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MILOVANOVIĆ-PEČARIĆ-FINK INEQUALITY FOR DIFFERENCE OF TWO INTEGRAL MEANS

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Abstract. In this paper we show some generalizations of estimations of difference of two integral means, Milovanović-Pečarić-Fink inequality.

1. Introduction

The following Ostrowski inequality is well known [8]:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,L,\quad x\in[a,b]\,, \tag{1.1}$$

where $f:[a,b]\to\mathbf{R}$ is a differentiable function such that $|f'(x)|\leq L$, for every $x\in[a,b]$.

Note that (1.1) can be given in the equivalent form

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{(x-a)^2+(b-x)^2}{2(b-a)}L. \tag{1.2}$$

The Ostrowski inequality has been generalized over the years in a number of ways. G. V. Milovanović and J. Pečarić [7] and A. M. Fink [5] have considered generalizations of (1.1) in the form

$$\left|\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k^{[a,b]}(x)\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq K(n,p,x)\|f^{(n)}\|_p \tag{1.3}$$

which is obtained from identity

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$$\begin{aligned}(1.4)\qquad &\frac{1}{n}\left(f(x)+\sum_{k=1}^{n-1}F_k^{[a,b]}(x)\right)-\frac{1}{b-a}\int_a^b f(t)dt\\ &=\frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k^{[a,b]}(t,x)f^{(n)}(t)dt,\end{aligned}$$

where

$$F_k^{[a,b]}(x)=\frac{n-k}{k!}\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a}$$

and

$$k^{[a,b]}(t,x)=\begin{cases} t-a, & a\le t\le x\le b;\\ t-b, & a\le x<t\le b.\end{cases} \tag{1.5}$$

In fact, G. V. Milovanović and J. Pečarić have proved that

$$K(n,\infty,x)=\frac{(x-a)^{n+1}+(b-x)^{n+1}}{n(n+1)!(b-a)}, \tag{1.6}$$

while A. M. Fink gave the following generalizations of this result:

Theorem 1. *Let $f^{(n-1)}$ be absolutely continuous on $[a,b]$ and let $f^{(n)}\in L_p[a,b]$. Then the inequality (1.3) holds with*

$$K(n,p,x)=\frac{\left[(x-a)^{nq+1}+(b-x)^{nq+1}\right]^{1/q}}{n!(b-a)}B((n-1)q+1,q+1)^{1/q} \tag{1.7}$$

where $1<p\le\infty,\ 1/p+1/q=1$, B is the Beta function, and

$$K(n,1,x)=\frac{(n-1)^{n-1}}{n^n n!(b-a)}\max[(x-a)^n,(b-x)^n]. \tag{1.8}$$

In this paper we use the formula (1.4) to generalize the results from [2] and [6] where is estimated the difference of the two integral means for absolutely continuous mappings whose first derivative is in $L_\infty[a,b]$. We will give the results for functions whose derivative of order $n,\ n\ge 1$, is from $L_p[a,b]$ spaces. See also [9] and [1]. We will make it in two cases: one is when $x\in[c,d]\subseteq[a,b]$ and the other when $x\in[a,b]\cap[c,d]=[c,b]$.

2. Some Integral Identities

Theorem 2. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous function on $[a,b]$ for some $n\ge 1$. Then if $a\le c<d\le b$, for every $x\in[c,d]$*

$$\begin{aligned}(2.1)\qquad &\frac{1}{d-c}\int_c^d f(t)dt-\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{n}\sum_{k=1}^{n-1}F_k(x)\\ &=\frac{1}{n!}\int_a^b (x-t)^{n-1}k_1(t,x)f^{(n)}(t)dt\end{aligned}$$

where

$$F_k(x) = F_k^{[a,b]}(x)-F_k^{[c,d]}(x)=\frac{n-k}{k!}\left(\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a}\right.$$
$$\left.-\frac{f^{(k-1)}(c)(x-c)^k-f^{(k-1)}(d)(x-d)^k}{d-c}\right)$$

and

$$(2.2)\qquad k_1(t,x)=\begin{cases}\dfrac{t-a}{b-a} & \text{if } t\in[a,c];\\ \dfrac{1}{b-a}k^{[a,b]}(t,x)-\dfrac{1}{d-c}k^{[c,d]}(t,x) & \text{if } t\in(c,d];\\ \dfrac{t-b}{b-a} & \text{if } t\in(d,b].\end{cases}$$

