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THE BEHAVIOR OF THE INTERPHASE HEAT-TRANSFER FOR THE FAST-IGNITING CATALYTIC CONVERTER

Yu-Hsien Chang and Guo-Chin Jau

Abstract. The main purpose of this paper is to study the thermal balance equations for the gas and solid interphase heat-transfer for the fast-igniting catalytic converter of automobiles:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = -\alpha\frac{\partial}{\partial x}u(t,x) + av(t,x) - au(t,x) & \text{for } t>0, 0<x<l; \\ \frac{\partial}{\partial t}v(t,x) = bu(t,x) - bv(t,x) + \lambda\exp(v(t,x)) & \text{for } t>0, 0<x<l; \\ u(t,0) = \eta & \text{for } t\geq 0; \\ u(0,x) = u_0(x) \text{ and } v(0,x) = v_0(x) & \text{for } x>0. \end{cases}$$

where u_0, v_0 are continuous functions on $[0,l]$ with $u_0(0)=\eta$. We establish some results concerning the existence and uniqueness of the mild solutions and classical solutions of the above differential system. The asymptotical behavior of the solution is also addressed.

1. Introduction and Notation

A catalytic converter is a device located in the exhaust system of an automobile. Pollutant gases flowing out of the engine pass through it and undergo chemical processes by which they are converted into relatively harmless gases. Gas flows through the passages and reacts on the surface of the tubular walls. Under the research works of many scientists (cf. [7, 10-12] and the references cited therein), D. T. Leighton and H. C. Chang [5] studied the thermal balance equations for the gas and solid interphase heat-transfer in the fast-igniting catalytic converter of

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automobiles. In their paper, they chose suitable variables to express the gas and solid interphase heat-transfer phenomenon by the following differential system:

$$(1.1)\quad \begin{cases} \pi a^2 (\rho c_\rho)_g \left[\frac{\partial T_g}{\partial t} + U \frac{\partial T_g}{\partial x}\right] = 2\pi a h (T_s - T_g) & \text{for all } t > 0 \text{ and } x > 0 \\ 2\pi a \Delta r (\rho c_\rho)_s \frac{\partial T_s}{\partial t} = 2\pi a h (T_g - T_s) & \\ \quad + \left(2\pi a \Delta r + \pi a^2\right) A e^{\beta\left(T_s - T_g^{in}\right)} & \text{for all } t > 0 \text{ and } x > 0 \\ T_g(0,x) = T_s(0,x) = T_s^0 \text{ and } T_g(t,0) = T_g^{in} & \text{for all } t \geq 0 \text{ and } x > 0 \end{cases}$$

where the functions and parameters used are as follows:

(1) a is the pore radius;

(2) A is the preexponential factor of zeroth order kinetics;

(3) c_p is the heat capacity;

(4) β is the inverse Frank-Kamenetskii temperature;

(5) T_g is the temperature of the gas and T_g^{in} is the inlet temperature of the gas;

(6) T_s is the temperature of the solid phases.

Let $u = \beta\left(T_g - T_g^{in}\right)$, $v = \beta\left(T_s - T_g^{in}\right)$, and $\eta = \beta\left(T_g^{in} - T_s^0\right)$. Then there are positive constants α, a, b, and λ such that

$$(1.2)\quad \begin{cases} \frac{\partial}{\partial t} u(t,x) = -\alpha \frac{\partial}{\partial x} u(t,x) + a v(t,x) - a u(t,x) & \text{for } t > 0, x > 0; \\ \frac{\partial}{\partial t} v(t,x) = b u(t,x) - b v(t,x) + \lambda \exp(v(t,x)) & \text{for } t > 0, x > 0; \\ u(t,0) = \eta & \text{for } t \geq 0; \\ u(0,x) = u_0(x) \quad \text{and} \quad v(0,x) = v_0(x) & \text{for } x > 0. \end{cases}$$

Since the length of the monolith has to be finite, we may assume that l is an arbitrary fixed constant. We will consider the uniqueness, existence and behavior of the solution for the differential system (1.2) on $[0,\infty) \times [0,l]$. Although the thermal increases very rapidly at the front edge of converter, the thermal spread is continuous. Under this consideration, we may further assume the initial conditions of u_0 and v_0 are as follows:

(H1) u_0 and v_0 are the continuous functions on $[0,l]$;

(H2) $u_0(0) = \eta$.

Recently, S. N. Ha, S. W. Roh, J. Park [3] and G. P. Cherepanov [1] also discussed the other models of a catalytic converter by the numerical method and routine methods, respectively.

The main result of this paper is Theorem 6 in Section 2. In this theorem we show that, followed by the results of Theorem 1 through Theorem 5, if the initial functions $u_0 \in C^1(0,l) \cap C[0,l]$ and $v_0 \in C[0,l]$ with $u_0(0) = \eta$, there is a constant $t_{\max} > 0$ such that $[0, t_{\max})$ is the maximal time interval for the unique solution (u, v), which is given by (2.3), of the differential system (1.2) on $[0, t_{\max}) \times [0, l]$. Moreover, if $t_{\max}$ is finite, then $\lim_{t \to t_{\max}} \sup_{x \in [0,l]} |v(t,x)| = \infty$. However, if the initial data is positive, one may show that $\lim_{t \to t_{\max}} \sup_{x \in [0,l]} v(t,x) = \infty$ provided that $t_{\max}$is finite, and $\lim_{t \to \infty} v(t,x) = \infty$ provide $t_{\max}$ is infinite. Since Theorem 6 gives the closed form of the solution, one may accurately estimate the ignition time. This offers an effective data for designing the catalytic converter. Furthermore, the fact that $v(t,x)$ goes to infinite (or $\sup_{x \in [0,l]} v(t,x)$ goes to infinity) as t goes to $t_{\max}$ indicates the difference of the temperature of solid phase and inlet temperature of the gas will rise over 300K after a certain time. This phenomenon guarantees the hydrocarbon and CO emissions from an automobile is hopefully reduced.

2. Main Results

Theorem 1. *Let $u_0(x) = u(t_0, x)$ and $v_0(x) = v(t_0, x)$ be continuous functions on $[0,l]$ with $u_0(0) = \eta$. If (u,v) is a solution of the differential equation (1.2) on $[t_0, \infty) \times [0, l]$, then (u, v) satisfies the following integral system:*

$$(2.1) \quad u(t,x) = \begin{cases} e^{a(t_0 - t)} u(t_0, x + \alpha t_0 - \alpha t) + a \int_{t_0}^{t} e^{a(\tau - t)} v(\tau, x + \alpha\tau - \alpha t)\, d\tau \\ \qquad \text{for all } (t,x) \in [t_0, \infty) \times [0,l] \text{ with } t_0 \le t < t_0 + x\alpha^{-1} \\ e^{-ax\alpha^{-1}} \eta + a \int_{0}^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)} v\left(t - x\alpha^{-1} + \tau, \alpha\tau\right) d\tau \\ \qquad \text{for all } (t,x) \in [t_0, \infty) \times [0,l] \text{ with } t_0 + x\alpha^{-1} \le t < \infty \end{cases}$$

and

$$v(t,x) = e^{b(t_0 - t)} v(t_0, x) + b \int_{t_0}^{t} e^{b(\tau - t)} u(\tau, x)\, d\tau + \lambda \int_{t_0}^{t} e^{b(\tau - t)} \exp(v(\tau, x))\, d\tau$$
$$\text{for all } (t,x) \in [t_0, \infty) \times [0,l].$$

