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ON THE CONVERGENCE ANALYSIS OF THE ITERATIVE METHOD WITH ERRORS FOR GENERAL MIXED QUASIVARIATIONAL INEQUALITIES IN HILBERT SPACES

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Abstract. The purpose of this paper is to investigate the iterative methods for a class of general mixed quasivariational inequalities in a Hilbert space. Utilizing the alternative equivalent formulation between general mixed quasivariational inequalities and implicit fixed-point problems, we suggest and analyze a new modified self-adaptive resolvent method with errors for solving this class of general mixed quasivariational inequalities in conjunction with a technique updating the solution. Moreover, we give the convergence analysis of this method in a Hilbert space. Since this class of general mixed quasivariational inequalities includes a number of known classes of variational inequalities as special cases, our results are more general than some earlier and recent ones in the literature.

1. Introduction

The theory of variational inequalities introduced by Stampacchia [1] in the early 1960s and later generalized and extended in various directions by others plays an important and fundamental role in the study of a wide class of problems arising in elasticity, fluid flow through porous media, finance, economics, transportation, circuit analysis, structural analysis and many other branches of mathematical and engineering science; see [1-17]. Among these generalizations of variational inequalities, a useful and significant generalization is called the mixed quasivariational inequality

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involving a nonlinear bifunction which enables us to study free,moving, unilateral and equilibrium problems; see e.g., [17].

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a closed convex subset of H and $T, g : H \to H$ be nonlinear operators. Let $\varphi(\cdot, \cdot) : H \times H \to R \cup \{+\infty\}$ be a bifunction continuous with respect to both arguments. Recently Noor [18] considered and studied the problem of finding $u \in H$ such that

$$(1) \qquad \langle Tu, g(v) - g(u) \rangle + \varphi(g(v), g(u)) - \varphi(g(u), g(u)) \geq 0, \quad \forall g(v) \in H$$

which is called the general mixed quasivariational inequality problem. If the bifunction $\varphi(\cdot, \cdot)$ is proper, convex and lower semicontinuous with respect to the first argument, then problem (1) is equivalent to finding $u \in H$ such that

$$0 \in Tu + \partial\varphi(g(u), g(u)),$$

which is known as a set-valued quasivariational inclusion problem where $\partial\varphi(\cdot, g(u)) : H \to 2^H$ is a maximal monotone operator. This problem has been studied extensively in recent years; see e.g., Zeng [19]

Special Cases.

(i) For $g \equiv I$, the identity operator, problem (1) reduces to

$$(2) \qquad \langle Tu, v - u \rangle + \varphi(v, u) - \varphi(u, u) \geq 0, \quad \forall v \in H.l$$

Problem (2) is called the mixed quasivariational inequality.

(ii) If $\varphi(u, v) = \varphi(v) \forall v \in H$, then problem (1)is equivalent to find $u \in H$ such that

$$(3) \qquad \langle Tu, g(v) - g(u) \rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0 \quad \forall v \in H,$$

which is called the general variational inequality.

(iii) If $\varphi(\cdot)$ is the indicator function of a closed and convex subset K in H, that is,

$$\varphi(u) = \begin{cases} 0, & \text{if} \quad u \in K \\ +\infty, & \text{otherwise}, \end{cases}$$

then problem (3) is equivalent to find $u \in H, g(u) \in K$ such that

$$(4) \qquad \langle Tu, g(v) - g(u) \rangle \geq 0, \quad \forall g(v) \in K$$

which is known as the general variational inequality. See [20,21].

(iv) For $g \equiv I$ the identity operator, we get the corresponding classical variational inequality.

From the above examples it is clear that for appropriate and suitable choice of the operators T, g and the bifunction φ, a number of known classes of variational inequalities can be obtained as special cases studied previously by many authors; see e.g., [1-18].

It is well known that there now exists a variety of techniques including projection method and its variant forms, auxiliary principle and resolvent equations to suggest and analyze various iterative algorithms for solving variational inequalities and related optimization problems. Moreover, it is also known that the projection method and its variant forms cannot be extended for mixed quasivariational inequalities due to the presence of the bifunction. However, if the bifunction is a proper, convex and lower semicontinuous with respect to the first argument, then it has been shown [11] that mixed quasivariational inequalities are equivalent to the fixed-point problems. Recently, utilizing the alternative equivalent formulation between mixed quasivariational inequalities and implicit fixed-point problems, Noor [18] proposed the following iterative method for solving problem (1) and proved its convergence in finite-dimensional Hilbert space H.