Proof. First we write the identity (1.4) for interval $[a,b]$ and for interval $[c,d]$. Then we subtract them to get the above statements. ■

Remark 1. If we put $c=d=x$ as a limit case in the identity (2.1), we get
$\frac{1}{d-c}\int_c^d f(t)dt\to f(x)$, $F_1^{[c,d]}(x)\to\frac{n-1}{n}f(x)$ and $F_k^{[c,d]}(x)\to 0$ for $k\geq 2$.
So, from the identity (1.4) for interval $[c,d]$ we have

$$\frac{1}{n!(d-c)}\int_c^d (x-t)^{n-1}k^{[c,d]}(t,x)f^{(n)}(t)dt\to 0$$

and consequently, for $c=d=x$ the identity (2.1) becomes the identity (1.4).

Theorem 3. *Let $f:[a,d]\to\mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous function on $[a,d]$ for some $n\geq 1$. Then if $a\leq c<b\leq d$, for every $x\in[c,b]$*

$$\begin{aligned}(2.3)\qquad &\frac{1}{d-c}\int_c^d f(t)dt-\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{n}\sum_{k=1}^{n-1}F_k(x)\\ &=\frac{1}{n!}\int_a^d (x-t)^{n-1}k_2(t,x)f^{(n)}(t)dt\end{aligned}$$

where $F_k(x)$ *is as in Theorem 2 and*

$$k_2(t,x) = \begin{cases} \dfrac{t-a}{b-a} & \text{if } \ t \in [a,c]; \\ \dfrac{1}{b-a}k^{[a,b]}(t,x) - \dfrac{1}{d-c}k^{[c,d]}(t,x) & \text{if } \ t \in (c,b]; \\ \frac{d-t}{d-c} & \text{if } \ t \in (b,d]. \end{cases} \tag{2.4}$$

Proof. Similar as the proof of Theorem 2. ∎

3. Estimations of the Difference of Two Integral Means

3.1 Case $[c,d] \subseteq [a,b]$

Theorem 4. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous on* $[a,b]$ *and let* $f^{(n)} \in L_p[a,b]$ *for some* $n \geq 1$. *Then for* $1 < p \leq \infty,\ 1/p + 1/q = 1,\ a \leq c < d \leq b$ *and* $x \in [c,d]$ *we have*

$$\begin{aligned} &\left| \frac{1}{d-c}\int_c^d f(t)dt - \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{n}\sum_{k=1}^{n-1} F_k(x) \right| \\ \leq\ & \frac{1}{n!(b-a)} \Big\{ (x-a)^{nq+1} B_{\frac{c-a}{x-a}}(q+1,(n-1)q+1) \\ & + \frac{(c-a+b-d)^q |x-s_0|^{nq+1}}{(d-c)^q} \left[\Psi_{r_1}(q+1,(n-1)q+1) \right. \\ & + B(q+1,(n-1)q+1) + \Psi_{r_2}((n-1)q+1,q+1) \big] \\ & \left. + (b-x)^{nq+1} B_{\frac{b-d}{b-x}}(q+1,(n-1)q+1) \right\}^{1/q} \cdot \|f^{(n)}\|_p, \end{aligned} \tag{3.1}$$

where $s_0 = (bc-ad)/(c-a+b-d)$ *and for* $s_0 \leq x \leq d$

$$r_1 = \frac{s_0 - c}{x - s_0} \ \ \textit{and} \ \ r_2 = \frac{d-x}{x-s_0},$$

while for $c \leq x \leq s_0$ *we have*

$$r_1 = \frac{d - s_0}{s_0 - x} \ \ \textit{and} \ \ r_2 = \frac{x-c}{s_0 - x}.$$

Here $B(\cdot,\cdot)$ and $B_r(\cdot,\cdot)$ are the Beta and the incomplete Beta function of Euler type given by

$$B(l,s)=\int_0^1 t^{l-1}(1-t)^{s-1}dt,\qquad B_r(l,s)=\int_0^r t^{l-1}(1-t)^{s-1}dt,\qquad l,s>0$$

and

$$\Psi_r(l,s)=\int_0^r t^{l-1}(1+t)^{s-1}dt$$

is a real positive valued integral.
For $p=1$ we have

$$\begin{aligned}&\left|\frac{1}{d-c}\int_c^d f(t)dt-\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{n}\sum_{k=1}^{n-1}F_k(x)\right|\\ &\le \frac{1}{n!(b-a)}\max\Big[(x-m)^{n-1}(m-a),\\ &\frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1}\frac{(c-a+b-d)|x-s_0|^n}{d-c},(M-x)^{n-1}(b-M)\Big]\|f^{(n)}\|_1,\end{aligned}\tag{3.2}$$

where $m=\min\{a+\frac{x-a}{n},c\}$ and $M=\max\{b-\frac{b-x}{n},d\}$.