Proof. For any fixed $(t,x) \in [t_0, \infty) \times [0,l]$, we define $s_0 = \max\left\{t_0 - t, -x\alpha^{-1}\right\}$. Let two functions $y : \left[s_0, (l-x)\alpha^{-1}\right] \to R$ and $z : \left[s_0, (l-x)\alpha^{-1}\right] \to R$ be defined by

$$y(s) = u(t + s, x + \alpha s) \text{ and } z(s) = v(t + s, x + \alpha s)$$

for all $s_0 \le s \le (l-x)\alpha^{-1}$. Since

$$y'(s) = az(s) - ay(s) \qquad \text{for all } s_0 < s < (l-x)\alpha^{-1},$$

$\frac{d}{ds}e^{as}y(s) = ae^{as}z(s)$ for all $s_0 < s < (l-x)\alpha^{-1}$. If $t_0 \le t < t_0 + x\alpha^{-1}$, then $s_0 = t_0 - t$ and

$$y(0) - e^{-a(t-t_0)}y(t_0 - t) = a\int_{-t+t_0}^{0} e^{as}z(s)\,ds.$$

This implies that $u(t,x) = e^{-a(t-t_0)}u(t_0, x+\alpha t_0 - \alpha t) + a\int_{-t+t_0}^{0} e^{as}v(t+s, x+\alpha s)ds$, or

$$u(t,x) = e^{-a(t-t_0)}u(t_0, x+\alpha t_0-\alpha t) + a\int_{t_0}^{t} e^{a(\tau-t)}v(\tau, x+\alpha\tau-\alpha t)\,d\tau.$$

If $t_0 + x\alpha^{-1} \le t < \infty$, then $s_0 = -x\alpha^{-1}$ and

$$y(0) - e^{-ax\alpha^{-1}}y\left(-x\alpha^{-1}\right) = a\int_{-x\alpha^{-1}}^{0} e^{as}z(s)\,ds.$$

This implies that $u(t,x) = e^{-ax\alpha^{-1}}u\left(t-x\alpha^{-1}, 0\right) + a\int_{-x\alpha^{-1}}^{0} e^{as}v(t+s, x+\alpha s)\,ds$, or

$$u(t,x) = e^{-ax\alpha^{-1}}\eta + a\int_{0}^{x\alpha^{-1}} e^{a\left(\tau-x\alpha^{-1}\right)}v\left(t-x\alpha^{-1}+\tau, \alpha\tau\right)d\tau.$$

On the other hand, since $v_t(t,x) = -bv(t,x) + bu(t,x) + \lambda e^{v(t,x)}$, we obtain

$$e^{bt}v_t(t,x) + be^{bt}v(t,x) = be^{bt}u(t,x) + \lambda e^{bt}e^{v(t,x)}.$$

Thus $\frac{\partial}{\partial t}e^{bt}v(t,x) = be^{bt}u(t,x) + \lambda e^{bt}e^{v(t,x)}$ and

$$e^{bt}v(t,x) - e^{bt_0}v(t_0,x) = b\int_{t_0}^{t} e^{b\tau}u(\tau,x)\,d\tau + \lambda\int_{t_0}^{t} e^{b\tau}e^{v(\tau,x)}d\tau.$$

This implies that

$$v(t,x) = e^{b(t_0-t)}v(t_0,x) + b\int_{t_0}^{t} e^{b(\tau-t)}u(\tau,x)\,d\tau + \lambda\int_{t_0}^{t} e^{b(\tau-t)}\exp(v(\tau,x))\,d\tau.$$

The proof of this theorem is complete now.

Remark. If (u,v) is a solution of the differential equation (1.2) on $[t_0,\infty)\times[0,l]$ and $(t,x) \in [t_0,\infty)\times[0,l]$ satisfies $t_0 + x\alpha^{-1} \le t < \infty$, then u can be expressed as

$$u(t,x) = e^{-ax\alpha^{-1}}\eta + a\int_{t-x\alpha^{-1}}^{t} e^{a(\tau-t)}v(\tau, x+\alpha\tau-\alpha t)\,d\tau.$$

Theorem 2. *Let $u_0(x) = u(t_0,x)$ and $v_0(x) = v(t_0,x)$ be continuous functions on $[0,l]$ with $u_0(x) = \eta$. Then there is a constant $\delta > 0$ such that integral system (2.1) has a unique solution (u,v) on the interval $[t_0, t_0+\delta]\times[0,l]$.*

Proof. In view of (2.1), $u(t,x)$ is depend on the function $v(t,x)$ only and it is determined as long as $v(t,x)$ does. For this sake, we need only to show the existence and uniqueness of $v(t,x)$. For any constant $K > \|v_0\|_{C[0,l]}$, one may choose a constant $\delta > 0$ such that

$$\delta < \min\left\{1, \frac{1}{b+\lambda e^K}, \frac{K - \|v_0\|_{C[0,l]}}{\frac{1}{2}abK + \lambda e^K + b\|u_0\|_{C[0,l]}}\right\}.$$

Let the operator $F : C([t_0, t_0+\delta]\times[0,l]) \to C([t_0, t_0+\delta]\times[0,l])$ which is given by

$$F(v)(t,x) = e^{b(t_0-t)}v_0(x) + b\int_{t_0}^{t} e^{b(\tau-t)}u(\tau,x)\,d\tau + \lambda\int_{t_0}^{t} e^{b(\tau-t)}\exp(v(\tau,x))\,d\tau,$$

where

$$u(t,x) = \begin{cases} e^{a(t_0-t)}u_0(x+\alpha t_0-\alpha t) + a\int_{t_0}^{t} e^{a(\tau-t)}v(\tau, x+\alpha\tau-\alpha t)\,d\tau \\ \qquad \text{for all } (t,x)\in[t_0,t_0+\delta]\times[0,l] \text{ with } t_0\le t< t_0+x\alpha^{-1} \\ e^{-ax\alpha^{-1}}\eta + a\int_0^{x\alpha^{-1}} e^{a\left(\tau-x\alpha^{-1}\right)}v\left(t-x\alpha^{-1}+\tau, \alpha\tau\right)d\tau \\ \qquad \text{for all } (t,x)\in[t_0,t_0+\delta]\times[0,l] \text{ with } t_0+x\alpha^{-1}\le t\le t_0+\delta \end{cases}.$$

Since u_0, v_0 are continuous on $[0,l]$ and $v\in C([t_0,t_0+\delta]\times[0,l])$, it is easy to prove that the operator F is well-defined.