Algorithm 1.1. (Algorithm 3.7 in [18]) Let T be g-pseudomonotone and g^{-1} exists. Let$\varphi(\cdot,\cdot) : H \times H \to R\bigcup\{+\infty\}$be a bifunction which not only is skew-symmetric and continuous with respect to both arguments but also is proper, convex and lower semicontinuous with respect to the first argument. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the following iterative scheme:

Step 1. (Predictor Step). Compute

$$g(y_n) = J_{\hat{\varphi}(u_n)}[g(u_n) - \rho_n T u_n], \quad n = 0, 1, 2, ...,$$

where ρ_n (prediction) satisfies

$$\rho_n \langle Tu_n - Tg^{-1} J_{\hat{\varphi}(u_n)}[g(y_n) - \rho_n T y_n], R(u_n)\rangle \le \sigma \|R(u_n)\|^2, \quad \sigma \in (0, 1).$$

Step 2. (Corrector Step). Compute

$$g(u_{n+1}) = g(u_n) - \alpha_n d(u_n), \quad n = 0, 1, 2, ...,$$

where $d(u_n) = R(u_n) + \rho_n T J_{\hat{\varphi}(u_n)}[g(y_n) - \rho_n T y_n], \alpha_n = \langle R(u_n), D(u_n)\rangle / \|d(u_n)\|^2$,

$R(u_n) = g(u_n) - J_{\hat{\varphi}(u_n)}[g(y_n) - \rho_n T y_n], J_{\hat{\varphi}(u_n)} = (I + \rho_n \partial\varphi(\cdot, g(u_n)))^{-1}$ and

$$D(u_n) = R(u_n) - \rho_n T u_n + \rho_n T g^{-1} J_{\hat{\varphi}(u_n)}[g(y_n) - \rho_n T y_n].$$

Theorem 1.1. (Theorem 3.3 in [18]). *Let $\bar{u} \in H$ be a solution of problem (1) and u_{n+1} be the approximate solution obtained from Algorithm 1.1. If H is a finite-dimensional space then* $\lim_{n\to\infty} u_n = \bar{u}$.

On the other hand, Xu [22] gave the new modified proximal point algorithms for finding a solution $u \in H$ of the equation $0 \in Tu$ where T is a maximal monotone operator and proved the strong and weak convergence of the approximate solutions generated by those algorithms, respectively.

In this paper motivated and inspired by Xu [22], we extend Noor's Algorithm 3.7 [18] to develop the new modified iterative algorithm for solving problem (1) in a real Hilbert space H and also give the convergence analysis of this method. Since this class of general mixed quasivariational inequalities includes a number of known classes of variational inequalities as special cases, our results are more general than some earlier and recent ones in the literature.

Throughout this paper, let Ω denote the solution set of problem (1).

2. Algorithms and Preliminaries

Let H be a real Hilbert space with inner product$\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let $T.g : H \to H$ be nonlinear operators. Let $\varphi(\cdot, \cdot) : H \times H \to R \bigcup \{+\infty\}$ be a bifunction which not only is continuous with respect to both arguments but also is proper, convex and lower semicontinuous with respect to the first argument.

We need the following well-known results and concepts.

Definition 2.1. The operator $T : H \to H$ is said to be

(i) g-monotone if

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq 0 \quad \forall u, v \in H;$$

(ii) g-pseudomonotone if for all $u, v \in H$,

$$\langle Tu, g(v) - g(u)\rangle \geq 0 \Rightarrow \langle Tv, g(u) - g(v)\rangle \leq 0.$$

Remark 2.1. If $g \equiv I$ the identity operator, then the concepts of g-monotonicity and g-pseudonmontonicity reduce to the ones of monotonicity and pseudomonotonity, respectively. It is well known [5] that monotonicity implies pseudomonotonicity but the converse is not true in general. This shows that pseudomonotonicity is a weaker condition than monotonicity.

