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UPPER GENERALIZED EXPONENTS OF MINISTRONG DIGRAPHS

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Abstract. We obtain upper bounds for the upper generalized exponents of digraphs in the class of ministrong digraphs and in the class of non-primitive ministrong digraphs, characterize the corresponding extremal digraphs, and discuss the numbers attainable as upper generalized exponents of ministrong digraphs.

1. INTRODUCTION

A digraph G is primitive if there exists a positive integer k such that there is a walk of length k from u to v for each ordered pair of vertices u and v (including $u = v$). The smallest such k is called the exponent of G, denoted by $\exp(G)$. Exponents for primitive digraphs have been studied extensively due to their intrinsic importance in graph theory, combinatorics, matrix theory and their applications in communication problems.

Brualdi and Liu [1] introduced the concept of upper generalized exponents for primitive digraphs as a generalization of exponents in 1990. Recently, Shao, Hwang and Wu [9, 10] extended this concept of upper generalized exponents from primitive digraphs to general digraphs.

Definition 1. [9, 10] Let G be a digraph and $X \subseteq V(G)$ be a subset of the vertex set $V(G)$. The exponent of the subset X, denoted by $\exp_G(X)$, is defined to be the smallest positive integer m such that for each vertex y of G, there is a walk of length m from at least one vertex in X to y. If no such m exists, then define $\exp_G(X) = \infty$.

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Definition 2. [9, 10] Let G be a digraph of order n with $1 \le k \le n$. Define $F(G,k) = \max\{\exp_G(X) : X \subseteq V(G)$, and $|X| = k\}$. $F(G,k)$ is called the k-th upper generalized exponent of G.

Definition 3. [10] A digraph G is called k-upper primitive if $F(G,k) < \infty$.

Clearly 1-upper primitive digraphs are just primitive digraphs and in this case $F(G,1) = \exp(G)$. We mention here that the upper generalized exponents also have an interpretation in the model of memoryless communication networks (see [1]).

Shao and Hwang [9] have obtained sharp upper bounds for the k-th upper generalized exponents of k-upper primitive symmetric digraphs and determined the corresponding set of upper generalized exponents, while Shao and Wu [10] have obtained a necessary and sufficient condition for a digraph to be k-upper primitive (See Lemma 1 below). In [14], we have recently obtained sharp upper bounds for the k-th upper generalized exponents of k-upper primitive digraphs and characterized the extremal case.

A strongly connected (or strong) digraph D is called ministrong provided each digraph obtained from D by removal of any arc is not strong. The set of all k-upper primitive ministrong digraphs of order n is denoted by $U(n,k)$. It is obvious that $U(n,1) \subseteq U(n,2) \subseteq \cdots \subseteq U(n,n)$, and that $U(n,1)$ is just the set of all primitive ministrong digraphs of order n. Let $E(n,k)$ be the set of k-th upper generalized exponents for the digraphs in $U(n,k)$, i.e., $E(n,k) = \{F(G,k) : G \in U(n,k)\}$.

The upper generalized exponents of primitive digraphs have been studied in [1, 4-6]. The exponents and upper generalized exponents of primitive ministrong digraphs have been studied in [2, 3, 7, 8, 11, 14]. In this paper, we obtain upper bounds for the k-th upper generalized exponents of digraphs in class $U(n,k)$ with $1 \le k \le n$ and digraphs in the class of non-primitive digraphs of $U(n,k)$ with $2 \le k \le n$ respectively, characterize completely the extremal digraphs, i.e., those digraphs whose k-th upper generalized exponents achieve the corresponding upper bounds, and investigate which numbers can be in $E(n,k)$.

If A is an $n \times n$ nonnegative matrix, then the digraph of A, $D(A) = (V,E)$, is the digraph with vertex set $V = \{1,2,\ldots,n\}$ and arc set $E = \{(i,j) : a_{ij} > 0\}$. It is well known that (see [7]):

(a) $D(A)$ is a primitive digraph if and only if A is a primitive matrix;

(b) $D(A)$ is strong if and only if A is an irreducible matrix;

(c) $D(A)$ is ministrong if and only A is a nearly reducible matrix.

Hence results in this paper can be expressed in their corresponding matrix versions.

2. Preliminary Results

Let G be a k-upper primitive digraph. We say vertex u is a t-in vertex of vertex v if there is a walk of length t from u to v, and the set of all t-in vertices of v in G is denoted by $R_G(t, v)$. Then $|R_G(t, v)| \geq n - k + 1$ for all $v \in V(G)$ implies $\exp_G(X) \leq t$ for any $X \subseteq V(G)$ with $|X| = k$ and hence $F(G, k) \leq t$. Denote the distance from vertex u to vertex v in a strong digraph G by $d_G(u, v)$, and the set of all distinct cycle lengths of G by $L(G)$.

Suppose G is a strong digraph with period $p(G) = p$, where $p(G)$ is the gcd (greatest common divisor) of all the cycle lengths of G. Then the vertices of G can be partitioned into p nonempty sets V_1, V_2, …, V_p where the arcs originating in V_i enter V_{i+1} (V_{p+1} is interpreted as V_1). Such a partition of $V(G)$ is called the imprimitive partition of G.

For a digraph G and a positive integer m, let G^m be the digraph with the same vertex set as G such that there is an arc from vertex x to vertex y in G^m if and only if there is a walk of length m from x to y in G. It is well known that if G is a strong digraph with period p then G^p is a disjoint union of p primitive subdigraphs with vertex sets V_1, V_2, $\cdots$, V_p respectively, where $V_1 \cup V_2 \cup \cdots \cup V_p$ is the imprimitive partition of G.