Let $B(0;K) = \{v\in C([t_0,t_0+\delta]\times[0,l]) : \|v\|_\infty \le K\}$ be a closed ball in the space $C([t_0,t_0+\delta]\times[0,l])$. For any $v\in B(0;K)$ and $(t,x)\in[t_0,t_0+\delta]\times[0,l]$, we have

$$|u(t,x)| \le \begin{cases} \|u_0\|_{C[0,l]} + a\displaystyle\int_{t_0}^{t} K\,d\tau & \text{if } t_0\le t<t_0+x\alpha^{-1}\le t_0+\delta \\ \|u_0\|_{C[0,l]} + a\displaystyle\int_{0}^{x\alpha^{-1}} K\,d\tau & \text{if } t_0+x\alpha^{-1}\le t\le t_0+\delta \end{cases}.$$

Then $|u(t,x)| \le \|u_0\|_{C[0,l]} + aK(t-t_0)$ for all $(t,x)\in[t_0,t_0+\delta]\times[0,l]$. This implies that

$$\begin{aligned} |F(v)(t,x)| &\le |v(t_0,x)| + b\int_{t_0}^{t}|u(\tau,x)|\,d\tau + \lambda\int_{t_0}^{t}\exp(v(\tau,x))\,d\tau \\ &\le \|v_0\|_{C[0,l]} + b\int_{t_0}^{t}\|u(t_0,\cdot)\|_\infty\,d\tau + b\int_{t_0}^{t} aK(\tau-t_0)\,d\tau + \lambda\int_{t_0}^{t} e^K\,d\tau \\ &\le \|v_0\|_{C[0,l]} + \left(b\|u(t_0,\cdot)\|_\infty + \frac{abK}{2} + \lambda e^K\right)\delta \\ &\le K. \end{aligned}$$

Thus the operator F maps $B(0;K)$ into itself.

Moreover, for any $v_1,\ v_2 \in B(0;K)$, we have

$$\begin{aligned}
&|u_1(t,x) - u_2(t,x)| \\
\leq\ & a\int_{t_0}^{t} e^{a(\tau-t)} |v_1(\tau, x+\alpha\tau-\alpha t) - v_2(\tau, x+\alpha\tau-\alpha t)|\, d\tau \\
\leq\ & a\|v_1 - v_2\|_\infty \textstyle\int_{t_0}^{t} e^{a(\tau-t)} d\tau \\
=\ & \|v_1 - v_2\|_\infty \left(1 - e^{a(t_0-t)}\right)
\end{aligned}$$

for all $(t,x) \in [t_0, t_0+\delta] \times [0,l]$ with $t_0 \leq t < t_0 + x\alpha^{-1} \leq t_0 + \delta$, and

$$\begin{aligned}
&|u_1(t,x) - u_2(t,x)| \\
\leq\ & a\int_{0}^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)} \left|v_1\left(t - x\alpha^{-1} + \tau, \alpha\tau\right) - v_2\left(t - x\alpha^{-1} + \tau, \alpha\tau\right)\right| d\tau \\
\leq\ & \|v_1 - v_2\|_\infty \left(1 - e^{-ax\alpha^{-1}}\right)
\end{aligned}$$

for all $(t,x) \in [t_0, t_0+\delta] \times [0,l]$ with $t_0 + x\alpha^{-1} \leq t \leq t_0 + \delta$. Then

$$|u_1(t,x) - u_2(t,x)| \leq \|v_1 - v_2\|_\infty \quad \text{for all} \quad (t,x) \in [t_0, t_0+\delta] \times [0,l].$$

This implies that

$$\begin{aligned}
&|F(v_1)(t,x) - F(v_2)(t,x)| \\
\leq\ & b\int_{t_0}^{t} |u_1(\tau,x) - u_2(\tau,x)|\, d\tau + \lambda \int_{t_0}^{t} |\exp(v_1(\tau,x)) - \exp(v_2(\tau,x))|\, d\tau \\
\leq\ & b\delta \|v_1 - v_2\|_\infty + \lambda e^K \delta \|v_1 - v_2\|_\infty
\end{aligned}$$

for all $(t,x) \in [t_0, t_0+\delta] \times [0,l]$. Then

$$\|F(v_1) - F(v_2)\|_\infty \leq \left(b\delta + \lambda e^K \delta\right) \|v_1 - v_2\|_\infty .$$

From the choice of δ, the operator F is a contraction mapping from $B(0;K)$ into itself. Thus, there is a unique function v such that $F(v) = v$. New, we many define the function $u(t,x)$ as

$$u(t,x) = \begin{cases} e^{a(t_0-t)} u_0(x + \alpha t_0 - \alpha t) + a\displaystyle\int_{t_0}^{t} e^{a(\tau-t)} v(\tau, x+\alpha\tau-\alpha t)\, d\tau \\ \qquad \text{for all } (t,x) \in [t_0, t_0+\delta] \times [0,l] \text{ with } t_0 \leq t < t_0 + x\alpha^{-1} \\ e^{-ax\alpha^{-1}}\eta + a\displaystyle\int_{0}^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)} v\left(t - x\alpha^{-1} + \tau, \alpha\tau\right) d\tau \\ \qquad \text{for all } (t,x) \in [t_0, t_0+\delta] \times [0,l] \text{ with } t_0 + x\alpha^{-1} \leq t \leq t_0+\delta \end{cases}.$$

Thus (u,v) is a unique solution of integral system (2.1) on the interval $[t_0, t_0+\delta] \times [0,l]$ and the assertion of this theorem is established.

From Theorem 2, there is a constant $\delta_1 > 0$ such that integral system (2.1) has a unique solution (u_1, v_1) on the interval $[0, \delta_1] \times [0, l]$ for the case $t_0 = 0$. The straight line $t = x\alpha^{-1}$ separates the interval $[0, \delta_1] \times [0, l]$ into two regions and the function has different exhibition on each region. In fact, it can be expressed as following two types.

Type 1: $u_1(t, x) = e^{a(t_0 - t)} u_0(x + \alpha t_0 - \alpha t) + a \int_{t_0}^{t} e^{a(\tau - t)} v(\tau, x + \alpha\tau - \alpha t)\, d\tau$

if $0 \le t < x\alpha^{-1}$;

Type 2: $u_1(t, x) = e^{-ax\alpha^{-1}}\eta + a \int_0^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)} v\left(t - x\alpha^{-1} + \tau, \alpha\tau\right) d\tau$

if $x\alpha^{-1} \le t \le \delta_1$.

Since $u_1(\delta_1, x)$ is a continuous function on $[0, l]$ with $u_1(\delta_1, 0) = \eta$, and $v_1(\delta_1, x)$ is a continuous function on $[0, l]$, there is a constant $\delta_2 > 0$ such that integral system (2.1) has a unique solution (u_2, v_2) on the interval $[\delta_1, \delta_1 + \delta_2] \times [0, l]$. Similarly, the interval $[\delta_1, \delta_1 + \delta_2] \times [0, l]$ is cut by the line $t = \delta_1 + x\alpha^{-1}$ into two regions. The expression of $u_2(t, x)$ is type 1 if $\delta_1 \le t < \delta_1 + x\alpha^{-1}$ and $u_2(t, x)$ is type 2 if $\delta_1 + x\alpha^{-1} \le t \le \delta_1 + \delta_2$. Let functions u, $v : [0, \delta_1 + \delta_2] \times [0, l] \to R$ be defined by

$$u(t, x) = \begin{cases} u_1(t, x) \text{ if } (t, x) \in [0, \delta_1] \times [0, l], \\ u_2(t, x) \text{ if } (t, x) \in [\delta_1, \delta_1 + \delta_2] \times [0, l], \end{cases}$$

and

$$v(t, x) = \begin{cases} v_1(t, x) \text{ if } (t, x) \in [0, \delta_1] \times [0, l], \\ v_2(t, x) \text{ if } (t, x) \in [\delta_1, \delta_1 + \delta_2] \times [0, l]. \end{cases}$$

