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PERIODIC ASPECTS OF SEQUENCES GENERATED BY TWO SPECIAL MAPPINGS

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Abstract. Let $\beta = \frac{q}{p}$ be a fixed rational number, where p and q are positive integers with $2 \le p < q$ and $\gcd(p,q) = 1$. Consider two real-valued functions $\sigma(x) = \beta^x \mod 1$ and $\tau(x) = \beta x \mod 1$. For each positive integer n, let $s(n) = \sigma(n) = \frac{s(n)_1}{p} + \cdots + \frac{s(n)_n}{p^n}$ and $t(n) = \tau^n(1) = \frac{t(n)_1}{p} + \cdots + \frac{t(n)_n}{p^n}$ be the p-ary representation. In this paper, we study the periods of both sequences $S_k = \{s(n+k)_n\}_{n=1}^{\infty}$ and $T_k = \{t(n+k)_n\}_{n=1}^{\infty}$ for any non-negative integer k.

1. INTRODUCTION

Given a real number $\beta > 1$, the function $\tau(x) = \beta x \mod 1$ (known as *beta transformation* whenever the domain is restricted to the unit interval $[0,1)$, Rényi [5], 1957) has been studied intensively. In this paper, we consider $\beta = \frac{q}{p}$ a rational number, where p and q are positive integers with $2 \le p < q$ and $\gcd(p,q) = 1$. We consider the iterates τ^n defined by $\tau^1 = \tau$ and $\tau^n = \tau(\tau^{n-1})$ for $n \ge 2$. The orbit of 1 is the infinite sequence $\{\tau^n(1)\}_{n=1}^{\infty}$ (Devaney [1], 1989). Each term of this sequence can be written as p-ary representation

$$t(n) = \tau^n(1) = \frac{t(n)_1}{p} + \cdots + \frac{t(n)_n}{p^n}. \tag{1.1}$$

Let T_k be the sequence $T_k = \{t(n+k)_n\}_{n=1}^{\infty}$ for any non-negative integer k. We show that these sequences, T_k with $k \ge 0$, exhibit a certain periodic behavior.

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Closely related to the sequence $\{t(n)\}_{n=1}^{\infty}$, of orbit is the sequence $\{s(n) = \sigma(n) = (\frac{q}{p})^n \mod 1\}_{n=1}^{\infty}$. The sequence $\{(\frac{3}{2})^n \mod 1\}_{n=1}^{\infty}$ is believed to be uniformly distributed modulo 1, but it is not known even to be dense in the closed interval $[0, 1]$. It is known that the sequence $\{(\frac{3}{2})^n \mod 1\}_{n=1}^{\infty}$ has infinitely many limit points in $[0, 1]$ (Vijayaraghavan [7], 1940), but it is not yet known whether $\{(\frac{3}{2})^n \mod 1\}_{n=1}^{\infty}$ has infinitely many limit points in $[0, 1/2)$ (see Flatto, Lagarias and Pollington [3], 1995). Mahler's famous $\frac{3}{2}$-problem (Mahler [4], 1968), still unsolved, asks whether there exits a real number $\eta > 0$ such that the sequence $\{\eta(\frac{3}{2})^n \mod 1\}_{n=1}^{\infty}$, is contained in the interval $[0, \frac{1}{2})$. Tijdeman [6] (1972) came close to solving Mahler's problem by showing that for every $\beta > 2$ there exists $\eta > 0$ such that the sequence $\{\eta(\frac{3}{2})^n \mod 1\}_{n=1}^{\infty}$ is contained in the interval $[0, \frac{1}{\beta-1}]$. For a recent reference to Mahler $\frac{3}{2}$-problem see Drmota and Tichy [2], (1997).

We can also write each $s(n) = (\frac{q}{p})^n \mod 1$ in the p-ary representation as

$$s(n) = \frac{s(n)_1}{p} + \cdots + \frac{s(n)_n}{p^n}. \tag{1.2}$$

So, we can also consider the sequences $S_k = \{s(n+k)_n\}_{n=1}^{\infty}$, $k \geq 0$. These sequences will also exhibit a certain periodic behavior. We give the proofs of our results only for the case of the sequences T_k, $k \geq 0$. The proofs for the sequences S_k, $k \geq 0$, are quite similar.