The following lemma was given in [10] for general digraphs. For completeness, however, we include a direct proof here.

Lemma 1. *Let G be a strong digraph of order n, and let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_p$ be the imprimitive partition of G. Then G is k-upper primitive if and only if $k > n - \min\{|V_i| : 1 \leq i \leq p\}$.*

Proof. If G is k-upper primitive, then for any $X \subseteq V(G)$ with $|X| = k$, $\exp_G(X) < \infty$, and hence $X \cap V_i \neq \emptyset$, $1 \leq i \leq p$, which implies $k > n - \min\{|V_i| : 1 \leq i \leq p\}$.

Conversely, suppose $k > n - \min\{|V_i| : 1 \leq i \leq p\}$. Let $X \subseteq V(G)$ with $|X| = k$. Then $X \cap V_i \neq \emptyset$ for any i with $1 \leq i \leq p$. Let $X' = \{x_1, x_2, \ldots, x_p\}$ with $x_i \in X \cap V_i$ for $1 \leq i \leq p$. Clearly, $X' \subseteq X$. Note that G^p is a disjoint union of p primitive digraphs with vertex sets $V_1, V_2, \ldots, V_p$ respectively. Let γ be the largest value of the exponents of these p primitive digraphs. Then in G every vertex of V_i can be reachable by a walk of length $p\gamma$ for $1 \leq i \leq p$. This implies $\exp_G(X) \leq \exp_G(X') \leq p\gamma < \infty$. ■

Suppose G is strong and non-primitive with imprimitive partition $V_1 \cup V_2 \cup \cdots \cup V_p$ where $p = p(G) \geq 2$. It follows from Lemma 1 that if G is k-upper primitive, then $|V_i| \geq n - k + 1$ for $1 \leq i \leq p$, and so $n = |V_1| + |V_2| + \cdots + |V_p| \geq 2(n - k + 1)$, i.e., $k \geq n/2 + 1$. Hence if we study the k-th upper generalized exponents for non-primitive digraphs, we only consider the case $n/2 + 1 \leq k \leq n$.

The following lemma is a generalization of [4, Lemma 2], where it was proved for primitive digraphs. Here we extend it to all k-upper primitive digraphs.

Lemma 2. *Let G be a strong k-upper primitive digraph of order n with $1 \leq k \leq n-1$, and let h be the smallest cycle length of G. Then $F(G,k) \leq h(n-k-1)+n$.*

Proof. Let $X \subseteq V(G)$ with $|X| = k$. By Lemma 1, we have $X \cap V_i \neq \emptyset$ for $1 \leq i \leq p$. Let $X \cap V_i = X_i$ and $|X_i| = k_i$, $1 \leq i \leq p$. For any vertex y in G, there exists a vertex, say $z \in V_j$ for some j, in a cycle of length h such that there is a walk of length $n-h$ from z to y.

Since $p|h$, G^h is a disjoint union of p primitive digraphs with vertex sets $V_1, \ldots, V_p$ respectively. Let P_j be the strong component of G^h with vertex set V_j. Then there is a loop on z in P_j. So there is a vertex $x \in X_j \subseteq X$ such that there is a walk of length $|V_j| - k_j$, and hence of length $n-k$ from x to z in P_j since $n-k \geq |V_i| - k_i$, which implies that there is a walk of length $h(n-k)$ from x to z in G. Hence there is a walk of length $n-h+h(n-k) = h(n-k-1)+n$ from x to y in G, and $F(G,k) = \max\{\exp_G(X)\colon X \subseteq V(G), \text{ and } |X| = k\} \leq s(n-k-1)+n$.

Let $G_{n,s}$ be the digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 1 \leq i \leq n-2\} \cup \{(n-1, 1), (n, 2), (s, n)\}$ where $2 \leq s \leq n-2$. Clearly $G_{n,n-3}$ is primitive if n is even. We have by Lemma 1 that $G_{n,n-3}$ is k-upper primitive if and only if $k \geq (n+3)/2$ when n is odd. If $n \equiv 1 \pmod 3$, then by Lemma 1 again $G_{n,n-4}$ is k-upper primitive if and only if $k \geq (2n+1)/3+1$. Let

$$F(n,k) = \begin{cases} n^2 - 4n + 6 & \text{if } k = 1, \\ (n-1)^2 - k(n-2) & \text{if } 2 \leq k \leq n. \end{cases}$$

Lemma 3. [14] *For $1 \leq k \leq n$, we have $F(G_{n,n-2}, k) = F(n,k)$.*

Lemma 4. *For $(n+2)/2 \leq k \leq n-2$, we have $F(G_{n,n-3}, k) = F(n,k) - (n-k-1)$.*

Proof. Let $G = G_{n,n-3}$, $t = F(n,k) - (n-k-1) = (n-1)(n-2) - k(n-3)$ and $k = n-r$. Then $r \leq (n-2)/2$. As may be verified, we have

$$R_G(t,1) = \{n, 1, 3, \ldots, 2r-1\},$$
$$R_G(t,2) = \{n-3, n-1, 2, 4, \ldots, 2r\},$$
$$R_G(t,3) = \{n, 1, 3, 5, \ldots, 2r+1\},$$
$$R_G(t,i) = \{i-2, i, i+2, \ldots, i+2r-2\}, \quad 4 \leq i \leq n-2r+1,$$
$$R_G(t, n-2r+j)$$

$$= \begin{cases} \{n-2r+j-2, n-2r+j, \ldots, n-2, 1, 3, \ldots, j-1\} \\ \qquad\qquad \text{if } n+j \text{ is odd and } 2 \le j \le 2r-2, \\ \{n-2r+j-2, n-2r+j, \ldots, n-1, 2, 4, \ldots, j-1\} \\ \qquad\qquad \text{if } n+j \text{ is even and } 3 \le j \le 2r-1, \end{cases}$$

$$R_G(t,n) = \{n-4, n-2, n, 1, \ldots, 2r-3\}.$$

Hence $|R_G(t,i)| \ge r+1 = n-k+1$ for all $i \in V(G)$. This implies $F(G,k) \le t$.