According to this definition, the interval $[0, \delta_1 + \delta_2] \times [0, l]$ should be divided into four regions (see Figure 1). The expression of $u(t, x)$ is Type 1 if $0 \le t < x\alpha^{-1} \le \delta_1$ or $\delta_1 \le t < \delta_1 + x\alpha^{-1} \le \delta_1 + \delta_2$, and the expression of function $u(t, x)$ is Type 2 if $x\alpha^{-1} \le t \le \delta_1$ or $\delta_1 + x\alpha^{-1} \le t \le \delta_1 + \delta_2$. The expression of $u(t, x)$ will be more complex as one continues to extend the interval of existence. However, we can show that the expression of $u(t, x)$ have only two types in the maximal interval of existence. In the following lemma, we first show that one needs only to consider the expression of $u(t, x)$ in two sub-regions of $[0, \delta_1 + \delta_2] \times [0, l]$ which were separated by the line $t = x\alpha^{-1}$. In fact, the expression of $u(t, x)$ is Type 1 provided that $0 \le t < x\alpha^{-1} \le \delta_1 + \delta_2$, and it is Type 2 as long as $x\alpha^{-1} \le t \le \delta_1 + \delta_2$ (see Figure 2). Therefore, the function (u, v) is the unique solution of the integral system:

$$(2.2)\quad u(t,x)=\begin{cases} e^{-at}u_0(x-\alpha t)+a\displaystyle\int_0^t e^{a(\tau-t)}v(\tau,x+\alpha\tau-\alpha t)\,d\tau \\ \qquad \text{for all } (t,x)\in[0,\delta_1+\delta_2]\times[0,l] \text{ with } 0\le t<x\alpha^{-1}, \\ e^{-ax\alpha^{-1}}\eta+a\displaystyle\int_0^{x\alpha^{-1}} e^{a\left(\tau-x\alpha^{-1}\right)}v\left(t-x\alpha^{-1}+\tau,\alpha\tau\right)d\tau \\ \qquad \text{for all } (t,x)\in[0,\delta_1+\delta_2]\times[0,l] \text{ with } x\alpha^{-1}\le t\le\delta_1+\delta_2; \end{cases}$$

and

$$v(t,x)=e^{-bt}v_0(x)+b\int_0^t e^{b(\tau-t)}u(\tau,x)\,d\tau+\lambda\int_0^t e^{b(\tau-t)}\exp(v(\tau,x))\,d\tau$$
$$\text{for all } (t,x)\in[0,\delta_1+\delta_2]\times[0,l].$$

To simplify the notation, we will use $u_i(j)$ (or $u(j)$) to indicate the expression of the function $u_i(t,x)$ (or $u(t,x)$) over the corresponding region is type j.

Lemma 3. *Let* $t_0=0$, $u_0(x)=u(0,x)$ *and* $v_0(x)=v(0,x)$ *be continuous functions on* $[0,l]$ *with* $u_0(0)=\eta$. *Suppose* (u_1,v_1) *and* (u_2,v_2) *are solutions of integral system (2.1) on the intervals* $[0,\delta_1]\times[0,l]$ *and* $[\delta_1,\delta_1+\delta_2]\times[0,l]$ *respectively. If we define functions* u, $v:[0,\delta_1+\delta_2]\times[0,l]\to R$ *as*

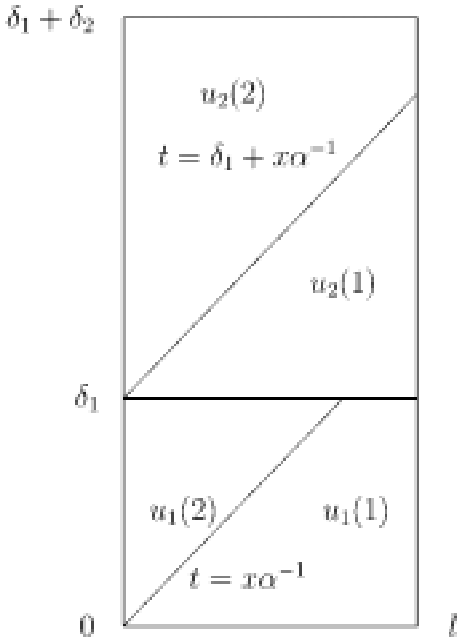

Fig. 1.

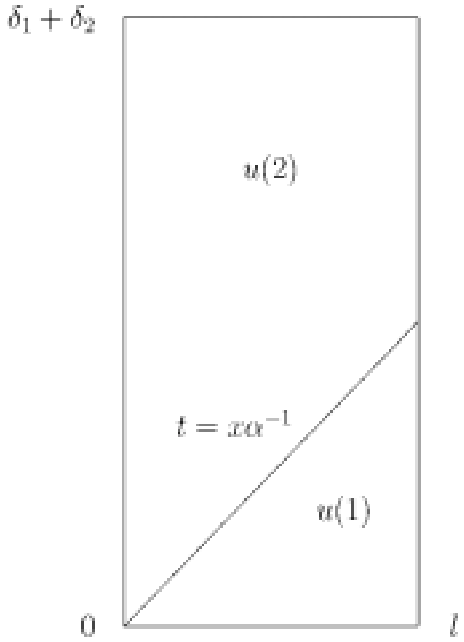

Fig. 2.

$$u(t,x)=\begin{cases} u_1(t,x) \ \text{if} \ (t,x)\in[0,\delta_1]\times[0,l], \\ u_2(t,x) \ \text{if} \ (t,x)\in[\delta_1,\delta_1+\delta_2]\times[0,l], \end{cases}$$

and

$$v(t,x)=\begin{cases} v_1(t,x) \ \text{if} \ (t,x)\in[0,\delta_1]\times[0,l], \\ v_2(t,x) \ \text{if} \ (t,x)\in[\delta_1,\delta_1+\delta_2]\times[0,l]. \end{cases}$$

Then the function (u,v) *is the unique solution of the integral system* (2.2) *on* $[0,\delta_1+\delta_2]\times[0,l]$.

Proof. As shown in Figure 3, we consider the rectangular region $R = [0,\delta_1+\delta_2]\times[0,l]$, which is to be separated into eight regions, (I)˜(VIII), by the lines $t=x\alpha^{-1}$, $t=\delta_1+x\alpha^{-1}$, $t=\delta_1$ and $t=\alpha\delta_1$. The eight regions are

$$\begin{aligned}
\text{(I)} &= \left\{(t,x)\in R: 0\le t<\delta_1, \delta_1<x\alpha^{-1}\right\} \\
\text{(II)} &= \left\{(t,x)\in R: \delta_1\le t<x\alpha^{-1}, \delta_1<x\alpha^{-1}\right\}, \\
\text{(III)} &= \left\{(t,x)\in R: x\alpha^{-1}\le t<\delta_1+x\alpha^{-1}, \delta_1<x\alpha^{-1}\right\}, \\
\text{(IV)} &= \left\{(t,x)\in R: \delta_1+x\alpha^{-1}\le t\le\delta_1+\delta_2, \delta_1<x\alpha^{-1}\right\}, \\
\text{(V)} &= \left\{(t,x)\in R: 0\le t<x\alpha^{-1}, \delta_1\ge x\alpha^{-1}\right\}, \\
\text{(VI)} &= \left\{(t,x)\in R: x\alpha^{-1}\le t<\delta_1, \delta_1\ge x\alpha^{-1}\right\}, \\
\text{(VII)} &= \left\{(t,x)\in R: \delta_1\le t<\delta_1+x\alpha^{-1}, \delta_1\ge x\alpha^{-1}\right\} \text{ and} \\
\text{(VII)} &= \left\{(t,x)\in R: \delta_1+x\alpha^{-1}\le t\le\delta_1+\delta_2, \delta_1\ge x\alpha^{-1}\right\}, \text{respectively.}
\end{aligned}$$