2. Main Result

Let T_k be the sequence $T_k = \{t(n+k)_n\}_{n=1}^{\infty}$ for any integer $k \geq 0$ as in the last section. For each integer $n \geq 1$, define a function $a : \mathbf{N} \times (\mathbf{N} \cup \{0\}) \longrightarrow \mathbf{R}$ by $a(n, i) = (\frac{q}{p})^i t(n) \mod 1$, where $\mathbf{N}$ is the set of all positive integers and $\mathbf{R}$ is the set of all real numbers. So, $a(n, 0) = t(n)$. For each integer $i \geq 0$, we write the p-ary representation of $a(n, i)$ as $a(n, i) = \frac{a(n,i)_1}{p} + \cdots + \frac{a(n,i)_{n+i}}{p^{n+i}}$, where $0 \leq a(n, i)_1, \ldots, a(n, i)_{n+i} < p$. We have the following relation between $a(n, i)$ and $t(n+i)$ for all integers $n \geq 1$ and $i \geq 0$.

Lemma 2.1. *For any positive integer n and for any non-negative integer i, $a(n, i)_j = t(n+i)_j$ for all $1 + i \leq j \leq n + i$.*

Proof. Fix $n \geq 1$. We prove this lemma by the induction on i. It is trivial that this lemma holds for $i = 0$ because $a(n, 0) = t(n)$.

For any $i \geq 0$, write $(\frac{q}{p})^i t(n) = a_i + \frac{a(n,i)_1}{p} + \cdots + \frac{a(n,i)_{n+i}}{p^{n+i}}$, where a_i is a non-negative integer, then

$$\left(\frac{q}{p}\right)^{i+1} t(n) = \left(a_i + \frac{a(n, i)_1}{p} + \cdots + \frac{a(n, i)_i}{p^i}\right)\frac{q}{p}$$

$$+\left(\frac{a(n,i)_{i+1}}{p^{i+1}}+\cdots+\frac{a(n,i)_{n+i}}{p^{n+i}}\right)\frac{q}{p}.$$

Therefore, for $2+i \le j \le n+i+1$, the number $a(n,i+1)_j$ is completely determined by the part $(\frac{a(n,i)_{i+1}}{p^{i+1}}+\cdots+\frac{a(n,i)_{n+i}}{p^{n+i}})\frac{q}{p}$, which is equal to $(\frac{t(n+i)_{i+1}}{p^{i+1}}+\cdots+\frac{t(n+i)_{n+i}}{p^{n+i}})\frac{q}{p}$ by the induction hypothesis. Since the number $t(n+i+1)_j$ is also completely determined by $(\frac{t(n+i)_{i+1}}{p^{i+1}}+\cdots+\frac{t(n+i)_{n+i}}{p^{n+i}})\frac{q}{p}$, we have $a(n,i+1)_j = t(n+i+1)_j$ for each $2+i \le j \le n+i+1$. This completes the proof. ■

We now define e_i to be the multiplicative order of q modulo p^{i+1}, $i \ge 0$.

Lemma 2.2. *For any positive integer n, $t(n)_n \equiv q^n$ mod p. Therefore, the sequence $T_0 = \{t(n)_n\}_{n=1}^{\infty}$ is purely periodic with the period e_0.*

Proof. From the definition, $t(1)_1 \equiv q \bmod p$. For any integer $n \ge 1$,

$$\frac{q}{p}t(n) = \left(\frac{t(n)_1}{p}+\cdots+\frac{t(n)_n}{p^n}\right)\frac{q}{p}$$

and so

$$t(n+1)_{n+1} \equiv qt(n)_n \equiv q^{n+1} \bmod p$$

by the assumption of the induction. Hence, the sequence T_0 is purely periodic with the period e_0. ■

The following theorem is our main result in this paper.