Moreover, for $p>1$ the inequality (3.1) is sharp and for $p=1$ the inequality (3.2) is best possible.

Proof. Use the identity (2.1) and apply the Hölder inequality to obtain

$$\begin{aligned}&\left|\frac{1}{d-c}\int_c^d f(t)dt-\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{n}\sum_{k=1}^{n-1}F_k(x)\right|\\ &\le\frac{1}{n!}\left(\int_a^b |x-t|^{(n-1)q}\,|k_1(x,t)|^q\,dt\right)^{1/q}\|f^{(n)}\|_p.\end{aligned}\tag{3.3}$$

The integral of the right hand side of (3.3) needs to be calculated. Write it as

$$\begin{aligned}&\int_a^c(x-t)^{(n-1)q}\left(\frac{t-a}{b-a}\right)^q dt+\int_c^x(x-t)^{(n-1)q}\left|\frac{t-a}{b-a}-\frac{t-c}{d-c}\right|^q dt\\ &+\int_x^d(t-x)^{(n-1)q}\left|\frac{t-b}{b-a}-\frac{t-d}{d-c}\right|^q dt+\int_d^b(t-x)^{(n-1)q}\left(\frac{b-t}{b-a}\right)^q dt.\end{aligned}\tag{3.4}$$

For the first of these let $t=a+s(x-a)$. Then

$$\begin{aligned}\int_a^c(x-t)^{(n-1)q}\left(\frac{t-a}{b-a}\right)^q dt &= \frac{(x-a)^{nq+1}}{(b-a)^q}\int_a^{\frac{c-a}{x-a}}(1-s)^{(n-1)q}s^q ds\\ &=\frac{(x-a)^{nq+1}}{(b-a)^q}B_{\frac{c-a}{x-a}}(q+1,(n-1)q+1).\end{aligned}$$

If for the last integral of (3.4) we put $t = b - u(b - x)$, we get

$$\begin{aligned}\int_d^b (t-x)^{(n-1)q}\left(\frac{b-t}{b-a}\right)^q dt &= \frac{(b-x)^{nq+1}}{(b-a)^q}\int_0^{\frac{b-d}{b-x}}(1-u)^{(n-1)q}u^q du\\ &= \frac{(b-x)^{nq+1}}{(b-a)^q}B_{\frac{b-d}{b-x}}(q+1,(n-1)q+1).\end{aligned}$$

Further

$$\int_c^x (x-t)^{(n-1)q}\left|\frac{t-a}{b-a}-\frac{t-c}{d-c}\right|^q dt = \frac{(c-a+b-d)^q}{(b-a)^q(d-c)^q}\int_c^x (x-t)^{(n-1)q}\left|t-s_0\right|^q dt$$

and

$$\int_x^d (t-x)^{(n-1)q}\left|\frac{b-t}{b-a}-\frac{d-t}{d-c}\right|^q dt = \frac{(c-a+b-d)^q}{(b-a)^q(d-c)^q}\int_x^d (t-x)^{(n-1)q}\left|t-s_0\right|^q dt.$$

We have that

$$s_0 - c = \frac{d-c}{c-a+b-d}(c-a) \geq 0,\ d - s_0 = \frac{d-c}{c-a+b-d}(b-d) \geq 0$$

and let first suppose that $s_0 \leq x$. Then with $t = s_0 - r(x - s_0)$ we get

$$\begin{aligned}\int_c^{s_0}(x-t)^{(n-1)q}(s_0-t)^q dt &= (x-s_0)^{nq+1}\int_0^{\frac{s_0-c}{x-s_0}}(1+r)^{(n-1)q}r^q dr\\ &= (x-s_0)^{nq+1}\Psi_{\frac{s_0-c}{x-s_0}}(q+1,(n-1)q+1).\end{aligned}$$