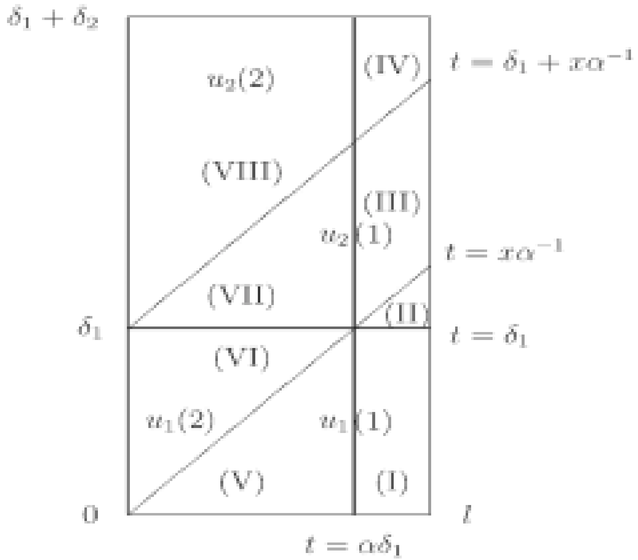

Fig. 3.

Since the expression of $u_1(t,x)$ is type 1 in both regions (I) and (V), and the expression of $u_2(t,x)$ is type 1 in the region (II). It follows that the expression of $u(t,x)$ is type 1 in these three regions. Similarly, since the expression of $u_1(t,x)$ is type 2 in the region (VI) and the expression of $u_2(t,x)$ is type 2 in both regions (IV) and (VIII), the expression of $u(t,x)$ is type 2 in these three regions. If we can show the expressions of $u(t,x)$ in both regions (VII) and (III) are type 2, then the assertion of this lemma is established. Choose any (t,x) in the region (VII), then $x\alpha^{-1}-t\le 0$ and $\delta_1\ge\alpha^{-1}(x+\alpha\delta_1-\alpha t)$. Denote $y=x+\alpha\delta_1-\alpha t$, then

$$\begin{aligned}
u(\delta_1,x+\alpha\delta_1-\alpha t)&=u(\delta_1,y)\\
&=e^{-ay\alpha^{-1}}\eta+a\int_0^{y\alpha^{-1}}e^{a\left(\tau-y\alpha^{-1}\right)}v\left(t-y\alpha^{-1}+\tau,\alpha\tau\right)d\tau\\
&=e^{-ay\alpha^{-1}}\eta+a\int_{\delta_1-y\alpha^{-1}}^{\delta_1}e^{a(\zeta-\delta_1)}v(\zeta,y+\alpha\zeta-\alpha\delta_1)\,d\zeta\\
&=e^{-ax\alpha^{-1}}e^{-a(\delta_1-t)}\eta+a\int_{t-x\alpha^{-1}}^{\delta_1}e^{a(\tau-\delta_1)}v(\tau,x+\alpha\tau-\alpha t)\,d\tau.
\end{aligned}$$

Hence,

$$\begin{aligned}
&u(t,x)\\
&=e^{-a(t-\delta_1)}u(\delta_1,x+\alpha\delta_1-\alpha t)+a\int_{\delta_1}^{t}e^{a(\tau-t)}v(\tau,x+\alpha\tau-\alpha t)\,d\tau\\
&=e^{-ax\alpha^{-1}}\eta+a\int_{t-x\alpha^{-1}}^{\delta_1}e^{a(\tau-t)}v(\tau,x+\alpha\tau-\alpha t)\,d\tau\\
&\quad+a\int_{\delta_1}^{t}e^{a(\tau-t)}v(\tau,x+\alpha\tau-\alpha t)\,d\tau\\
&=e^{-ax\alpha^{-1}}\eta+a\int_{t-x\alpha^{-1}}^{t}e^{a(\tau-t)}v(\tau,x+\alpha\tau-\alpha t)\,d\tau\\
&=e^{-ax\alpha^{-1}}\eta+a\int_0^{x\alpha^{-1}}e^{a\left(\tau-x\alpha^{-1}\right)}v\left(t-x\alpha^{-1}+\tau,\alpha\tau\right)d\tau.
\end{aligned}$$

This implies the expression of $u(t,x)$ is Type 2 for any (t,x) in the region (VII). Similarly, one may show that the expression of $u(t,x)$ is Type 2 for all (t,x) in the region (III). The proof of this lemma is complete now.

From Theorem 2 and Lemma 3, we immediately obtain the following theorem.

Theorem 4. *Suppose* $u_0(x)=u(0,x)$ *and* $v_0(x)=v(0,x)$ *are continuous functions on* $[0,l]$ *with* $u_0(0)=\eta$*. Then there is a* $t_{\max}$ *(* $t_{\max}$ *is either a finite number or infinite) such that* $[0,t_{\max})$ *is the maximal time interval for the unique solution* (u,v) *of the integral system*

$$(2.3) \quad u(t,x) = \begin{cases} e^{-at}u_0(x-\alpha t) + a\int_0^t e^{a(\tau-t)} v(\tau, x+\alpha\tau-\alpha t)\,d\tau \\ \quad \text{for all } (t,x)\in[0,t_{\max})\times[0,l] \text{ with } 0\le t< x\alpha^{-1}, \\ e^{-ax\alpha^{-1}}\eta + a\int_0^{x\alpha^{-1}} e^{a\left(\tau-x\alpha^{-1}\right)} v\left(t-x\alpha^{-1}+\tau,\alpha\tau\right)d\tau \\ \quad \text{for all } (t,x)\in[0,t_{\max})\times[0,l] \text{ with } x\alpha^{-1}\le t<t_{\max}, \end{cases}$$

$$v(t,x) = e^{-bt}v_0(x) + b\int_0^t e^{b(\tau-t)}u(\tau,x)\,d\tau + \lambda\int_0^t e^{b(\tau-t)}\exp(v(\tau,x))\,d\tau$$
$$\text{for all } (t,x)\in[0,t_{\max})\times[0,l].$$

Theorem 5. *Suppose $u_0(x) = u(0,x)$ and $v_0(x) = v(0,x)$ are continuous functions on $[0,l]$ with $u_0(0)=\eta$. Let (u,v) be the unique solution of integral system (2.3) on $[0,t_{\max})\times[0,l]$, where $[0,t_{\max})$ is the maximal time interval. If $t_{\max}$ is finite, then*

$$\lim_{t\to t_{\max}} \sup_{x\in[0,l]} |v(t,x)| = \infty.$$

Proof. We will first prove that if $t_{\max}$ is finite, then

$$\overline{\lim_{t\to t_{\max}}}\left(\sup_{x\in[0,l]}|u(t,x)| + \sup_{x\in[0,l]}|v(t,x)|\right) = \infty.$$