On the other hand, let $X_0 = V(G) \backslash \{2,4,\ldots,2r\}$. Clearly $|X_0| = k$. Since $R_G(t-1,1) = \{2,4,\ldots,2r\}$, there is no walk of length $t-1$ from any vertex in X_0 to vertex 1 and hence $F(G,k) \ge \exp_G(X_0) \ge t$. It follows that $F(G,k) = t = F(n,k) - (n-k-1)$. ■

Lemma 5. *For $(2n+1)/3 + 1 \le k \le n-2$, we have $F(G_{n,n-4}, k) = F(n,k) - 2(n-k-1)$.*

Proof. Let $G = G_{n,n-4}$, $t = (n-1)(n-3) - k(n-4)$. By similar arguments as in Lemma 4, we have $|R_G(t,i)| \ge n-k+1$ for all $i \in V(G)$, $\exp_G(X_0) \ge t$ where $X_0 = V(G) \backslash \{2,5,\ldots,3(n-k)-1\}$ and hence $F(G,k) = t = F(n,k) - 2(n-k-1)$, as desired. ■

Lemma 6. *Let G be a strong $(n-1)$-upper primitive digraph of order n with $|L(G)| \ge 2$, and let h and t be respectively the smallest and the largest cycle lengths of G. Then $F(G,n-1) \le \max\{n-h, t\}$.*

Proof. Let $X \subseteq V(G)$ with $|X| = n-1$.

Case 1. X contains a cycle C. Suppose the length of C is r, where $h \le r \le t$. Then every vertex of G is reachable from some vertex of C, and hence from some vertex in X by a walk of length $n-h$ since $n-h \ge n-r$.

Case 2. X contains no cycle. Let $V(G) \backslash X = \{u\}$. Then u lies on every cycle of G. Take a cycle C_t of length t. Then all vertices except u are reachable from some vertex in $V(C_t) \backslash \{u\}$, and hence from some vertex in X, by a walk of length t. By Lemma 1, G must contain a cycle with length less than t. Suppose G contains a cycle C' with length q where $h \le q < t$. Write $t = mq + r$, where r is an integer with $1 \le r \le q$. Clearly, there is a vertex $x \in V(C_t) \backslash \{u\}$ such that there is a path of length r from x to u in C_t. By attaching the cycle C' m times to this path, we get a walk of length t from $x \in X$ to u.

It follows that every vertex of G is reachable by a walk of length $\max\{n-s, t\}$ from some vertex in X, which implies that $\exp_G(X) \le \max\{n-s, t\}$, and hence $F(G, n-1) \le \max\{n-s, t\}$. ■

Let $H_{n,s}$ where $3 \le s \le n-1$ be the digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 1 \le i \le n-1\} \cup \{(2,1), (s,2), (n,3)\}$. Clearly $L(H_{n,s}) = \{2, s-1, n-2\}$. If $H_{n,s}$ is non-primitive, then n is even, s is odd, and $p(H_{n,s}) = 2$. By Lemma 1, $H_{n,s}$ is k-upper primitive if and only if $k \ge n/2+1$. Let H_n^1 where $n \ge 5$ be the digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 2 \le i \le n-1\} \cup \{(1,3), (3,1), (3,2), (n,3)\}$, and let H_n^2 where $n \ge 6$ be the digraph with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 1 \le i \le n-1\} \cup \{(3,1), (n,3)\}$.

Lemma 7. *If G is one of the digraph $H_{n,s}$ ($n \ge 4$), H_n^1 ($n \ge 5$) or H_n^2 ($n \ge 6$), then $F(G, n-1) = n-2$.*

Proof. It follows from Lemma 6 that $F(G, n-1) \le n-2$. Conversely, Since there is no walk of length $n-3$ from any vertex in $X_0 = V(G)\backslash\{3\}$ to vertex n, we have $F(G, n-1) \ge \exp_G(X_0) \ge n-2$. ∎

Lemma 8. *For any $G \in U(n,k)$ with $1 \le k \le n-1$, $F(G,k) \ge 2$.*

Proof. Let $G \in U(n,k)$. Then there exists a vertex $v \in V(G)$ such that its indegree (also outdegree) is 1. Let (u,v) be the unique arc incident to v. Take $X_0 \subseteq V(G)\backslash\{u\}$ with $|X_0| = k$, we have $F(G,k) \ge \exp_G(X_0) \ge 2$. ∎

Lemma 9 is a generalization of [8, Lemma 2.3].

Lemma 9. $E(n,k) \subseteq E(n+1, k+1)$.

