily regular). Then each α-evaluating homomorphism π on an algebra A on X is point evaluating. To see this: Suppose that is not true. Then there exits an α-evaluating homomorphism π on an algebra A such that π is not point evaluating. Then for each $x \in X$ there exits $f_x \in A$ such that $\pi(f_x) \neq f_x(x)$. Let

$$g_x := (f_x - \pi(f_x)\mathbf{1})^2 \quad (x \in X).$$

Then $g_x \in A$. As $0 < g_x(x)$, for each $x \in X$ there exits an open set O_x with $x \in O_x$ such that $0 < g_x(y)$ for each $y \in O_x$. As $\{O_x : x \in X\}$ is an open cover of X there exits a subset I of X with cardinality at most α such that $X = \cup_{x \in I} O_x$. Then there exits $x_0 \in X$ such that $\pi(f_x) = f_x(x_0)$ for each $x \in I$. Also there exists $a \in I$ with $x_0 \in O_a$. In particular $\pi(f_a) = f_a(x_0)$. This contradicts the fact $0 < g_a(y)$ for each $y \in O_a$ as $x_0 \in O_a$.

By combining the above theorem with the previous remark we have the following theorem.

Theorem 2. *Let X be a completely regular Hausdorff space. Then the followings are equivalent.*

(i) *X is Lindelöf.*

(ii) *Each countable evaluating homomorphism on any $*$-algebra on X is point evaluating.*

(iii) *Each countable evaluating homomorphism on any algebra on X is point evaluating.*

References

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