Definition 2.2. The bifunction $\varphi(\cdot, \cdot)$ is said to be skew-symmetric if

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0, \quad \forall u, v \in H.$$

Clearly if the skew-symmetric bifunction $\varphi(\cdot,\cdot)$ is linear in both arguments, then

$$\varphi(u,u) \geq 0 \quad \forall u \in H.$$

Definition 2.3. Let A be a maximal monotone operator. Then the resolvent operator associated with A is defined as $J_A(u) = (I+\rho A)^{-1}(u)\ \forall u \in H$ where $\rho > 0$ is a constant and I is the identity operator.

Remark 2.2. [18]. Let $\varphi(\cdot,\cdot) : H \times H \to R \bigcup \{+\infty\}$ be a bifunction. If for every fixed $u \in H, \varphi(\cdot,u) : H \to R \bigcup \{+\infty\}$is proper, convex and lower semicontinuous, then the subdifferential $\partial\varphi(\cdot,u) : H \to 2^H$ is maximal monotone and its resolvent is defined by

$$J_{\varphi(u)} = (I+\rho\partial\varphi(\cdot,u))^{-1} \equiv (I+\rho\partial\varphi(u))^{-1}$$

where $\partial\varphi(u) \equiv \partial\varphi(\cdot,u)$ unless otherwise specified.

The resolvent operator $J_{\varphi(u)}$ has the following characterization.

Lemma 2.1. *For a given $u \in H, z \in H$ satisfies the inequality*

$$\langle u-z, v-u\rangle + \rho\varphi(v,u) - \rho\varphi(u,u) \geq 0\ \forall v \in H,$$

if and only if $u = J_{\varphi(u)}z$ where $J_{\varphi(u)}$ is resolvent operator and $\rho > 0$ is a constant.

Lemma 2.2. *Let$\varphi(\cdot,\cdot) : H \times H \to R \bigcup\{+\infty\}$ be a bifunction and $g : H \to H$ be a homeomorphism. If for every fixed $u \in H, \varphi(\cdot,u) : H \to R\bigcup\{+\infty\}$ is proper,convex and lower semicontinuous, then the following statements are equivalent:*

(i) $u \in H$ *is a solution of problem (1);*

(ii) $u \in H$ *satisfies the equation*

$$0 \in T(u) + \partial\varphi(g(u), g(u));$$

(iii) $u \in H$*satisfies the relation*

$$g(u) = J_{\hat{\varphi}(u)}[g(u-\rho Tu)]\ \rho > 0, \tag{5}$$

where $J_{\hat{\varphi}(u)} := J_{\varphi(g(u))} = (I+\rho\partial\varphi(\cdot,g(u)))^{-1}.$

Proof. Observe that

$$\begin{aligned}
&\langle -Tu, g(v)-g(u)\rangle \leq \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \quad \forall g(v) \in H\\
\Leftrightarrow\ & -Tu \in \partial\varphi(g(u),g(u))\\
\Leftrightarrow\ & g(u) - \rho Tu \in (I+\rho\partial\varphi(\cdot,g(u)))(g(u))\\
\Leftrightarrow\ & g(u) = (I+\rho\partial\varphi(\cdot,g(u)))^{-1}(g(u)-\rho Tu) = J_{\hat{\varphi}(u)}(g(u)-\rho Tu)
\end{aligned}$$

from which the result follows. ■

Lemma 2.3. ([23], p. 303). *Let* $\{a_n\}$*and*$\{b_n\}$ *be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty$*, then* $\lim_{n\to\infty} a_n$ *exists.*

According to Noor ([18, p. 128), we can rewrite (5) in the following form:

$$g(u) = J_{\hat{\varphi}(u)}[g(w) - \rho Tw],$$

$$g(w) = J_{\hat{\varphi}(u)}[g(y) - \rho Ty],$$

$$g(y) = J_{\hat{\varphi}(u)}[g(u) - \rho Tu].$$

In this paper, we suggest a new modified self-adaptive resolvent method with errors for solving problem (1) by modifying a technique updating the solution. To this end, we define the residue vector $R(u)$ by

$$R(u) := g(u) - g(w) = g(u) - J_{\hat{\varphi}(u)}[g(y) - \rho Ty]$$

where $g(w) = J_{\hat{\varphi}(u)}[g(y) - \rho Ty]$, and $g(y) = J_{\hat{\varphi}(u)}[g(u) - \rho Tu]$.