Indeed, if $t_{\max}$ is finite and $\overline{\lim\limits_{t\to t_{\max}}}\left(\sup\limits_{x\in[0,l]}|u(t,x)| + \sup\limits_{x\in[0,l]}|v(t,x)|\right) < \infty$, then there is a constant $K_1>0$ such that

$$\sup_{x\in[0,l]}|u(t,x)| + \sup_{x\in[0,l]}|v(t,x)| < K_1 \qquad \text{for all } 0\le t<t_{\max}.$$

Since the function $\exp(-bt)$ is the uniformly continuous on $[0,t_{\max}]$. For any $\varepsilon>0$, there is a constant $0<\delta<\frac{\varepsilon}{2\lambda e^{K_1}+2bK_1}$ such that

$$|\exp(-bt') - \exp(-bt)| < \frac{\varepsilon}{2K_1 + 2\left(\lambda b^{-1}e^{K_1}+K_1\right)e^{bt_{\max}}}$$

for all t', $t\in[0,t_{\max}]$ with $|t'-t|<\delta$. This implies that

$$\begin{aligned}
&\left|v\left(t',x\right) - v(t,x)\right| \\
&\le \left|e^{-bt'} - e^{-bt}\right| |v_0(x)| + b\int_0^t e^{b\tau}\left|e^{-bt'}-e^{-bt}\right| |u(\tau,x)|\,d\tau + b\int_t^{t'} e^{b(\tau-t')}|u(\tau,x)|\,d\tau \\
&\quad + \lambda\int_0^t e^{b\tau}\left|e^{-bt'}-e^{-bt}\right| e^{|v(\tau,x)|}d\tau + \lambda\int_t^{t'} e^{b(\tau-t')}e^{|v(\tau,x)|}d\tau \\
&\le \left|e^{-bt'}-e^{-bt}\right|\left\{K_1 + e^{bt_{\max}}K_1 + \lambda b^{-1}e^{bt_{\max}}e^{K_1}\right\} + \left\{bK_1+\lambda e^{K_1}\right\}\left|t'-t\right| \\
&< \varepsilon
\end{aligned}$$

for any fixed $x \in [0,l]$ and all $0 \le t \le t' < t_{\max}$ such that $0 < t' - t < \delta$. Hence, $\sup_{x\in[0,l]} |v(t',x) - v(t,x)| \le \varepsilon$ for all $0 \le t \le t' < t_{\max}$ which satisfy $0 < t' - t < \delta$. This implies that $\lim_{t\to t_{\max}} v(t,x)$ exists for each x in $[0,l]$ and $\lim_{t'\to t} \sup_{x\in[0,l]} |v(t',x) - v(t,x)| = 0$ uniformly on $[0, t_{\max})$. To prove that $\lim_{t\to t_{\max}} u(t,x)$ exists for each $x \in [0,l]$, we need to consider following two cases: $t_{\max} > x\alpha^{-1}$ and $x\alpha^{-1} \ge t_{\max}$ separately. At first, we consider the case $t_{\max} > x\alpha^{-1}$. Since $\lim_{t'\to t} \sup_{x\in[0,l]} |v(t',x) - v(t,x)| = 0$ uniformly on the interval $[0, t_{\max})$, for any $\varepsilon > 0$, there is a constant $\sigma_1 > 0$ such that

$$\sup_{x\in[0,l]} \left|v\left(t',x\right) - v(t,x)\right| < \varepsilon \qquad \text{for all } t,\ t' \in [0,t_{\max}) \text{ with } |t'-t| \le \sigma_1.$$

Let $0 < \sigma_2 < \min\left\{\sigma_1, 2^{-1}\left(t_{\max} - x\alpha^{-1}\right)\right\}$. Since

$$t_{\max} - \sigma_2 > t_{\max} - 2^{-1}\left\{t_{\max} - x\alpha^{-1}\right\} = 2^{-1}\left\{t_{\max} + x\alpha^{-1}\right\} \ge x\alpha^{-1},$$

we obtain that $x\alpha^{-1} \le t \le t' < t_{\max}$ as long as $t_{\max} - \sigma_2 \le t \le t' < t_{\max}$. If t and t' satisfy $0 \le t_{\max} - \sigma_2 \le t \le t' < t_{\max}$, then

$$\begin{aligned}
&\left|u\left(t',x\right) - u(t,x)\right| \\
&\le a\int_0^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)} \left|v\left(t' - x\alpha^{-1} + \tau, \alpha\tau\right) - v\left(t - x\alpha^{-1} + \tau, \alpha\tau\right)\right| d\tau \\
&\le \varepsilon.
\end{aligned}$$

Secondly, we consider the case $t_{\max} \le x\alpha^{-1}$. For any $\varepsilon > 0$, there is a constant $0 < \sigma_3 < \min\{\varepsilon, \sigma_1\}$ such that $\left|e^{-at'}u_0(x-\alpha t') - e^{-at}u_0(x-\alpha t)\right| < \varepsilon$, for all $t,\ t' \in [0, t_{\max})$, and $|t'-t| \le \sigma_3$.

$$\begin{aligned}
&\left|u\left(t',x\right) - u(t,x)\right| \\
&\le \left|e^{-at'}u_0\left(x - \alpha t'\right) - e^{-at}u_0(x-\alpha t)\right| + \int_t^{t'} \left|v\left(\tau, x + \alpha\tau - \alpha t'\right)\right| d\tau \\
&\qquad + \int_0^t \left|v\left(\tau, x + \alpha\tau - \alpha t'\right) - v(\tau, x + \alpha\tau - \alpha t)\right| d\tau \\
&\le \varepsilon + K_1\left|t' - t\right| + \varepsilon t \\
&\le \varepsilon + K_1\varepsilon + \varepsilon t_{\max}.
\end{aligned}$$

This implies that $\lim_{t\to t_{\max}} u(t,x)$ exists for each fixed $x \in [0,l]$. Denote $u(t_{\max},x) = \lim_{t\to t_{\max}} u(t,x)$ and $v(t_{\max},x) = \lim_{t\to t_{\max}} v\,(t,x)$ for each x in $[0,l]$. With the same technique used in the proof of Theorem 2, the solution (u,v) can be extended beyond $t_{\max}$. This contradicts to the choice of $t_{\max}$. Therefore, we obtain that

$$\overline{\lim_{t\to t_{\max}}}\left(\sup_{x\in[0,l]}|u(t,x)| + \sup_{x\in[0,l]}|v(t,x)|\right) = \infty.$$

Now, we show that

$$\lim_{t\to t_{\max}}\left(\sup_{x\in[0,l]}|u(t,x)| + \sup_{x\in[0,l]}|v(t,x)|\right) = \infty.$$

If this is false, then there are constants $M > 0$ and a sequence $\{t_n : n \in N\}$ in $[0, t_{\max})$ such that $t_n \nearrow t_{\max}$ and

$$\sup_{x\in[0,l]}|u(t_n,x)| + \sup_{x\in[0,l]}|v(t_n,x)| \le M \qquad \text{for all } n \in N.$$

This implies that $|u(t_n,x)| + |v(t_n,x)| \le M$ for all $x \in [0,l]$ and $n \in N$. Since the function $t \longmapsto \sup_{x\in[0,l]}|u(t,x)| + \sup_{x\in[0,l]}|v(t,x)|$ is continuous on $[0, t_{\max})$ and

$$\overline{\lim_{t\to t_{\max}}}\left(\sup_{x\in[0,l]}|u(t,x)| + \sup_{x\in[0,l]}|v(t,x)|\right) = \infty,$$

without loss of generality, we may assume that there exists a sequence $\{h_n : n \in N\}$ which satisfies that $h_n \to \infty$ as $n \to \infty$,

$$\sup_{x\in[0,l]}|u(t_n+h_n,x)| + \sup_{x\in[0,l]}|v(t_n+h_n,x)| = 2M+1$$

and

$$\sup_{x\in[0,l]}|u(t,x)| + \sup_{x\in[0,l]}|v(t,x)| < 2M+1 \qquad \text{for all } t_n \le t < t_n + h_n.$$