Proof. Let $m \in E(n,k)$. Then there exists a digraph $G \in U(n,k)$ with $F(G,k) = m$. Hence for any subset $X \subseteq V(G)$ with $|X| = k$ we have $\exp_G(X) \le m$, and there exists a subset $X_0 \subseteq V(G)$ with $|X_0| = k$ such that $\exp_G(X_0) = m$. Adding a new vertex u to G such that u has the same adjacency relations as some vertex in X_0, we get a digraph G_1. Clearly G_1 is ministrong. Since $G \in U(n,k)$, we know that $G_1 \in U(n+1, k+1)$.

Let $X_1 \subseteq V(G_1)$ be any subset of $V(G_1)$ with $|X_1| = k+1$. Then we have $\exp_{G_1}(X_1) \le \exp_G(X_1\backslash\{u\}) \le m$ and $\exp_{G_1}(X_0 \cup \{u\}) = \exp_G(X_0) = m$. It follows that $m = F(G_1, k+1) \in E(n+1, k+1)$. ∎

3. Upper Bounds and Extremal Digraphs

In this section, we give upper bounds and corresponding extremal digraphs for the k-th upper generalized exponents of digraphs in $U(n,k)$ and digraphs in $U(n,k)\backslash U(n,1)$ respectively.

Theorem 1. *For $1 \le k \le n$, $\max\{F(G,k) : G \in U(n,k)\} = F(n,k)$.*

Proof. Let h and t be respectively the smallest and the largest cycle lengths of G and $p(G) = p$. Suppose that $V_1 \cup V_2 \cup \cdots \cup V_p$ is the imprimitive partition of G.

Case 1. $k \geq n-1$ or $k = 1$. It is obvious that $F(G, n) = 1 = F(n, n)$. If $k = n-1$, we have $t \leq n-1$ by Lemma 1 and hence $F(G, n-1) \leq \max\{n-h, t\} \leq n-1 = F(n, n-1)$ by Lemma 6. If $k = 1$ (i.e., G is a primitive ministrong digraph), it has been proved in [2] that $F(G, 1) = \exp(G) \leq n^2 - 4n + 6 = F(n, 1)$.

Case 2. $2 \leq k \leq n-2$.

First suppose that G is non-primitive. By Lemma 1, $h \leq n-2$. If $h = n-2$, then $n-1, n \notin L(G)$ since G is non-primitive and ministrong. Hence $p = p(G) = n-2$ and $\min\{|V_i| : 1 \leq i \leq p\} \leq 2$. By Lemma 1, $k > n - \min\{|V_i| : 1 \leq i \leq n-2\} \geq n-2$, a contradiction. Hence we have $h \leq n-3$ if G is non-primitive.

Suppose that G is primitive. Then $h \leq n-2$. If $h = n-2$, then it can be easily checked that G must be isomorphic to $G_{n,n-2}$.

It follows that $h \leq n-3$ or G is isomorphic to $G_{n,n-2}$ for $2 \leq k \leq n-2$.

Case 2.1. $h \leq n-3$. By Lemma 2,

$$\begin{aligned} F(G,k) &\leq n + h(n-k-1) \leq n + (n-3)(n-k-1) \\ &\leq (n-1)^2 - k(n-2) = F(n,k). \end{aligned}$$

Case 2.2. G is isomorphic to $G_{n,n-2}$. By Lemma 3, we have $F(G, k) = F(G_{n,n-2}, k) = F(n, k)$.

Combining Cases 1 and 2, we have $F(G, k) \leq F(n, k)$ for $1 \leq k \leq n$. From Case 2.2, the upper bound $F(n, k)$ can be attained for all n, k with $1 \leq k \leq n$. ■

Since $F(G, n) = 1$ for any ministrong digraph G of order n, we only consider the case $1 \leq k \leq n-1$. Recall that if a non-primitive G of order n is k-upper primitive, then we have $k \geq n/2 + 1$.

Theorem 2. *Let $G \in U(n,k) \backslash U(n,1)$ for $n/2 + 1 \leq k \leq n-2$. Then*

$$F(G,k) \leq \begin{cases} F(n,k) - (n-k-1) & \text{if } n \text{ is odd,} \\ F(n,k) - 2(n-k-1) & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, equality holds in the above two cases if and only if G is isomorphic to $G_{n,n-3}$ or $G_{n,n-4}$ respectively.

Proof. Let h be the smallest cycle length of G. Note that G is non-primitive. From the proof of Theorem 1, we have $h \leq n-3$.

Case 1. $h \le n-5$. By Lemma 2,

$$\begin{aligned} F(G,k) &\le n + h(n-k-1) \le n + (n-5)(n-k-1) \\ &= F(n,k) - 2(n-k-1) - (n-k-2) \le F(n,k) - 2(n-k-1). \end{aligned}$$

Case 2. $h = n-3$. Then $h \ge 2$ and $n \ge 5$. Since G is non-primitive and ministrong, we have $n-2$, $n \notin L(G)$. By Lemma 1, we have $L(G) \neq \{n-3\}$ and hence $L(G) = \{n-3, n-1\}$ where n is odd. It can be easily checked that G must be isomorphic to $G_{n,n-3}$.

Case 3. $h = n-4$. First suppose $n = 6$. We have $k \ge 6/2+1 = 4$, and so $k = 4$. Since $h = 2$, it follows that G is symmetric and hence $F(G,4) \le 2(6-4) = 4 < 7 = F(6,4) - 2(6-4-1)$ by [9, Lemma 4.1], or G is isomorphic to $D^{(1)}$ or $D^{(2)}$, where $V(D^{(1)}) = V(D^{(2)}) = \{i : 1 \le i \le 6\}$, $E(D^{(1)}) = E \cup \{(3,6),(6,3)\}$ and $E(D^{(2)}) = E \cup \{(5,6),(6,5)\}$ with $E = \{(1,2),(2,3),(3,4),(4,1),(2,5),(5,2)\}$, and it can be easily checked that $F(D^{(1)},4) = 4$, $F(D^{(2)},4) = 5$. In the following we suppose $n \ge 7$. By Lemma 1, we have $|L(G)| \ge 2$. Note that $G \in U(n, n-2) \backslash U(n,1)$.