Theorem 2.3. *For each non-negative integer k, the sequence T_k is purely periodic with the period m_k dividing e_k. Furthermore, for $k \ge 1$, let $d_k = \frac{e_k}{e_{k-1}}$ and write $p = p_{k,1}p_{k,2}$ where $\gcd(d_k, p_{k,2}) = 1$ and a prime π divides d_k if and only if π divides $p_{k,1}$. Moreover, let μ_k be the largest factor of e_0 so that $q^{e_0/\mu_k} \equiv 1 \bmod p_{k,1}$ and $\gcd(\mu_k, e_0/\mu_k) = 1 = \gcd(\mu_k, d_k)$. Then either $\frac{e_k}{2\mu_k}$ divides m_k, whenever $k \ge 2$, $e_k \equiv \cdots \equiv e_1 \equiv 2 \equiv p \mod 4$, and $e_0 \equiv 1 \mod 2$, or $\frac{e_k}{\mu_k}$ divides m_k, otherwise.*

Proof. Let $k \ge 1$. For any integer $n \ge 1$,

$$\begin{aligned} a(n+k, e_k) &= \frac{a(n+k,0)q^{e_k}}{p^{e_k}} \bmod 1 \\ &= \frac{a(n+k,0)(cp^{k+1}+1)}{p^{e_k}} \bmod 1 \end{aligned}$$

for some integer c because e_k is the multiplicative order of q modulo p^{k+1}. Hence, $t(n+e_k+k)_{n+e_k} = a(n+k,0)_n = t(n+k)_n$ by Lemma 2.1. Since n is arbitrary, T_k is purely periodic with period m_k dividing e_k.

If $e_k = e_{k-1}$, then $d_k = 1$ and so, $p_{k,1} = 1$ and $\mu_k = e_0$. In this case, the last assertion of the theorem is true trivially.

From now on, let $e_k \neq e_{k-1}$. Since $q^{e_{k-1}} \equiv 1 \bmod p^k$, we can write $q^{e_{k-1}} = h_{k+1}p^{k+1} + h_k p^k + 1$ for some non-negative integers h_{k+1} and h_k with $0 \le h_k < p$. In fact, $1 \le h_k < p$ because $e_k \neq e_{k-1}$. Notice that the number d_k is the smallest positive integer satisfying $h_k d_k \equiv 0 \bmod p$ (i.e., $h_k = c_k p/d_k$ for some $1 \le c_k < d_k$ with $\gcd(c_k, d_k) = 1$) and so, $q^{ie_{k-1}} \equiv (h_k p^k + 1)^i \equiv ih_k p^k + 1 \bmod p^{k+1}$ for all $0 \le i < d_k$. For $n \ge 1$ and $d_k > i \ge 0$, $a(n+k, ie_{k-1}) = \frac{t(n+k)q^{ie_{k-1}}}{p^{ie_{k-1}}} \bmod 1$ by the definition. From Lemma 2.1,

$$\begin{aligned} t(n+ie_{k-1}+k)_{n+ie_{k-1}} &= a(n+k, ie_{k-1})_{n+ie_{k-1}} \\ &\equiv a(n+k,0)_{n+k} ih_k + a(n+k,0)_n \quad \bmod p \\ &\equiv t(n+k)_{n+k} ih_k + t(n+k)_n \quad \bmod p. \end{aligned}$$

So, for any $n \ge 1$, numbers $t(n+ie_{k-1}+k)_{n+ie_{k-1}}$, $0 \le i < d_k$, are all distinct because $\gcd(t(n+k)_{n+k}, p) = 1$.