Since for each positive integer n,

$$|v(t_n+h_n,x)|$$

$$\leq e^{-bh_n}\left|v\left(t_n,x\right)\right|+b\int_{t_n}^{t_n+h_n} e^{b(\tau-t_n-h_n)}\left|u\left(\tau,x\right)\right|d\tau$$
$$+\lambda\int_{t_n}^{t_n+h_n} e^{b(\tau-t_n-h_n)}e^{|v(\tau,x)|}d\tau$$

and

$$\left|u\left(t_n+h_n,x\right)\right|$$
$$\leq\begin{cases} e^{-ah_n}\left|u\left(t_n,x-\alpha h_n\right)\right|+a\int_{t_n}^{t_n+h_n} e^{a(\tau-t_n-h_n)}\left|v\left(\tau,x+\alpha\tau-\alpha t_n-\alpha h_n\right)\right|d\tau \\ \quad\text{for all } (t,x)\in[t_n,t_{\max})\times[0,l] \text{ with } t_n\leq t<t_n+x\alpha^{-1}, \\ e^{-ax\alpha^{-1}}\eta+a\int_0^{x\alpha^{-1}} e^{a\left(\tau-x\alpha^{-1}\right)}\left|v\left(t_n+h_n-x\alpha^{-1}+\tau,\alpha\tau\right)\right|d\tau \\ \quad\text{for all } (t,x)\in[t_n,t_{\max})\times[0,l] \text{ with } t_n+x\alpha^{-1}\leq t<t_{\max}, \end{cases}$$

this implies that

$$\sup_{x\in[0,l]}\left|v\left(t_n+h_n,x\right)\right|\leq M+\left\{b\left(2M+1\right)+\lambda e^{(2M+1)}\right\}h_n \quad \text{for all } n\in N,$$

and

$$\sup_{x\in[0,l]}\left|u\left(t_n+h_n,x\right)\right|\leq M+a\left(2M+1\right)h_n \qquad \text{for all } n\in N.$$

Hence,

$$2M+1=\sup_{x\in[0,l]}\left|u\left(t_n+h_n,x\right)\right|+\sup_{x\in[0,l]}\left|v\left(t_n+h_n,x\right)\right|$$
$$\leq 2M+\left\{(a+b)\left(2M+1\right)+\lambda e^{(2M+1)}\right\}h_n$$

which is impossible as $n\to\infty$. Therefore, we have

$$\lim_{t\to t_{\max}}\left(\sup_{x\in[0,l]}\left|u\left(t,x\right)\right|+\sup_{x\in[0,l]}\left|v\left(t,x\right)\right|\right)=\infty.$$

To complete the proof we need only to show that $\lim_{t\to t_{\max}}\sup_{x\in[0,l]}\left|v\left(t,x\right)\right|=\infty$. Assume by contradiction, $\lim_{t\to t_{\max}}\sup_{x\in[0,l]}\left|v\left(t,x\right)\right|$ is finite. Then there is a constant $\widetilde{M}>0$ such that $\sup_{x\in[0,l]}\left|v\left(t,x\right)\right|\leq\widetilde{M}$ for all $t\in[0,t_{\max})$. This implies that

$$\left|u\left(t,x\right)\right|\leq\left\|u_0\right\|_\infty+\widetilde{M} \qquad \text{for all } (t,x)\in[0,t_{\max})\times[0,l].$$

Hence,

$$\lim_{t\to t_{\max}}\left(\sup_{x\in[0,l]}|u(t,x)|+\sup_{x\in[0,l]}|v(t,x)|\right)\le\left(\|u_0\|_\infty+\widetilde{M}\right)+\widetilde{M}<\infty.$$

This is contrary to the previous proof. Therefore, we have $\lim_{t\to t_{\max}}\sup_{x\in[0,l]}|v(t,x)|=\infty$ and the assertion of this theorem seems to be true.

Now, we can prove that the solution (u,v) of the integral system (2.3) is also the solution of the differential equation (1.2) on $[0,t_{\max})\times[0,l]$ provided that the initial function $u_0(x)=u(0,x)\in C^1(0,l)\cap C[0,l]$.

Theorem 6. *Suppose $u_0(x)=u(0,x)\in C^1(0,l)\cap C[0,l]$ and $v_0(x)=v(0,x)\in C[0,l]$ with $u_0(0)=\eta$. There is a constant $t_{\max}>0$ such that $[0,t_{\max})$ is the maximal time interval for the unique solution (u,v) of the differential equation (1.2) on the interval $[0,t_{\max})\times[0,l]$.*

Moreover, if $t_{\max}$ is finite, then $\lim_{t\to t_{\max}}\sup_{x\in[0,l]}|v(t,x)|=\infty$.

Proof. From Theorem 4, there is a constant $t_{\max}>0$ such that $[0,t_{\max})$ is the maximal time interval for the unique solution (u,v) of the integral system

$$u(t,x)=\begin{cases} e^{-at}u_0(x-\alpha t)+a\displaystyle\int_0^t e^{a(\tau-t)}v(\tau,x+\alpha\tau-\alpha t)\,d\tau \\ \qquad\text{for all }(t,x)\in[0,t_{\max})\times[0,l]\text{ with }0\le t<x\alpha^{-1} \\ e^{-ax\alpha^{-1}}\eta+a\displaystyle\int_0^{x\alpha^{-1}} e^{a\left(\tau-x\alpha^{-1}\right)}v\left(t-x\alpha^{-1}+\tau,\alpha\tau\right)d\tau \\ \qquad\text{for all }(t,x)\in[0,t_{\max})\times[0,l]\text{ with }x\alpha^{-1}\le t<t_{\max}\end{cases}$$

and

$$v(t,x)=e^{-bt}v_0(x)+b\int_0^t e^{b(\tau-t)}u(\tau,x)\,d\tau+\lambda\int_0^t e^{b(\tau-t)}\exp(v(\tau,x))\,d\tau$$
$$\text{for all }(t,x)\in[0,t_{\max})\times[0,l].$$

Since $u(t,x)$ and $v(t,x)$ are continuous on $[0,t_{\max})\times[0,l]$, $v(t,x)$ is differentiable respect to t on $(0,t_{\max})$ and

$$v_t(t,x)=-bv(t,x)+bu(t,x)+\lambda e^{v(t,x)}\qquad\text{for all }t>0,\ 0<x<l.$$