It is clear form Lemma 2.2 that $u \in H$ is a solution of problem (1) if and only if $u \in H$ is the zero of the equation

$$R(u) = g(u) - J_{\hat{\varphi}(u)}[g(y) - \rho Ty] = 0. \tag{6}$$

By suitable rearrangement of terms, we can rewrite (6) as

$$R(u) - \rho Tu + \rho Tg^{-1} J_{\hat{\varphi}(u)}[g(y) - \rho Ty] = 0. \tag{7}$$

Motivated and inspired by Xu [22], we extend Noor's Algorithm 3.7 [18] to develop a new modified iterative algorithm for solving problem (1) in an arbitrary real Hilbert space H.

Algorithm 2.1. Let T be g-pseudomonotone and g be a homeomorphism. Let $\varphi(\cdot,\cdot) : H \times H \to R \bigcup \{+\infty\}$ be a bifunction which not only is skew-symmetric and continuous with respect to both arguments but also is proper, convex and lower semicontinuous with respect to the first argument. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the following iterative scheme:

(i) Compute

$$g(y_n) = J_{\hat{\varphi}(u_n)}[g(u_n) - \rho_n Tu_n], \quad n = 0, 1, 2, ... \tag{8}$$

where ρ_n (prediction) satisfies

$$\rho_n\langle Tu_n - Tg^{-1}J_{\hat{\varphi}(u_n)}[g(y_n) - \rho_n Ty_n], R(u_n)\rangle \leq \sigma\|R(u_n)\|^2, \quad \sigma \in (0,1)\,.$$

(ii) Compute

$$g(\tilde{u}_{n+1}) = g(u_n) - \alpha_n d(u_n), \quad n = 0, 1, 2, ..., \tag{9}$$

where $d(u_n) = R(u_n) + \rho_n TJ_{\hat{\varphi}(u_n)}[g(y_n) - \rho Ty_n], \alpha_n = \langle R(u_n), D(u_n)\rangle / \|d(u_n)\|^2$,

$R(u_n) = g(u_n) - J_{\hat{\varphi}(u_n)}[g(y_n) - \rho_n Ty_n], J_{\hat{\varphi}(u_n)} = (I + \rho_n\partial\varphi(\cdot, g(u_n)))^{-1}$ and

$$D(u_n) = R(u_n) - \rho_n Tu_n + \rho_n Tg^{-1}J_{\hat{\varphi}(u_n)}[g(y_n) - \rho_n Ty_n].$$

(iii) Select two relaxation parameters $\beta_n, \gamma_n \in [0,1]$ with $\beta_n + \gamma_n \leq 1$ and compute the $(n+1)$th iterate

$$g(u_{n+1}) := (1 - \beta_n - \gamma_n)g(\tilde{u}_{n+1}) + \beta_n g(u_n) + \gamma_n e_n \tag{10}$$

where $\{e_n\}$ is an error sequence in H introduced to take into account possible inexact computation.

Theorem 2.1. [18]. *Let $\bar{u} \in H$ be a solution of problem (1). If $T : H \to H$ is g-pseudomonotone and the bifunction $\varphi(\cdot,\cdot) : H \times H \to R\bigcup\{+\infty\}$ is skew-symmetric, then*

$$\langle g(u) - g(\bar{u}), d(u)\rangle \geq \langle R(u), D(u)\rangle \geq (1-\sigma)\|R(u)\|^2 \;\forall u \in H. \tag{11}$$

Proof. See inequality (22) in the proof of Noor ([18], Theorem 3.1). ■

Theorem 2.2. *Let $\bar{u} \in H$ be a solution of problem (1) and let $\tilde{u}_{n+1}$ be the approximate solution obtained from Algorithm 2.1. Then,*

$$\|g(\tilde{u}_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - (1-\sigma)^2\|R(u_n)\|^4/\|d(u_n)\|^2. \tag{12}$$