We have seen that $t(n+e_{k-1}+k)_{n+e_{k-1}} \equiv a(n+k,0)_n + a(n+k,0)_{n+k}h_k$ $\bmod p$ for arbitrary positive integer n. We also have $t(n+m_k+e_{k-1}+k)_{n+m_k+e_{k-1}} \equiv a(n+k, m_k)_{n+m_k} + a(n+k, m_k)_{n+m_k+k}h_k \bmod p$. Since m_k is the period of T_k, we have $t(n+e_{k-1}+k)_{n+e_{k-1}} = t(n+m_k+e_{k-1}+k)_{n+m_k+e_{k-1}}$ and $a(n+k,0)_n = t(n+k)_n = t(n+m_k+k)_{n+m_k} = a(n+k, m_k)_{n+m_k}$. These imply $a(n+k,0)_{n+k}h_k \equiv a(n+k, m_k)_{n+m_k+k}h_k \bmod p$. But from Lemmas 2.1 and 2.2, we have $a(n+k,0)_{n+k} = t(n+k)_{n+k} \equiv q^{n+k} \bmod p$ and $a(n+k, m_k)_{n+m_k+k} = t(n+m_k+k)_{n+m_k+k} \equiv q^{n+k+m_k} \bmod p$. So, $h_k q^{n+k+m_k} \equiv h_k q^{n+k} \bmod p$ and thus $h_k q^{m_k} \equiv h_k \bmod p$. This implies that q^{m_k} is of the form $q^{m_k} = 1 + rd_k$ for some integer $r > 0$. So, if $\mu_{k,1}$ is the multiplicative order of q modulo $p_{k,1}$, then $\mu_{k,1}$ divides m_k.

Write $u_k = \frac{e_k}{m_k}$ because m_k divides e_k. Let $v_k = \gcd(u_k, d_k)$ and $w_k = \frac{d_k}{v_k}$. Then, $m_k | w_k e_{k-1}$ and so, $t(n+k)_n = t(n+k+w_k e_{k-1})_{n+w_k e_{k-1}}$. If $v_k > 1$, then $1 \le w_k < d_k$, and the last equality contradicts that $t(n+k+ie_{k-1})_{n+ie_{k-1}}$, $0 \le i < d_k$, are all distinct for arbitrary $n \ge 1$. Hence, $v_k = 1$ and so, $w_k = d_k$. We have shown that $\frac{e_k}{m_k}$ and d_k are relatively prime. Combining this with the fact that $\mu_{k,1}$ divides m_k from the last paragraph, we have that the multiplicative order of q modulo $p_{k,1}^{k+1}$ divides m_k.

Let u be a prime with $u|u_k$ and let ℓ be the positive integer satisfying $u^\ell || e_k$. From $\gcd(d_k, u_k) = 1$ and $u_k | e_k$, we have $u^\ell || e_{k-1}$. Let i_0 be the smallest integer satisfying $u^\ell || e_{i_0}$, then $i_0 \le k-1$. If $i_0 = 0$ for every prime factor u of u_k, then

$u_k|e_0$ and thus $u_k|\mu_k$ from the definition of μ_k. Finally, consider $i_0 \neq 0$ for some u. Then u divides p and moreover, $u = 2$. In this case, $\ell = 1 = i_0$ and so $e_0 \equiv 1 \mod 2$. Trivially, we also have $e_k \equiv \cdots \equiv e_1 \equiv 2 \equiv p \mod 4$ and $u_k|2\mu_k$. This completes the proof. ∎

The following is the most important case for $m_k = e_k$.

Corollary 2.4. *If every prime factor of p divides $\frac{e_k}{e_{k-1}}$ (in particular, $\frac{e_k}{e_{k-1}} = p$), then the period m_k of T_k equals e_k.*

Proof. Since every prime factor of p divides $d_k = \frac{e_k}{e_{k-1}}$, we have $\mu_k = 1$ from the definition of μ_k in the last theorem. This implies $m_k = \frac{e_k}{\mu_k} = e_k$. ∎