With $t = s_0 + w(x - s_0)$ we have

$$\begin{aligned}\int_{s_0}^x (x-t)^{(n-1)q}(t-s_0)^q dt &= (x-s_0)^{nq+1}\int_0^1 (1-w)^{(n-1)q}w^q dw\\ &= (x-s_0)^{nq+1}B(q+1,(n-1)q+1)\end{aligned}$$

and with $t = x + v(x - s_0)$

$$\begin{aligned}\int_x^d (t-x)^{(n-1)q}(t-s_0)^q dt &= (x-s_0)^{nq+1}\int_0^{\frac{d-x}{x-s_0}} v^{(n-1)q}(1+v)^q dv\\ &= (x-s_0)^{nq+1}\Psi_{\frac{d-x}{x-s_0}}((n-1)q+1,q+1).\end{aligned}$$

If $x \leq s_0$, similarly we get

$$\int_c^x (x-t)^{(n-1)q}(s_0-t)^q dt = (s_0-x)^{nq+1}\Psi_{\frac{x-c}{s_0-x}}((n-1)q+1,q+1),$$

$$\int_x^{s_0} (t-x)^{(n-1)q}(s_0-t)^q dt = (s_0-x)^{nq+1} B(q+1,(n-1)q+1)$$

and

$$\int_{s_0}^{d} (t-x)^{(n-1)q}(t-s_0)^q dt = (s_0-x)^{nq+1} \Psi_{\frac{d-s_0}{s_0-x}}(q+1,(n-1)q+1).$$

So, the inequality (3.1) is proved. Equality holds in (3.1) when

$$f^{(n)}(t) = |(x-t)^{n-1}k^1(t,x)|^{q-1}\mathrm{sgn}\{(x-t)^{n-1}k^1(t,x)\}.$$

For $p=1$, (3.3) is replaced by

$$\begin{aligned}&\left|\frac{1}{d-c}\int_c^d f(t)dt - \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{n}\sum_{k=1}^{n-1} F_k(x)\right| \\ \le\ & \frac{1}{n!}\sup_{t\in[a,b]} |x-t|^{(n-1)}\,|k_1(x,t)|\,\|f^{(n)}\|_1.\end{aligned}$$

By an elementary exercise we have

$$\max_{t\in[a,c]} (x-t)^{n-1}\frac{t-a}{b-a} = \frac{(x-m)^{n-1}(m-a)}{b-a}$$

and

$$\max_{t\in[d,b]} (t-x)^{n-1}\frac{b-t}{b-a} = \frac{(M-x)^{n-1}(b-M)}{b-a}.$$

If $x \ge s_0$ we get

$$\max_{t\in[c,s_0]} (x-t)^{n-1}(s_0-t) = (x-c)^{n-1}(s_0-c),$$

$$\max_{t\in[s_0,x]} (x-t)^{n-1}(t-s_0) = \frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1}(x-s_0)^n$$

and

$$\max_{t\in[x,d]} (t-x)^{n-1}(t-s_0) = (d-x)^{n-1}(d-s_0).$$

Also, for $x \le s_0$ we have

$$\max_{t\in[c,x]} (x-t)^{n-1}(s_0-t) = (x-c)^{n-1}(s_0-c),$$

$$\max_{t\in[x,s_0]} (t-x)^{n-1}(s_0-t) = \frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1}(s_0-x)^n$$

and

$$\max_{t\in[s_0,d]}(t-x)^{n-1}(t-s_0)=(d-x)^{n-1}(d-s_0).$$

So, the inequality (3.2) holds. To argue that this is best possible one should take

$$f_{\varepsilon}^{(n)}(t)=\begin{cases}\varepsilon^{-1}, & t\in(t_0-\varepsilon,t_0);\\ 0, & else,\end{cases}$$

where t_0 is the point that gives maximum. ∎

Remark 2. If we put $c=d=x$ as a limit case in (3.1) and (3.2), we get the Ostrowski inequality (see (1.7) and (1.8)).

Corollary 1. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n\geq 1$. Then for $a\leq c<d\leq b$, we have*

$$\begin{aligned}&\left|\frac{1}{d-c}\int_c^d f(t)dt-\frac{1}{b-a}\int_a^b f(t)dt+\frac{1}{n}\sum_{k=1}^{n-1}F_k(x)\right|\\ &\leq \frac{L}{n(n+1)!}\left[\frac{(x-a)^{n+1}}{b-a}+\frac{(b-x)^{n+1}}{b-a}\right.\\ &\left.-\frac{(x-c)^{n+1}}{d-c}-\frac{(d-x)^{n+1}}{d-c}+\frac{2(c-a+b-d)|x-s_0|^{n+1}}{(b-a)(d-c)}\right],\end{aligned}\tag{3.5}$$

for every $x\in[c,d]$.