Proof. Following the idea of the proof of Noor ([18], Theorem 3.2), we give the proof of the theorem. From (9), (11) and the definition of α_n, we obtain

$$\|g(\tilde{u}_{n+1}) - g(\bar{u})\|^2 = \|g(u_n) - g(\bar{u}) - \alpha_n d(u_n)\|^2$$
$$= \|g(u_n) - g(\bar{u})\|^2 - 2\alpha_n\langle g(u_n) - g(\bar{u}), d(u_n)\rangle + \alpha_n^2\|d(u_n)\|^2$$
$$\leq \|g(u_n) - g(\bar{u})\|^2 - 2\alpha_n\langle R(u_n), D(u_n)\rangle + \alpha_n^2\|d(u_n)\|^2 \quad \text{(using (11))}$$

$$
\begin{aligned}
&= \|g(u_n) - g(\bar{u})\|^2 - 2\alpha_n \langle R(u_n), D(u_n)\rangle \\
&\quad +\alpha_n \cdot \frac{\langle R(u_n), D(u_n)\rangle}{\|d(u_n)\|^2} \cdot \|d(u_n)\|^2 \quad \text{(using the definition of } \alpha_n) \\
&= \|g(u_n) - g(\bar{u})\|^2 - \alpha_n \langle R(u_n), D(u_n)\rangle \\
&= \|g(u_n) - g(\bar{u})\|^2 - \frac{[\langle R(u_n), D(u_n)\rangle]^2}{\|d(u_n)\|^2} \quad \text{(using the definition of } \alpha_n) \\
&\leq \|g(u_n) - g(\bar{u})\|^2 - \frac{(1-\sigma)^2 \|R(u_n)\|^4}{\|d(u_n)\|^2} \quad \text{(using (11))}
\end{aligned}
$$

from which the result follows. ∎

3. Convergence Analysis

Now, we are ready to establish the sufficient and necessary condition for the convergence of the approximate solutions x_n generated by Algorithm 2.1 to an exact solution of problem (1) in an arbitrary real Hilbert space$H$. Let Ω denote the solution set of problem (1) and define

$$g(\Omega) = \{g(u) : u \in \Omega\}.$$

Theorem 3.1. *Let $\{u_n\}$ be a sequence of approximate solutions generated by Algorithm 2.1. Let $\{e_n\}$ be a bounded sequence in H and $\{\beta_n\}, \{\gamma_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\{\beta_n + \gamma_n\}$ is bounded away from 1, namely $0 \leq \beta_n + \gamma_n \leq 1 - \delta$ for some $\delta \in (0, 1)$;

(ii) $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Assume that $\Omega \neq \emptyset$ and that $g(\Omega)$ is bounded. Then $\{u_n\}$ converges to a solution of problem (1) if and only if $\liminf_{n\to\infty} d(g(u_n), g(\Omega)) = 0$ where $d(y, D)$ denotes the distance of y to set D; i.e., $d(y, D) = \inf_{x\in D} d(y, x)$.

Proof. "Necessity". Suppose that $\{u_n\}$ converges to a solution $\bar{u} \in \Omega$ of problem (1). Then from the continuity of g,we have

$$d(g(u_n), g(\Omega)) = \inf_{u\in\Omega} d(g(u_n), g(u)) \leq d(g(u_n), g(\bar{u})) \to 0 \quad \text{as} \quad n \to \infty.$$

This shows that $\liminf_{n\to\infty} d(g(u_n), g(\Omega)) = \lim_{n\to\infty} d(g(u_n), g(\Omega)) = 0$.

"Sufficiency." Suppose that $\liminf_{n\to\infty} d(g(u_n), g(\Omega)) = 0$. In order to show the convergence of $\{u_n\}$ to a solution of problem (1), we divide the proof into several steps.