Case 3.1. $n-1 \in L(G)$. We can readily show that $L(G) = \{n-4, n-1\}$, $n \equiv 1 \pmod 3$ and G is isomorphic to the digraph $G_{n,n-4}$.

Case 3.2. Case 3.2. $n-1 \notin L(G)$. Then $L(G) = \{n-4, n-2\}$ and n is even. Take a cycle C of length $n-2$. Then there are exactly two vertices, say x and y, lying outside C.

Case 3.2.1. G contains one of the arcs (x,y) or (y,x), say (x,y). Since $n > 6$, (y,x) is not an arc of G. Since G is strong, there exist vertices u and v such that (u,x) and (y,v) are both arcs of G. Note that G is ministrong and $L(G) = \{n-4, n-2\}$. It follows that G is isomorphic the digraph D with $V(D) = \{1,2,\ldots,n\}$ and $E(D) = \{(i,i+1) : 1 \le i \le n-3\} \cup \{(n-2,1),(n-5,n-1),(n-1,n),(n,2)\}$. Suppose $G = D$. It can be easily seen that $|R_D(F(n,k) - 2(n-k-1) - 1, i)| \ge n-k-1$ for all $i \in V(D)$, which implies that $F(G,k) \le F(n,k) - 2(n-k-1) - 1$.

Case 3.2.2. Neither (x,y) nor (y,x) is an arc of G. Then here exist vertices u, v, u', v' in C such that (u,x), (x,v), (u',y), (y,v') are all arcs of G. Let r_1 and r_2 be the distances in C from u to v and from u' to v' respectively. Note that $L(G) = \{n-4, n-2\}$ and G is ministrong. It is easy to see that $r_1 = 4$ or $r_2 = 4$. Suppose $r_2 = 4$. Then the subdigraph induced by vertices in $V(G) \backslash \{x\}$ is isomorphic to $G_1 = G_{(n-1),(n-1)-3}$. Suppose G_1 is a subdigraph of G with $V(G) = V(G_1) \cup \{n\}$, $x = n$, where (u,n), (n,v) are arcs of G with u, $v \in V(C)$. Let $X \subseteq V(G)$ with $|X| = k$ and $t = F(n,k) - 2(n-k-1) - 1$. Every vertex $i \in V(G) \backslash \{n\}$ can be reachable from some vertex in $X \backslash \{n\}$ by a walk of length $\exp_{G_1}(X \backslash \{n\})$ and hence of length t. This is because $\exp_{G_1}(X \backslash \{n\}) \le F(G_1, k-1) = (n-2)^2 - (k-1)(n-3) - (n-2-(k-1)) = t$. Let (u, u_1) be the unique arc in C incident from vertex u. If $u \neq 1$, then vertex n can be reachable from some vertex in $X\backslash\{n\}$) by a walk of length t. This is because any walk to u_1 must pass the arc (u, u_1). Suppose $u = 1$. Then we must have $v = 3$ or $v = 5$. If $v = 3$, then it is easy to see that $R_G(t, n) = \{n, 2, 4, \ldots, 2(n-k)\}$; if $v = 5$, then $R_G(t, n) = \{n, n-2, 4, 6, \ldots, 2(n-k)\}$. In either case, we have $|R_G(t, n)| = n - k + 1$, implying that vertex n can be reachable from some vertex in X by a walk of length t. Hence $F(G, k) = \exp_G(X) \leq t = F(n, k) - 2(n - k - 1) - 1$.

Combining Cases 1, 2 and 3, we have $F(G, k) \leq F(n, k) - 2(n-k-1) - 1 < F(n, k) - 2(n - k - 1)$ or G is isomorphic to $G_{n,n-3}$ or $G_{n,n-4}$ for $n/2 + 1 \leq k \leq n - 3$.

Suppose $k = n - 2$. If $h \leq n - 6$, then $F(G, n - 2) \leq n + h \leq 2n - 6 < F(n, k) - 2(n - k - 1)$. If $h = n - 3$ or $n - 4$, we have proved in Cases 2 and 3 that G is isomorphic to $G_{n,n-3}$ or $G_{n,n-4}$. We only need to consider the case $h = n - 5$. By similar arguments as in Case 3, we have $F(G, n - 2) \leq 2n - 6 < F(n, k) - 2(n - k - 1)$.

By Lemmas 4 and 5, the theorem is proved. ∎

Theorem 3. *Let $G \in U(n, k)$, $1 \leq k \leq n - 2$. Then $F(G, k) = F(n, k)$ if and only if G is isomorphic to $G_{n,n-2}$.*

Proof. The case $k = 1$ is proved in [2]. Suppose $k > 1$. If G is isomorphic to $G_{n,n-2}$, then $F(G, k) = F(G_{n,n-2}, k) = F(n, k)$ by Lemma 3.