Proof. For integrable function $F:[a,b]\to\mathbf{R}$ we have

$$\left|\int_a^b F(t)df^{(n-1)}(t)\right|\leq L\int_a^b|F(t)|\,dt,$$

since $f^{(n-1)}$ is L-Lipschitzian function. So, we use the Theorem 4 with $p=\infty$ and by integration by parts we have

$$B(2,n)=\frac{1}{n(n+1)},\; B_r(2,n)=-\frac{r(1-r)^n}{n}-\frac{(1-r)^{n+1}}{n(n+1)}+\frac{1}{n(n+1)},$$

$$\Psi_r(2,n)=\frac{r(1+r)^n}{n}-\frac{(1+r)^{n+1}}{n(n+1)}+\frac{1}{n(n+1)},\;\Psi_r(n,2)=\frac{r^n(1+r)}{n}-\frac{r^{n+1}}{n(n+1)}.$$

Then with easy calculation we get the inequality (3.5). ∎

Remark 3. If we put $c=d=x$ as a limit case in the inequality (3.5), we get the Ostrowski inequality for L-Lipschitzian function (see (1.6)).

Now we will consider the inequalities from Theorem 4 and Corollary 1 in case when $n = 1$.

Corollary 2. *Let $f : [a,b] \to \mathbf{R}$ be such that $f' \in L_p[a,b]$. Then for $1 < p \le \infty,\ 1/p + 1/q = 1$ and $a \le c < d \le b$, we have inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{d-c}\int_c^d f(t)dt\right| \le \left[\frac{(c-a)^{q+1} + (b-d)^{q+1}}{(q+1)(b-a)^{q-1}(c-a+b-d)}\right]^{1/q} \|f'\|_p, \tag{3.6}$$

while for $p = 1$ we have

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{d-c}\int_c^d f(t)dt\right| \le \frac{c-a+b-d+|c-a-b+d|}{2(b-a)}\|f'\|_1. \tag{3.7}$$

Proof. Put $n=1$ in the inequalities from Theorem 4. See also [3], [9] and [1].

Remark 4. If $p = \infty$ we get

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{d-c}\int_c^d f(t)dt\right| \le \frac{(c-a)^2 + (b-d)^2}{2(c-a+b-d)}L,$$

which is the inequality (1.2) for $c = d = x$. See also [6] and [2].

3.2 Case $[a,b] \cap [c,d] = [c,b]$.

Theorem 5. *Let $f : [a,d] \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous and let $f^{(n)} \in L_p[a,d]$. Then for $1 < p \le \infty,\ 1/p + 1/q = 1,\ a \le c < b \le d$ and $x \in [c,b]$ we have*

$$\begin{aligned}
&\left|\frac{1}{d-c}\int_c^d f(t)dt - \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{n}\sum_{k=1}^{n-1} F_k(x)\right| \\
\le\ & \frac{1}{n!}\left\{\frac{(x-a)^{nq+1}}{(b-a)^q} B_{\frac{c-a}{x-a}}(q+1,(n-1)q+1)\right. \\
&+ \frac{(c-a+b-d)^q (s_0-x)^{nq+1}}{(b-a)^q (d-c)^q}\left[\Psi_{\frac{x-c}{s_0-x}}((n-1)q+1,q+1)\right. \\
&\left.+ B_{\frac{b-x}{s_0-x}}((n-1)q+1,q+1)\right] \\
&\left.+ \frac{(d-x)^{nq+1}}{(d-c)^q} B_{\frac{d-b}{d-x}}(q+1,(n-1)q+1)\right\}^{1/q} \cdot \|f^{(n)}\|_p.
\end{aligned} \tag{3.8}$$

For $p = 1$ we have

$$\left| \frac{1}{d-c} \int_c^d f(t)dt - \frac{1}{b-a} \int_a^b f(t)dt + \frac{1}{n} \sum_{k=1}^{n-1} F_k(x) \right|$$

$$(3.9) \quad \leq \frac{1}{n!} \max \left\{ \frac{(x-m)^{n-1}(m-a)}{b-a}, \frac{(c-a+b-d)}{(b-a)(d-c)}(m_1-x)^{n-1}(s_0-m_1), \frac{(M_1-x)^{n-1}(d-M_1)}{d-c} \right\} \|f^{(n)}\|_1,$$

where m as in Theorem 4, $m_1 = \min\{x + \frac{s_0-x}{n}, b\}$ and $M_1 = \max\{d - \frac{d-x}{n}, b\}$.