Step 1. We claim that for each $\bar{u} \in \Omega, \lim_{n\to\infty} \|g(u_n) - g(\bar{u})\|$ exists and in particular, $\{g(u_n)\}$ is bounded. Indeed, since it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$, we have

$$\begin{aligned} &\|g(u_{n+1}) - g(\bar{u})\|^2 \\ (13)\qquad &= \|(1-\beta_n-\gamma_n)(g(\tilde{u}_{n+1})-g(\bar{u}))+\beta_n(g(u_n)-g(\bar{u}))+\gamma_n(e_n-g(\bar{u}))\|^2 \\ &\le (1-\beta_n-\gamma_n)\|g(\tilde{u}_{n+1})-g(\bar{u})\|^2+\beta_n\|g(u_n)-g(\bar{u})\|^2+\gamma_n\|e_n-g(\bar{u})\|^2. \end{aligned}$$

Substituting (12) into (13) and from condition (i), we get

$$\begin{aligned} &\|g(u_{n+1}) - g(\bar{u})\|^2 \\ &\le (1-\beta_n-\gamma_n)[\|g(u_n)-g(\bar{u})\|^2 - \frac{(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2}] \\ &\quad +\beta_n\|g(u_n)-g(\bar{u})\|^2 + \gamma_n\|e_n - g(\bar{u})\|^2 \\ (14)\qquad &= (1-\gamma_n)\|g(u_n)-g(\bar{u})\|^2 - (1-\beta_n-\gamma_n)\cdot\frac{(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2} \\ &\quad +\gamma_n\|e_n-g(\bar{u})\|^2 \\ &\le \|g(u_n)-g(\bar{u})\|^2 - \frac{\delta(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2} + \gamma_n\|e_n-g(\bar{u})\|^2 \end{aligned}$$

and hence

$$\begin{aligned} \|g(u_{n+1}) - g(\bar{u})\|^2 &\le \|g(u_n)-g(\bar{u})\|^2 + \gamma_n\|e_n-g(\bar{u})\|^2 \\ (15)\qquad &\le \|g(u_n)-g(\bar{u})\|^2 + \gamma_n[\|e_n\| + \|g(\bar{u})\|]^2 \\ &\le \|g(u_n)-g(\bar{u})\|^2 + M^2\gamma_n, \end{aligned}$$

where $M = \sup_{n\ge 0}\|e_n\| + \sup\{\|g(u)\| : u \in \Omega\} < \infty$. Since $\{e_n\}$ is bounded, it follows from $\sum_{n=0}^{\infty}\gamma_n < \infty$ that $\lim_{n\to\infty}\|g(u_n) - g(\bar{u})\|$exists. Thus, $\{g(u_n)\}$ is bounded.

Step 2. We claim that $\{u_n\}$ converges to some $\tilde{u} \in H$. Indeed, from (15) we can see that

$$\|g(u_{m+n}) - g(\bar{u})\|^2 \le \|g(u_m) - g(\bar{u})\|^2 + M^2 \cdot \sum_{j=m}^{m+n-1}\gamma_j \quad \forall m \ge 0, n \ge 1, \bar{u} \in \Omega.$$

Since $\liminf_{n\to\infty} d(g(u_n), g(\Omega)) = 0$, there must exist a subsequence $\{u_{n_i}\} \subset \{u_n\}$ such that

$$\lim_{n_i\to\infty} d(g(u_{n_i}), g(\Omega)) = \lim_{n\to\infty} \inf d(g(u_n), g(\Omega)) = 0$$

which together with $\sum_{j=0}^{\infty} \gamma_j < \infty$ implies that there must exist a positive integer $N_0 \geq 1$ such that when $n_i, n \geq N_0$,

$$d(g(u_{n_i}), g(\Omega)) < \frac{\varepsilon}{8} \quad \text{and} \quad M^2 \cdot \sum_{j=n}^{\infty} \gamma_j < \frac{\varepsilon}{8}. \tag{16}$$

Thus, there must exist $u^* \in \Omega$ such that $d(g(u_{N_0}, g(u^*)) < \frac{\varepsilon}{8}$. Hence this implies that whenever $n \geq N_0 + 1$ and $m \geq 1$,

$$\begin{aligned} \|g(u_n) - g(u^*)\|^2 &\leq \|g(u_{N_0}) - g(u^*)\|^2 + M^2 \cdot \sum_{j=N_0}^{n-1} \gamma_j \\ &\leq \|g(u_{N_0}) - g(u^*)\|^2 + M^2 \cdot \sum_{j=N_0}^{\infty} \gamma_j \\ &< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}, \end{aligned}$$

and

$$\begin{aligned} \|g(u_{m+n}) - g(u^*)\|^2 &\leq \|g(u_{N_0}) - g(u^*)\|^2 + M^2 \cdot \sum_{j=N_0}^{m+n-1} \gamma_j \\ &\leq \|g(u_{N_0}) - g(u^*)\|^2 + M^2 \cdot \sum_{j=N_0}^{\infty} \gamma_j \\ &< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}. \end{aligned}$$