Suppose $F(G, k) = F(n, k)$. Then G is primitive; otherwise $F(G, k) \leq F(n, k) - (n - k - 1) < F(n, k)$ by Theorem 2, which is a contradiction. Now it follows from [14, Theorem 2] that G is isomorphic to $G_{n,n-2}$. ∎

Theorem 4. *Let $G \in U(n, n-1)$. Then $F(G, n-1) = F(n, n-1) = n-1$ if and only if G is isomorphic to some digraph $G_{n,s}$ with $2 \leq s \leq n - 2$.*

Proof. Suppose G is isomorphic to some digraph $G_{n,s}$ with $2 \leq s \leq n - 2$. Take $X_0 = V(G)\backslash\{2\}$. It can be verified as in [14] that there exists no walk of length $n - 2$ from any vertex in X_0 to vertex 1, which implies that $F(G, n - 1) \geq \exp_G(X_0) \geq n - 1$. By Theorem 1, we have $F(G, n - 1) = n - 1$.

Now suppose $F(G, n-1) = \exp_G(X) = n-1$ with $V(G)\backslash X = \{u\}$. If there is a cycle C of length r not containing u, then for any $v \in V(G)$, there is a walk of length $n-r$ from a vertex in X to v. Note that $r > 1$. We have $F(G, n-1) < n-1$, a contradiction. Hence u is contained in all cycles of G. It follows from Lemma 1 that $|L(G)| \geq 2$. Let h and t be respectively the smallest and the largest cycle lengths of G. By Lemma 6, we have $n - 1 = F(G, n - 1) \leq \max\{n - h, t\}$. So $t = n - 1$. Assume $(1, 2, \ldots, n - 1, 1)$ is a cycle of length $n - 1$ of G. Since G is strong, there exist v and w (v and w may be equal) in $\{1, 2, \ldots, n - 1\}$ such that (v, n) and (n, w) are arcs of G. Suppose $w = 2$ and $v = s$. Then G contains a subdigraph $G_{n,s}$. Since G is ministrong, it is clear that G has no arcs other than those in $G_{n,s}$ and $s \neq 1$. Note that $|L(G)| \geq 2$. We have $s \neq n-1$. Hence G is isomorphic to some $G_{n,s}$ with $2 \leq s \leq n-2$. ■

Corollary 1. *The numbers of non-isomorphic digraphs and primitive digraphs of order n with the $(n-1)$-th upper generalized exponents equal to $n-1$ are $n-3$ and $\varphi(n-1)-1$ respectively, where φ is the Euler's totient function.*

Theorem 5. *Let $G \in U(n, n-1)$, $n \geq 4$. Then $F(G, n-1) = F(n, n-1) - 1 = n-2$ if and only if G is isomorphic to some digraph $H_{n,s}$ with $3 \leq s \leq n-1$, H_n^1 or H_n^2.*

Proof. Suppose G is isomorphic to some digraph $H_{n,s}$ with $3 \leq s \leq n-1$, H_n^1 or H_n^2, we have $F(G, n-1) = n-2$ by Lemma 7.

Conversely, suppose $F(G, n-1) = \exp_G(X) = n-2$ with $X = V(G)\backslash\{u\}$. By Lemma 1, we have $L(G) \neq \{n-1\}$ and $L(G) \neq \{n\}$. If $|L(G)| \geq 2$, then G has no cycles of length n, and G has no cycles of length $n-1$ by Theorem 4 and the fact $F(G, n-1) = n-2 < n-1$. Hence for any cycle of G with length r, we have $2 \leq r \leq n-2$.

Case 1. X contains a cycle C of length r with $2 \leq r \leq n-2$. Then $n-2 = F(G, n-1) = \exp_G(X) \leq n-r$. We have $r \leq 2$, and hence $r = 2$. Suppose $V(C) = \{x_1, x_2\}$. Let $d_i = \max\{d_G(x_i, y) : y \in V(G)\backslash V(C)\}$ for $i = 1$, 2. Then $d = \min\{d_1, d_2\} \leq n-2$. We have $d = n-2$; otherwise we have $d \leq n-3$, and hence $F(G, n-1) = \exp_G(X) \leq \exp_G\{x_1, x_2\} \leq n-3$, a contradiction. It follows that G contains a subdigraph which is isomorphic to the digraph D with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 1 \leq i \leq n-1\} \cup \{(2, 1)\}$. Suppose D is a subdigraph of G with $x_1 = 1$, $x_2 = 2$, $d_2 = d_G(2, n) = n-2$. Clearly there is no arc from i to j in G with $j - i > 1$.

We have $3 \notin X$; otherwise $F(G, n-1) = \exp_G(X) \leq \exp_G(\{1, 2, 3\}) \leq n-3$, a contradiction. Also vertex n is on some cycle with length $n-2$; otherwise $F(G, n-1) = \exp_G(X) \leq \exp_G(\{1, 2, n\}) \leq n-3$, a contradiction. Hence there is an arc from vertex n to vertex 3. To ensure that G is ministrong, there is also an arc from some vertex s to vertex 2 with $3 \leq s \leq n-1$ and no other arcs in G. Hence G is isomorphic to some digraph $H_{n,s}$ with $3 \leq s \leq n-1$.

Case 2. X does not contain any cycle. Then u is on every cycle of G. Let t be the length of a longest cycle C of G. As the proof in Theorem 4, we have $|L(G)| \geq 2$. Suppose G contains a cycle C_1 of length $q < t$. Write $t = mq + r$ where m and r are both integers with $1 \leq r \leq q$. There exists a vertex $x \in V(C)\backslash\{u\} \subseteq X$ such that there is a path of length r from x to u in the cycle C. Attaching the cycle C_1 to this path m times, we obtain a walk of length t from x to u. Clearly any vertex except u of G is reachable from itself by a walk of length t. Hence $n-2 = F(G, n-1) \le t$. Note that $t \le n-2$. We have $t = n-2$. It follows that G contains a subdigraph which is isomorphic to the digraph D' with vertex set $\{1, 2, \ldots, n\}$ and arc set $\{(i, i+1) : 3 \le i \le n-1\} \cup \{(n, 3)\}$. Suppose D' is a subdigraph of G with $u = 3$.