In the Theorem 2.3, the period m_k of T_k satisfies either $\frac{e_k}{\mu_k}|m_k$ or $\frac{e_k}{2\mu_k}|m_k$, but m_k may not equal it (respectively). For instance, consider $\frac{q_1}{p_1} = \frac{55}{6}$ and $\frac{q_2}{p_2} = \frac{271}{6}$. Then both of them have the same orders $e_0 = 1$, $e_1 = 2 = e_2$, and $e_3 = 6$ and thus both have $\mu_3 = 1$. From Theorem 2.3, $\frac{e_3}{2\mu_3} = 3$ divides both m_3, but they are not equal. Indeed, periods of the first four sequences generated by $\frac{q_1}{p_1}$ are $m_0 = 1$, $m_1 = 2 = m_2$, and $m_3 = 3 = \frac{e_3}{2\mu_3}$, while periods of the first four sequences generated by $\frac{q_2}{p_2}$ are $m_0 = 1$, $m_1 = 2 = m_2$, and $m_3 = 6 = 2\frac{e_3}{2\mu_3}$.

We now consider the sequences S_k generated by the function $\sigma(n) = (\frac{q}{p})^n \mod 1$ as described before. It is easy to see that Lemma 2.2 is also true for S_0, i.e., the period of S_0 is e_0. In general, we have the following theorem for S_k which is an analogous result of Theorem 2.3 for T_k. The proof of the following theorem is omitted because it is similar to the proof of Theorem 2.3 with a suitable modification.

Theorem 2.5. *For each non-negative integer k, the sequence S_k is purely periodic with the period m_k dividing e_k. The period m_0 of the sequence S_0 is e_0. For $k \geq 1$, let $d_k = \frac{e_k}{e_{k-1}}$ and write $p = p_{k,1}p_{k,2}$ where $\gcd(d_k, p_{k,2}) = 1$ and a prime π divides d_k if and only if π divides $p_{k,1}$. Moreover, let μ_k be the largest factor of e_0 so that $q^{e_0/\mu_k} \equiv 1 \mod p_{k,1}$ and $\gcd(\mu_k, e_0/\mu_k) = 1 = \gcd(\mu_k, d_k)$. Then either $\frac{e_k}{2\mu_k}$ divides m_k, if $k \geq 2$, $e_k \equiv \cdots \equiv e_1 \equiv 2 \equiv p \mod 4$, and $e_0 \equiv 1 \mod 2$, or $\frac{e_k}{\mu_k}$ divides m_k, otherwise.*

Notice that Corollary 2.4 does also hold for S_k from the last theorem.

3. Special Cases

We still consider first the sequence T_k for any integer $k > 0$. It is trivial that if $e_k = 1$, then the period length of T_k is 1. But if $e_k = e_{k-1} \geq 2$, the period length of T_k may not be equal to e_k. For instance, if $\frac{q}{p} = \frac{809}{6}$, then $e_3 = e_2 = e_1 = e_0 = 2$ and the period length of T_3 is $1 \neq e_3$. Note that $e_k = e_{k-1}$ cannot occur anywhere. The following proposition gives a constraint for k with $e_k = e_{k-1}$.

Proposition 3.1. *Let p and q be positive integers with $p \geq 2$ and $\gcd(p, q) = 1$. For each integer $n \geq 0$, let e_n be the multiplicative order of q modulo p^{n+1}. Let $k \geq 0$ be a fixed integer. If $e_{k+2} = e_{k+1} > e_k$, then $k = 0$, $p \equiv 2 \mod 4$, and $e_2 = e_1 = 2e_0$ with e_0 odd.*

Proof. Since $e_{k+1} > e_k$ and $q^{e_k} \equiv 1 \mod p^{k+1}$, we can write $q^{e_k} = h_{k+1}p^{k+1} + 1$ for some non-negative integer $h_{k+1} \not\equiv 0 \mod p$. Since $e_k | e_{k+1}$, we write $d_{k+1} = \frac{e_{k+1}}{e_k}$, then d_{k+1} is the smallest positive integer satisfying $h_{k+1}d_{k+1} \equiv 0 \mod p$ and $d_{k+1}|p$. So, $1 < d_{k+1} \leq p$ and $h_{k+1}d_{k+1} \not\equiv 0 \mod p^2$. Now,

$$\begin{aligned} q^{e_{k+1}} &= q^{e_k d_{k+1}} = (h_{k+1}p^{k+1} + 1)^{d_{k+1}} \\ &\equiv 1 + d_{k+1}h_{k+1}p^{k+1} + \frac{d_{k+1}(d_{k+1} - 1)}{2} h_{k+1}^2 p^{2k+2} \mod p^{k+3}. \end{aligned}$$