Therefore, we derive

$$\begin{aligned} |g(u_{m+n}) - g(u_n)\|^2 &= \|g(u_{m+n}) - g(u^*) + g(u^*) - g(u_n)\|^2 \\ &\leq 2\|g(u_{m+n}) - g(u^*)\|^2 + 2\|g(u_n) - g(u^*)\|^2 \\ &< 2 \cdot \frac{\varepsilon}{4} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon, \quad \forall n \geq N_0 + 1, m \geq 1. \end{aligned}$$

This shows that $\{g(u_n)\}$ is a Cauchy sequence in H which is complete. Thus $\lim_{n\to\infty} g(u_n)$ exists. Since $g : H \to H$ is a homeomorphism from H onto itself, there is no doubt that $g^{-1} : H \to H$ is continuous. Thus $\lim_{n\to\infty} u_n$ exists and hence we may assume that

$$\lim_{n\to\infty} u_n = \tilde{u} \in H.$$

Step 3. We claim that $\tilde{u} \in H$ is a solution of problem (1). Indeed, from (14) it follows that

$$\begin{aligned} &\frac{\delta(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2} \\ &\le \|g(u_n)-g(\bar{u})\|^2 - \|g(u_{n+1})-g(\bar{u})\|^2 + \gamma_n\|e_n - g(\bar{u})\|^2 \\ &\le \|g(u_n)-g(\bar{u})\|^2 - \|g(u_{n+1})-g(\bar{u})\|^2 + M^2\cdot\gamma_n. \end{aligned} \tag{17}$$

Since $\lim_{n\to\infty}\|g(u_n)-g(\bar{u})\|$ exists and $\lim_{n\to\infty}\gamma_n = 0$, from (17) we get

$$\lim_{n\to\infty}\frac{\delta(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2} = 0,$$

and hence $\lim_{n\to\infty}\|R(u_n)\| = 0$. Note that $R(u)$ is continuous. Thus from $\lim_{n\to\infty} u_n = \tilde{u}$, it follows that $R(\tilde{u}) = 0$. Therefore, $\tilde{u} \in H$ is a solution of problem (1) by invoking Lemma 2.2. ■

Theorem 3.2. *Let $\{u_n\}$ be a sequence of approximate solutions generated by Algorithm 2.1. Let $\{e_n\}$ be a bounded sequence in H and $\{\beta_n\}, \{\gamma_n\}$ be real sequence in $[0,1]$ satisfying conditions (i) and (ii) in Theroem 3.1. Assume that $\Omega \neq \emptyset$. If H is a finite-dimensional space, then $\{u_n\}$ converges to a solution of problem(1).*

Proof. At first, take an arbitrary $\bar{u} \in \Omega$. Then from(14), it follows that

$$\begin{aligned} &\|g(u_{n+1})-g(\bar{u})\|^2 \\ &\le \|g(u_n)-g(\bar{u})\|^2 - \frac{\delta(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2} + \gamma_n\|e_n - g(\bar{u})\|^2 \\ &\le \|g(u_n)-g(\bar{u})\|^2 - \frac{\delta(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2} + M^2\cdot\gamma_n. \end{aligned} \tag{18}$$

From (18) and Lemma 2.3, we know that $\lim_{n\to\infty}\|g(u_n)-g(\bar{u})\|$ exists and hence $\{g(u_n)\}$ is bounded. This implies that $\{u_n\}$ is bounded. Also, from (18) we conclude that

$$\sum_{n=0}^{\infty}\frac{\delta(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2} \le \|g(u_0)-g(\bar{u})\|^2 + M^2\cdot\sum_{n=0}^{\infty}\gamma_n < \infty$$

which hence implies that

$$\lim_{n\to\infty} R(u_n) = 0.$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and let the subsequence $\{u_{n_j}\}$ converge to $\hat{u}$. Since $R(u)$ is continuous, we have

$$R(\hat{u}) = \lim_{n_j \to \infty} R(u_{n_j}) = 0$$

and $\hat{u}$ is a solution of problem (1) by invoking Lemma 2.2. Thus, it follows from the continuity of g that

$$\lim_{n \to \infty} \|g(u_n) - g(\hat{u})\| = \lim_{n_j \to \infty} \|g(u_{n_j}) - g(\hat{u})\| = 0.$$