If vertices 1 and 2 are on a cycle, then vertices 1, 2 and 3 form a cycle of length 3, G is isomorphic to H_n^2; otherwise vertices 1 and 3, 2 and 3 form two cycles of length 2, G is isomorphic to H_n^1. ■

Let Ω_n be the family of digraphs $H_{n,s}$ $(n \ge 4)$, H_n^1 $(n \ge 5)$ and H_n^2 $(n \ge 6)$. Let $f(n) = |\Omega_n|$ and $g(n) = |\Omega_n \cap U(n,1)|$. It can be easily seen that $f(4) = 1$, $f(5) = 3$, $f(n) = n-3+2 = n-1$ for $n \ge 6$, $g(4) = 0$, $g(5) = 3$ and for $n \ge 6$

$$g(n) = \begin{cases} n-1 & \text{if } n \text{ is odd and } n \not\equiv 2 \pmod 3, \\ n-2 & \text{if } n \text{ is odd and } n \equiv 2 \pmod 3, \\ (n-2)/2 & \text{if } n \text{ is even and } n \not\equiv 2 \pmod 3, \\ (n-4)/2 & \text{if } n \text{ is even and } n \equiv 2 \pmod 3. \end{cases}$$

Corollary 2. *The numbers of non-isomorphic digraphs and primitive digraphs of order n with the $(n-1)$-th upper generalized exponents equal to $n-2$ are $f(n)$ and $g(n)$ respectively.*

It follows from Corollary 2 that there are $n-2-\varphi(n-1)$ non-isomorphic, non-primitive digraphs in $U(n, n-1)$ whose $(n-1)$-th upper generalized exponents achieve $n-1$ if $n-1$ ($n \ge 5$) is not prime, there are $f(n) - g(n)$ non-isomorphic, non-primitive digraphs in $U(n, n-1)$ whose $(n-1)$-th upper generalized exponents achieve $n-2$ if $n-1$ ($n \ge 4$) is prime.

By Theorems 4 and 5, we have the following.

Theorem 6. *If $G \in U(n, n-1) \backslash U(n, 1)$ for $n \ge 4$, then*

$$F(G, n-1) \le \begin{cases} n-1 & \text{if } n-1 \text{ is not prime}, \\ n-2 & \text{otherwise}. \end{cases}$$

Equality in the above two cases holds if and only if G is respectively isomorphic to

(1) *some $G_{n,s}$ with $2 \le s \le n-2$ and $\gcd(s, n-1) > 1$;*

(2) *some $H_{n,s}$ $(n \ge 4)$ with $3 \le s \le n-1$ where s is odd, H_n^1 $(n \ge 5)$ or H_n^2 $(n \equiv 2 \pmod 3$ and $n \ge 8)$.*

The numbers of such digraphs in (1) *and* (2) *are $n-2-\varphi(n-1)$ and $f(n) - g(n)$ respectively.*

4. Set of Upper Generalized Exponents

In this section we study the set of k-th upper generalized exponents of digraphs in $U(n,k)$. Clearly $E(n,n)=\{1\}$. We consider the case $1\le k\le n-1$.

Theorem 7. *For* $1\le k\le n-4$*, and any integer* m *with* $F(n,k)-(n-k-2)+1\le m\le F(n,k)-1$*, we have* $m\notin E(n,k)$*.*

Proof. Let $G\in U(n,k)$ and let h be the length of a shortest cycle of G. By the proof of Theorem 1, either $F(G,k)=F(n,k)$ (G is isomorphic to $G_{n,n-2}$) or $h\le n-3$. Suppose $h\le n-3$. If $k=1$, by a result of [7], we have $F(G,1)\le n+h(n-3)\le n^2-5n+9=F(n,1)-(n-1-2)$. If $2\le k\le n-4$, then by Lemma 2, $F(G,k)\le n+h(n-k-1)\le F(n,k)-(n-k-2)$.

Hence for any $G\in U(n,k)$, we have either $F(G,k)=F(n,k)$ or $F(G,k)\le F(n,k)-(n-k-2)$. ■

Theorem 8. $E(4,1)=\{6\}$, $E(5,2)=\{4,5,6,10\}$. *For* $n\ge 5$, $3n-5\in E(n,n-3)$*,* $3n-6\notin E(n,n-3)$*.*

Proof. If $G\in U(4,1)$, then it can be easily checked that G is isomorphic to $G_{4,2}$. Hence $E(4,1)=\{F(G_{4,2},1)\}=\{6\}$.

Suppose $G\in U(5,2)$. Since $2<5/2+1$, G is primitive. As is proved in [14], G is isomorphic to $G_{5,3}$, D_1, D_2 or D_3, where $V(D_1)=V(D_2)=V(D_3)=\{1,2,3,4,5\}$, $E(D_1)=E(G_{4,2})\cup\{(2,5),(5,2)\}$, $E(D_2)=E(G_{4,2})\cup\{(1,5),(5,1)\}$ and $E(D_3)=E(G_{4,2})\cup\{(4,5),(5,4)\}$. It can checked readily that $F(D_1,2)=4$, $F(D_2,2)=5$ and $F(D_3,2)=6$. Note that $F(G_{5,3},2)=10$. We have $E(5,2)=\{4,5,6,10\}$.