Since $e_{k+2} = e_{k+1}$, $q^{e_{k+1}} \equiv 1 \mod p^{k+3}$, this implies $k = 0$ because $h_{k+1}d_{k+1} \not\equiv 0 \mod p^2$. So, $d_1h_1 + \frac{d_1(d_1-1)h_1^2 p}{2} \equiv 0 \mod p^2$. Since $h_1d_1 \equiv 0 \mod p$ and $h_1d_1 \not\equiv 0 \mod p^2$, we have $\frac{d_1(d_1-1)h_1^2}{2} \not\equiv 0 \mod p$, and thus $p \equiv 0 \equiv d_1 \mod 2$ and $h_1 \equiv 1 \mod 2$. From $h_1d_1 \equiv 0 \mod p$ again, we have $\frac{d_1(d_1-1)h_1^2}{2} \equiv \frac{p}{2} \mod p$ and so, $d_1h_1 \equiv \frac{p}{2}p \mod p^2$. This implies $d_1 \equiv 2 \mod 4$. If there were an odd prime u dividing d_1, then u would be an odd prime factor of p and thus would divide $\frac{e_2}{e_1}$. So, $d_1 = 2$ and thus $p \equiv 2 \mod 4$ and $h_1 \equiv \frac{p^2}{4} \mod p^2$. Hence, $e_2 = e_1 = 2e_0$. ∎

If there exists a positive integer k satisfying $e_k = e_{k-1}$, then we have either $e_k = e_{k-1} = \cdots = e_1 = e_0$ or $e_k = e_{k-1} = \cdots = e_1 = 2e_0$ with e_0 odd and $p \equiv 2 \mod 4$ from Proposition 3.1. Unfortunately, we are unable to determine the period of sequences T_i for each $1 \leq i \leq k$ with these conditions. However, we can determine some special cases. Indeed, we are going to study periods of sequences T_k (and S_k) whenever either $e_k = e_{k-1} = \cdots = e_1 = e_0 = 2$ or $e_k = e_{k-1} = \cdots = e_1 = 2$ and $e_0 = 1$.

Now, let k_0 be the largest positive integer of k such that $e_k = e_1 = 2$, then for any integer $k > k_0$, we have $e_k > e_{k-1}$. For determining the period of T_k with $1 \leq k \leq k_0$, we need the following lemma, which is stated in a general situation.

Lemma 3.2. *For any positive integer k, $t(ie_k + k)_{ie_k} = 0$ for all positive integers i.*

Proof. From $t(k+1) = \frac{q}{p}t(k) \mod 1 = \left(\frac{t(k)_1}{p} + \cdots + \frac{t(k)_k}{p^k}\right)\frac{q}{p} \mod 1$, we have $t(k+1)_1 p^k + \cdots + t(k+1)_k p + t(k+1)_{k+1} \equiv (t(k)_1 p^{k-1} + \cdots + t(k)_k)q \mod p^{k+1}$. Since $a(k+1, ie_k - 1) = t(k+1)(\frac{q}{p})^{ie_k - 1} \mod 1 = \frac{t(k+1)q^{ie_k-1}}{p^{ie_k-1}} \mod 1$, we have, from Lemma 2.1, that

$$\begin{aligned}
& t(ie_k + k)_{ie_k} p^k + \cdots + t(ie_k + k)_{ie_k + k} \\
&= a(k+1, ie_k - 1)p^k + \cdots + a(k+1, ie_k - 1)_{ie_k + k} \\
&\equiv (t(k+1)_1 p^k + \cdots + t(k+1)_k p + t(k+1)_{k+1})q^{ie_k - 1} \mod p^{k+1} \\
&\equiv (t(k)_1 p^{k-1} + \cdots + t(k)_k)q^{ie_k} \mod p^{k+1}.
\end{aligned}$$

Hence, we have $t(ie_k + k)_{ie_k} = 0$ because $q^{ie_k} \equiv 1 \mod p^{k+1}$. ■

The following proposition is easy to see from Lemma 3.2 and its proof is omitted.