Moreover, for $p > 1$ the inequality (3.8) is sharp and for $p = 1$ the inequality (3.9) is best possible.

Proof. Because we have that

$$s_0 - c = \frac{d-c}{c-a+b-d}(c-a) \geq 0 \quad \text{and} \quad s_0 - b = \frac{b-a}{c-a+b-d}(d-b) \geq 0,$$

without loss of generality we can take that $c-a+b-d > 0$ and then $s_0 - c > 0$ and $s_0 - b > 0$. Now, similarly as in the Theorem 4 we have

$$\int_a^c (x-t)^{(n-1)q} \left(\frac{t-a}{b-a}\right)^q dt = \frac{(x-a)^{nq+1}}{(b-a)^q} B_{\frac{c-a}{x-a}}(q+1, (n-1)q+1),$$

$$\int_c^x (x-t)^{(n-1)q} \left| \frac{t-a}{b-a} - \frac{t-c}{d-c} \right|^q dt$$
$$= \frac{(c-a+b-d)^q}{(b-a)^q(d-c)^q} \int_c^x (x-t)^{(n-1)q}(s_0-t)^q dt$$
$$= \frac{(c-a+b-d)^q(s_0-x)^{nq+1}}{(b-a)^q(d-c)^q} \Psi_{\frac{x-c}{s_0-x}}((n-1)q+1, q+1),$$

$$\int_x^b (t-x)^{(n-1)q} \left| \frac{t-b}{b-a} - \frac{t-d}{d-c} \right|^q dt$$
$$= \frac{(c-a+b-d)^q}{(b-a)^q(d-c)^q} \int_x^b (t-x)^{(n-1)q}(s_0-t)^q dt$$
$$= \frac{(c-a+b-d)^q(s_0-x)^{nq+1}}{(b-a)^q(d-c)^q} B_{\frac{b-x}{s_0-x}}((n-1)q+1, q+1)$$

and

$$\int_b^d (t-x)^{(n-1)q} \left(\frac{d-t}{d-c}\right)^q dt = \frac{(d-x)^{nq+1}}{(d-c)^q} B_{\frac{d-b}{d-x}}(q+1, (n-1)q+1).$$

For $p = 1$ we have

$$\max_{t\in[a,c]}(x-t)^{n-1}\left(\frac{t-a}{b-a}\right) = \frac{(x-m)^{n-1}(m-a)}{b-a},$$

$$\begin{aligned}\max_{t\in[c,x]}(x-t)^{n-1}\left|\frac{t-a}{b-a}-\frac{t-c}{d-c}\right| &= \frac{c-a+b-d}{(b-a)(d-c)}\max_{t\in[c,x]}(x-t)^{n-1}(s_0-t)\\ &= \frac{(x-c)^{n-1}(c-a)}{b-a},\end{aligned}$$

$$\begin{aligned}\max_{t\in[x,b]}(t-x)^{n-1}\left|\frac{t-b}{b-a}-\frac{t-d}{d-c}\right| &= \frac{c-a+b-d}{(b-a)(d-c)}\max_{t\in[x,b]}(t-x)^{n-1}(s_0-t)\\ &= \frac{(c-a+b-d)}{(b-a)(d-c)}(m_1-x)^{n-1}(s_0-m_1)\end{aligned}$$

and

$$\max_{t\in[b,d]}(t-x)^{n-1}\left(\frac{d-t}{d-c}\right) = \frac{(M_1-x)^{n-1}(d-M_1)}{d-c}.$$

Proof of sharpness and best possibility is the same as in the Theorem 4. ∎

Remark 5. If we put $c = b = x$ in inequalities (3.8) and (3.9) we get

$$\begin{aligned}&\left|\frac{1}{d-x}\int_x^d f(t)dt - \frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{n}\sum_{k=1}^{n-1}\hat{F}_k(x)\right|\\ &\le \frac{1}{n!}\left\{\left[(x-a)^{(n-1)q+1}+(d-x)^{(n-1)q+1}\right]B(q+1,(n-1)q+1)\right\}^{1/q}\cdot\|f^{(n)}\|_p\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{d-x}\int_x^d f(t)dt - \frac{1}{x-a}\int_a^x f(t)dt + \frac{1}{n}\sum_{k=1}^{n-1}\hat{F}_k(x)\right|\\ &\le \frac{(n-1)^{n-1}}{n^n n!}\max\left[(x-a)^{n-1},(d-x)^{n-1}\right]\|f^{(n)}\|_1,\end{aligned}$$

where

$$\hat{F}_k(x) = \frac{n-k}{k!}\left(\frac{f^{(k-1)}(a)(x-a)^k}{x-a} + \frac{f^{(k-1)}(d)(x-d)^k}{d-x}\right).$$