Now suppose $n\ge 6$. Let $G\in U(n,k)$ and let h be the smallest cycle length of G. Then $h\le n-2$. If $h=n-2$, then G is isomorphic to $G_{n,n-2}$ and $F(G,n-3)=F(n,n-3)=3n-5\in E(n,n-3)$. If $h\le n-4$, by Lemma 2, $F(G,k)\le n+2h\le n+2(n-4)=3n-8$. We are left with the case $h=n-3$. By Lemma 1, $L(G)\ne\{n-3\}$. Hence $|L(G)|\ge 2$.

Case 1. $n-1\in L(G)$. As is proved in [14], G is isomorphic to $G_{n,n-3}$. By Lemma 4, we have $F(G,n-3)=3n-7$.

Case 2. $n-1\notin L(G)$. As is proved in [14], G is isomorphic the digraph D^1_{n-3} with vertex set $\{1,2,\dots,n\}$ and arc set $\{(i,i+1):1\le i\le n-3\}\cup\{(n-2,1),(n-4,n-1),(n-1,n),(n,2)\}$ where $n\ge 6$ or G contains a subdigraph which is isomorphic $G_1 = G_{(n-1),(n-1)-2}$. In the former case, suppose $G = D^1_{n-3}$. It can be checked as in [14] that $|R_G(3n-7, i)| \geq 4$ for all i. Hence $F(G, n-3) \leq 3n-7$. Now suppose G_1 is a subdigraph of G and $V(G) = V(G_1) \cup \{n\}$. Let $X \subseteq V(G)$ with $|X| = n-3$. Every vertex $i \in V(G)\backslash\{n\}$ can be reachable from some vertex in $X\backslash\{n\}$ by a walk of length $\exp_{G_1}(X\backslash\{n\})$ and hence of length $3n-8$. This is because $\exp_{G_1}(X\backslash\{n\}) \leq F(G_1, n-4) = (n-2)^2 - (n-4)(n-3) = 3n-8$. It follows that every vertex of G can be reachable from some vertex in $X\backslash\{n\}$ by a walk of length $3n-8+1 = 3n-7$, which implies $F(G, n-3) \leq \exp_{G_1}(X)$ $\leq \exp_{G_1}(X\backslash\{n\}) \leq 3n-7$.

Now it follows that for any $G \in U(n, n-3)$ with $F(G, n-3) \neq 3n-5$, we have $F(G, n-3) \leq 3n-7$. ∎

By Theorems 7 and 8, there are gaps in the set $E(n, k)$ for $1 \leq k \leq n-3$.

Theorem 9. *For* $n \geq 4$*,* $E(n, n-1) = \{2, 3, \ldots, n-1\}$.

Proof. By Lemma 8 and Theorem 1, we have $E(n, n-1) \subseteq \{2, 3, \ldots, n-1\}$.

By Theorems 4 and 5, we have $i-2$, $i-1 \in E(i, i-1)$ for $i = 4, 5, \ldots, n$. Using Lemma 9, we have $\{2, 3, \ldots, n-1\} \subseteq E(n, n-1)$. ∎

Theorem 10. *For* $n \geq 4$*, we have* $E(4, 2) = \{5\}$*,* $E(5, 3) = \{4, 5, 7\}$ *and for* $n \geq 6$*,* $E(n, n-2) = \{2, 3, \ldots, 2n-3\}$.

Proof. If $G \in U(4, 2)$, then G is primitive by Lemma 1. Hence $E(4, 2) = \{F(G_{4,2}, 2)\} = \{5\}$. By similar arguments as in Theorem 8, we have $E(5, 3) = \{4, 5, 7\}$ since $F(D_1, 3) = 4$, $F(D_2, 3) = 5$, $F(D_3, 3) = 4$ and $F(G_{5,3}, 3) = 7$.

Now suppose $n \geq 6$. By Lemma 8 and Theorem 1, we have $E(n, n-2) \subseteq \{2, 3, \ldots, 2n-3\}$. We only need to prove the reverse inclusion.

By [9, Theorem 4.1], we have $\{2, 3, 4\} \subseteq E(n, n-2)$.

Let G be the digraph with vertex set $\{i : 1 \leq i \leq 6\}$ and arc set $\{(i, i+1) : 2 \leq i \leq 5\} \cup \{(1, 3), (3, 2), (4, 1), (6, 4)\}$. Clearly $G \in U(6, 1) \subseteq U(6, 4)$. It can be easily seen that $|R_G(6, i)| \geq 4$ for all $i \in V(G)$, which implies that $F(G, 4) \leq 6$. Note that there is no walk of length 5 from any vertex in $\{1, 2, 5, 6\}$ to vertex 6. Hence $F(G, 4) \geq \exp_G(\{1, 2, 5, 6\}) \geq 6$. We have $6 = F(G, 4) \in E(6, 4)$. Note also that $5 \in E(4, 2)$ and by Lemmas 3 and 4, we have $2i-4 \in E(i, i-2)$ for $i \geq 6$ and $2i-3 \in E(i, i-2)$ for $i \geq 5$. Hence we have by Lemma 9 that $\{5, 6 \ldots, 2n-3\} \subseteq E(n, n-2)$.

It follows that $\{2, 3, \ldots, 2n-3\} \subseteq E(n, n-2)$. ∎

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