According to the continuity of g^{-1}, we deduce that $\{u_n\}$ converges to the solution $\hat{u}$ of problem (1). ∎

References

1. G. Stampacchia, Forms Bilineaires Coercitives sur les Ensembles Convexes, *Comptes Rendus de l'Academie des Sciences, Paris,* **258** (1964), 4413-4416.
2. C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities*, John Wiley and Sons, New York, NY, 1984.
3. H. Brezis, *Operateur Maximaux Monotones et Semegroups de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam, Holland, 1973.
4. J. Crank, *Free and Moving Boundary Problems*, Clarendon Press, Oxford, UK, 1984.
5. F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, NY, 1995.
6. F. Giannessi, A. Maugeri and P. M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, Kluwer Academic Publishers, Dordrecht, Holland, 2001.
7. R. Glowinski, J. L. Lions and R. Tremolieres, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, Holland, 1981.
8. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, SIAM, Philadelphia, Pennsylvania, 2000.
9. N. Kikuchi and J. T. Oden, *Contact Problems in Elasticity, SIAM, Philadelphia, Pennsylvania*, 1988.
10. M. Patriksson, *Nonlinear Programming and Variational Inequalities: A Unified Approach*, Kluwer Academic Publishers, Dordrecht, Holland, 1998.
11. M. A. Noor, Set-Valued Mixed Quasivariational Inequalities and Implicit Resolvent Equations, *Mathematical and Computer Moderlling*, **29** (1999), 1-11.
12. R. Glowinski and P. Le Tallec, *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia, Pennsylvania, 1989.

13. L. C. Zeng, Iterative Algorithms for Finding Approximate Solutions to General Strongly Nonlinear Variational Inequalities, *Journal of Mathematical Analysis and Applications*, **187** (1994), 352-360.

14. L. C. Zeng, Iterative Algorithm for Finding Approximate Solutions to Completely Generalized Strongly Nonlinear Quasivariational Inequalities, *Journal of Mathematical Analysis and Applications*, **201** (1996), 180-194.

15. L. C. Zeng, On a General Projection Algorithm for Variational Inequalities, *Journal of Optimization Theory and Applications*, **97** (1998), 229-235.

16. L. C. Zeng, A Note on General Algorithm for Variational Inequalities, *Journal of Mathematical Analysis and Applications*, **223** (1998), 354-363.

17. L. C. Zeng, Existence and Algorithm of Solutions for General Multivalued Mixed Implicit Quasivariational Inequalities, *Applied Mathematics and Mechanics*, **24** (2003), 1324-1333.

18. M. A. Noor, Iterative Methods for General Mixed Quasivariational Inequalities, *Journal of Optimization Theory and Applications*, **119** (2003), 123-136.

19. L. C. Zeng, Perturbed Proximal Point Algorithm for Generalized Nonlinear Set-Valued Mixed Quasivariational Inclusions, *Acta Mathematica Sinica*, **47** (2004), 11-18.

20. A. H. Siddiqi and Q. H. Ansari, Strongly Nonlinear Quasivariational Inequalities, *Journal of Mathematical Analysis and Applications*, **149** (1990), 444-450.

21. A. H. Siddiqi and Q. H. Ansari, General Strongly Nonlinear Variational Inequalities, *Journal of Mathematical Analysis and Applications*, **166** (1992), 386-392.

22. H. K. Xu, Iterative Algorithms for Nonlinear Operators, *Journal of London Mathematical Society*, **66** (2002), 240-256.

23. K. K. Tan and H. K. Xu, Approximating Fixed Points of Nonexpansive Mappings by the Ishikawa Iteration Process, *Journal of Mathematical Analysis and Applications*, **178** (1993), 301-308.

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