Corollary 3. *Let $f : [a,d] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,d]$ for some $n \ge 1$. Then for $a \le c < b \le d$, we have*

$$\begin{aligned}&\left|\frac{1}{d-c}\int_c^d f(t)dt - \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{n}\sum_{k=1}^{n-1}F_k(x)\right|\\ &\le \frac{L}{n(n+1)!}\left[\frac{(x-a)^{n+1}}{b-a}-\frac{(b-x)^{n+1}}{b-a}-\frac{(x-c)^{n+1}}{d-c}+\frac{(d-x)^{n+1}}{d-c}\right]\end{aligned} \tag{3.10}$$

for every $x \in [c, b]$.

Proof. We use Theorem 5 with $p = \infty$ and by partial integration and with easy calculation we get the inequality (3.10). ∎

Remark 6. If we put $c = b = x$ in the inequality (**??**), we get

$$\left| \frac{1}{d-x} \int_x^d f(t)dt - \frac{1}{x-a} \int_a^x f(t)dt + \frac{1}{n} \sum_{k=1}^{n-1} \hat{F}_k(x) \right| \leq \frac{L}{n(n+1)!} \left[(x-a)^n + (d-x)^n \right].$$

Corollary 4. *Let* $f : [a, d] \to \mathbf{R}$ *be such that* $f' \in L_p[a, d]$. *Then for* $1 < p \leq \infty,\ 1/p + 1/q = 1,\ a \leq c < b \leq d$ *we have inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{d-c} \int_c^d f(t)dt \right| \leq \left\{ \frac{1}{(q+1)(c-a+b-d)} \left[\frac{(c-a)^{q+1}}{(b-a)^{q-1}} - \frac{(d-b)^{q+1}}{(d-c)^{q-1}} \right] \right\}^{1/q} \|f'\|_p, \tag{3.11}$$

while for $p = 1$ *we have*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{d-c} \int_c^d f(t)dt \right| \leq \frac{1}{2} \left[\frac{c-a}{b-a} + \frac{d-b}{d-c} + \left| \frac{c-a}{b-a} - \frac{d-b}{d-c} \right| \right] \|f'\|_1. \tag{3.12}$$

Proof. Put $n = 1$ in the inequalities in Theorem 5. See also [1]. ∎

Remark 7. For $c = b = x$ we have

$$\left| \frac{1}{x-a} \int_a^x f(t)dt - \frac{1}{d-x} \int_x^d f(t)dt \right| \leq \left[\frac{d-a}{(q+1)} \right]^{1/q} \|f'\|_p,$$

and

$$\left| \frac{1}{x-a} \int_a^x f(t)dt - \frac{1}{d-x} \int_x^d f(t)dt \right| \leq \|f'\|_1.$$

See also [1].

Remark 8. If $p = \infty$ we get

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{d-c} \int_c^d f(t)dt \right| \leq \frac{c-a+d-b}{2} L$$

and for $c = b = x$ we have (see [1])

$$\left|\frac{1}{x-a}\int_a^x f(t)dt - \frac{1}{d-x}\int_x^d f(t)dt\right| \leq \frac{d-a}{2}L.$$

4. On Some Further Generalizations

In [4] Lj. Dedić, J. Pečarić and N. Ujević generalized Milovanović-Pečarić-Fink inequality using harmonic sequence of polynomials, where (P_n) is a harmonic sequence of polynomials, if $P_n' = P_{n-1},\ n \geq 1,\ P_0 = 1.$

Theorem 6. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 1$ and let $f^{(n)} \in L_p[a,b]$ for $1 \leq p \leq \infty,\ 1/p + 1/q = 1.$ Then the inequality*

$$\left|\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1}\tilde{F}_k^{[a,b]}\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq C(n,p,x)\|f^{(n)}\|_p \tag{4.1}$$

holds for $x \in [a,b]$, and

$$C(n,p,x) = \frac{1}{n(b-a)}\|P_{n-1}k^{[a,b]}(\cdot,x)\|_q, \tag{4.2}$$

where is

$$\tilde{F}_k^{[a,b]} = \frac{(-1)^k(n-k)}{b-a}[P_k(a)f^{(k-1)}(a) - P_k(b)f^{(k-1)}(b)].$$