Proposition 3.3. *Let k_0 be the largest positive integer such that for all integers $1 \le k \le k_0$, $e_k = e_1 = 2$ with either $e_0 = 2$ or $e_0 = 1$ and $p \equiv 2 \mod 4$, then for each $1 \le k \le k_0$, the period m_k of the sequence T_k is either 1 or 2 and m_k is 1 if and only if $t(1+k)_1 = 0$.*

In the Proposition 3.3, the case $e_1 = 2$ can be determined explicitly, namely the period m_1 of T_1 is 2 whenever $e_1 = 2$. Indeed, write $q = q_2 p^2 + q_1 p + q_0$, where $0 \le q_1, q_0 < p$ and $q_2 \ge 0$. If $e_0 = 1$ and $p \equiv 2 \mod 4$, then $q_0 = 1$ and $q_1 = p/2$. In this case, $t(2)_1 = p/2$ and so, $m_1 = 2$. If $e_0 = 2$, then $q^2 \equiv 2q_1 q_0 p + q_0^2 \mod p^2$. Notice that q_0^2 can be written as $q_0^2 = a_1 p + 1$ with $1 \le a_1 < p$. From $q^2 \equiv 1 \mod p^2$, we have $p | (2q_1 q_0 + a_1)$ and $p \nmid (q_1 q_0 + a_1)$. So, in the case $e_1 = 2 = e_0$, $0 \ne t(2)_1 \equiv q_1 + a_1 \mod p$ and thus m_1 equals 2.

Notice also that it can occur that the period m_k of T_k equals 1 when $k \ge 2$ and $e_k = e_{k-1} = 2$. For instance, let $\frac{q}{p} = \frac{487}{6}$. It is easy to check that $e_0 = 1$, $e_1 = e_2 = e_3 = 2$ and $e_4 = 4$. And T_0 has the period 1, both T_1 and T_2 have the same period 2, and T_3 has the period 1. Indeed, $t(0) = \frac{1}{6}$, $t(1) = \frac{3}{6} + \frac{1}{6^2}$, $t(2) = \frac{5}{6} + \frac{0}{6^2} + \frac{1}{6^3}$, and $t(3) = \frac{0}{6} + \frac{0}{6^2} + \frac{3}{6^3} + \frac{1}{6^4}$.

We now study the periods of the sequences S_k in these special cases. We state them in the following proposition without proof because its proof follows easily from the fact that $s(ie_k)_{ie_k - k} = 0$ for all positive integers k and i with $ie_k > k$.

Proposition 3.4. *Let k_0 be the largest positive integer so that for all $1 \le k \le k_0$, $e_k = e_1 = 2$ with either $e_0 = 2$ or $e_0 = 1$ and $p \equiv 2 \mod 4$. Write $q = q_0 + q_1 p + \cdots + q_{k_0} p^{k_0} + q_{k_0+1} p^{k_0+1}$, where $0 \le q_{k_0+1}$ and $0 \le q_k < p$ for each $0 \le k \le k_0$, then for each $1 \le k \le k_0$, the period of the sequence S_k is either 1 or 2 and the period of S_k is 1 if and only if $q_k = 0$.*

It should be noted that the period of S_k can be 1 when $e_k = e_{k-1} = 2$. For example, let $\frac{q}{p} = \frac{33615}{14}$, then $e_0 = 1$, $e_1 = e_2 = e_3 = e_4 = 2$, and $e_5 = 14$. The period of S_0 is 1, the periods of S_1, S_2, and S_3 are all equal to 2, and the period of S_4 is 1. Indeed, $33615 = 1 + 7 \times (14) + 3 \times (14)^2 + 12 \times (14)^3$.

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