This inequality follows from the identity

$$\frac{1}{n}\left[f(x)+\sum_{k=1}^{n-1}(-1)^k P_k(x) f^{(k)}(x) + \sum_{k=1}^{n-1}\tilde{F}_k^{[a,b]}\right] - \frac{1}{b-a}\int_a^b f(t)dt = \frac{(-1)^{n-1}}{n(b-a)}\int_a^b P_{n-1}(t)k^{[a,b]}(t,x)f^{(n)}(t)dt. \tag{4.3}$$

Now we will give two theorems which generalize Theorem 2 and the Theorem 3.

Theorem 7. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous function on $[a,b]$ for some $n \geq 1$. Then if $a \leq c < d \leq b$, for every $x \in [c,d]$*

$$\frac{1}{d-c}\int_c^d f(t)dt - \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{n}\sum_{k=1}^{n-1}\tilde{F}_k = \frac{1}{n}\int_a^b P_{n-1}(t)k_1(t,x)f^{(n)}(t)dt \tag{4.4}$$

where

$$\tilde{F}_k = (-1)^k(n-k)\left(\frac{P_k(a)f^{(k-1)}(a) - P_k(b)f^{(k-1)}(b)}{b-a} - \frac{P_k(c)f^{(k-1)}(c) - P_k(d)f^{(k-1)}(d)}{d-c}\right)$$

and $k_1(t,x)$ *as in Theorem* 2.

Proof. First we write the identity (4.3) for interval $[a,b]$ and for interval $[c,d]$. Then we subtract them and get the above statements. ∎

Theorem 8. *Let* $f : [a,d] \to \mathbf{R}$ *be such that* $f^{(n-1)}$ *is absolutely continuous function on* $[a,d]$ *for some* $n \geq 1$. *Then if* $a \leq c < b \leq d$, *for every* $x \in [c,b]$

$$\frac{1}{d-c}\int_c^d f(t)dt - \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{n}\sum_{k=1}^{n-1}\tilde{F}_k = \frac{1}{n}\int_a^d P_{n-1}(t)k_2(t,x)f^{(n)}(t)dt \tag{4.5}$$

where $\tilde{F}_k$ *is as in Theorem* 7 *and* $k_2(t,x)$ *as in Theorem* 3.

Proof. Similar to the proof of Theorem 7.

Remark 9. From Theorem 7 and Theorem 8 now follow the generalizations of Theorem 4 and Theorem 5. Also we can note that Theorems 2 and 3 are special case of Theorems 7 and 8 when $P_k(t) = (t-x)^k/k!$.

References

1. A. Aglić Aljinović, J. Pečarić, and I. Perić, Estimations of the difference of two weighted integral means via Montgomery identities, *Math. Inequal. Appl.*, **7(3)** (2004), 315-336.
2. N. S. Barnett, P. Cerone, S. S. Dragomir, and A. M. Fink, Comparing two integral means for absolutely continuous mappings whose derivatives are in $L_\infty[a,b]$ and applications, *Comput. Math. Appl.*, **44** (2002), 241-251.
3. P. Cerone and S. S. Dragomir, Diferences between means with bounds from a Riemann-Stieltjes integral, *RGMIA Res. Rep. Coll.*, **4(2)** (2001).
4. Lj. Dedić, J. Pečarić, and N. Ujević, On generaliations of Ostrowski inequality and some related results, *Czechoslovak Math. J.*, **53(128)** (2003), 173-189.

5. A. M. Fink, Bounds of the deviation of a function from its avereges, *Czechoslovak Math. J.*, **42(117)** (1992), 289-310.

6. M. Matić, J. Pečarić, Two-point Ostrowski inequality, *Math. Inequal. Appl.*, **4** (2001), 215-221.

7. G. V. Milovanović, and J. Pečarić, On generalizations of the inequality of A. Ostrowski and some related applications, *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz.*, **544-576** (1976), 155-158.

8. A. Ostrowski, Über die Absolutabweichung einer differentiebaren Funktion von ihren Integralmittelwert, *Comment. Math. Helv.*, **10** (1938), 226-227.

9. J. Pečarić, I. Perić, and A. Vukelić, Estimations of the difference of two integral means via Euler-type identities, *Math. Inequal. Appl.*, (to appear).

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