If $(t,x)\in(0,t_{\max})\times(0,l)$ such that $0\le t<x\alpha^{-1}$, then

$$u_t(t,x)=\frac{\partial}{\partial t}\left\{e^{-at}u_0(x-\alpha t)+a\int_{x\alpha^{-1}-t}^{x\alpha^{-1}}e^{a\left(\tau-x\alpha^{-1}\right)}v\left(\tau+t-x\alpha^{-1},\alpha\tau\right)d\tau\right\}$$

$$= -ae^{-at}u_0\left(x-\alpha t\right) - \alpha e^{-at}u_0'\left(x-\alpha t\right) + ae^{-at}v\left(0, x-\alpha t\right)$$
$$+a\int_{x\alpha^{-1}-t}^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)}v_t\left(\tau + t - x\alpha^{-1}, \alpha\tau\right)d\tau$$

and

$$u_x\left(t,x\right) = \frac{\partial}{\partial x}\left\{e^{-at}u_0\left(x-\alpha t\right) + a\int_{x\alpha^{-1}-t}^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)}v\left(\tau + t - x\alpha^{-1}, \alpha\tau\right)d\tau\right\}$$
$$= e^{-at}u_0'\left(x-\alpha t\right) + a\alpha^{-1}\left\{v\left(t,x\right) - e^{-at}v\left(0, x-\alpha t\right)\right\}$$
$$-a^2\alpha^{-1}\int_{x\alpha^{-1}-t}^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)}v\left(\tau + t - x\alpha^{-1}, \alpha\tau\right)d\tau$$
$$-a\alpha^{-1}\int_{x\alpha^{-1}-t}^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)}v_t\left(\tau + t - x\alpha^{-1}, \alpha\tau\right)d\tau.$$

This implies that $u_t\left(t,x\right) = -\alpha u_x\left(t,x\right) + av\left(t,x\right) - au\left(t,x\right)$ for $\left(t,x\right) \in \left(0, t_{\max}\right) \times \left(0, l\right)$ with $0 \le t < x\alpha^{-1}$. On the other hand, if $\left(t,x\right) \in \left(0, t_{\max}\right) \times \left(0, l\right)$ such that $x\alpha^{-1} < t < t_{\max}$, then

$$u_t\left(t,x\right) = a\int_0^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)}v_t\left(t - x\alpha^{-1} + \tau, \alpha\tau\right)d\tau$$

and

$$u_x\left(t,x\right) = -a\alpha^{-1}e^{-ax\alpha^{-1}}\eta + av\left(t,x\right) - a^2\alpha^{-1}$$
$$\int_0^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)}v\left(t - x\alpha^{-1} + \tau, \alpha\tau\right)d\tau$$
$$-a\alpha^{-1}\int_0^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)}v_t\left(t - x\alpha^{-1} + \tau, \alpha\tau\right)d\tau.$$

This implies that $u_t\left(t,x\right) = -\alpha u_x\left(t,x\right) + av\left(t,x\right) - au\left(t,x\right)$ for all $\left(t,x\right) \in \left(0, t_{\max}\right) \times \left(0, l\right)$ with $x\alpha^{-1} < t < t_{\max}$. From continuity of $u\left(t,x\right)$ and $v\left(t,x\right)$, we have

$$u_t\left(t,x\right) = -\alpha u_x\left(t,x\right) + av\left(t,x\right) - au\left(t,x\right)$$

for all $\left(t,x\right) \in \left(0, t_{\max}\right) \times \left(0, l\right)$ with $x\alpha^{-1} = t$. Therefore, $[0, t_{\max})$ is the maximal time interval for the unique solution of $\left(u,v\right)$ for the differential equation (1.2) on the interval $[0, t_{\max}) \times [0, l]$. From Theorem 5, if $t_{\max}$ is finite, then $\lim_{t \to t_{\max}} \sup_{x \in [0,l]} \left|v\left(t,x\right)\right| = \infty$. The proof of this theorem is complete now.

Theorem 6 says that $\lim_{t\to t_{\max}} \sup_{x\in[0,l]} |v(t,x)| = \infty$ as long as $t_{\max}$ is finite. However, $t_{\max} < \infty$ is not a crucial condition for $|v(t,x)| \to \infty$. We can show that $\lim_{t\to\infty} v(t,x) = \infty$ under the assumption of positive initial conditions, when the existence of $v(\cdot,x)$ is globally. In fact, we have following result.

Theorem 7. *Suppose* $u_0(x) = u(0,x) \in C^1(0,l) \cap C[0,l]$ *and* $v_0(x) = v(0,x) \in C[0,l]$ *with satisfy* $u_0(x) \geq 0$, $v_0(x) \geq 0$ *on* $[0,l]$ *and* $u_0(0) = \eta > \max\left\{0, e^{a\alpha^{-1}l}\left[\ln\left(b\lambda^{-1}\right) - 1\right]\right\}$. *If* (u,v) *is the solution of the differential equation (1.2) on the interval* $[0,\infty)\times[0,l]$, *then*

$$\lim_{t\to\infty} v(t,x) = \infty.$$

Proof. At first, we consider the following differential system:

$$(2.4)\qquad \begin{cases} \dfrac{\partial}{\partial t}u(t,x) = -\alpha\dfrac{\partial}{\partial x}u(t,x) + av(t,x) - au(t,x) \ \text{ for } t>0,\ x>0; \\ \dfrac{\partial}{\partial t}v(t,x) = bu(t,x) - bv(t,x) \ \text{ for } t>0,\ x>0; \\ u(t,0) = \eta \ \text{for } t \geq 0; \\ u(0,x) = u_0(x) \ \text{ and } v(0,x) = v_0(x) \ \text{ for } x>0. \end{cases}$$

Let $\varphi_0(t,x) = u_0(x)$ and $\phi_0(t,x) = v_0(x)$ for all $(t,x)\in[0,\infty)\times[0,l]$. For each positive integer n, we define functions φ_n and ϕ_n as

$$\varphi_n(t,x) = \begin{cases} e^{-at}u_0(x-\alpha t) + a\displaystyle\int_0^t e^{a(\tau-t)}\phi_{n-1}(\tau, x+\alpha\tau-\alpha t)\,d\tau \\ \qquad \text{for all } (t,x)\in[0,\infty)\times[0,l] \text{ with } 0\leq t< x\alpha^{-1}, \\ e^{-ax\alpha^{-1}}\eta + a\displaystyle\int_0^{x\alpha^{-1}} e^{a\left(\tau - x\alpha^{-1}\right)}\phi_{n-1}\left(t - x\alpha^{-1}+\tau, \alpha\tau\right)d\tau \\ \qquad \text{for all } (t,x)\in[0,\infty)\times[0,l] \text{ with } x\alpha^{-1}\leq t<\infty, \end{cases}$$

and

$$\phi_n(t,x) = e^{-bt}v_0(x) + b\int_0^t e^{b(\tau-t)}\varphi_{n-1}(\tau,x)\,d\tau \text{ for all } (t,x)\in[0,\infty)\times[0,l].$$

Then φ_n and ϕ_n are continuous on $[0,\infty)\times[0,l]$ for all $n = 0,1,2,3,\ldots$. Let $k = \max\{a,b\}$ and $K = 2\|u_0\|_\infty + \|v_0\|_\infty$. Then

$$\begin{aligned} |\varphi_1(t,x) - \varphi_0(t,x)| &\leq 2\|u_0\|_\infty + a\int_0^t e^{a(\tau-t)}\|v_0\|_\infty\,d\tau \\ &\leq 2\|u_0\|_\infty + \|v_0\|_\infty = K, \end{aligned}$$

and

$$\begin{aligned} \left|\phi_1(t,x)-\phi_0(t,x)\right| &\le \left(1-e^{-bt}\right)\|v_0\|_\infty + b\int_0^t e^{b(\tau-t)}\|u_0\|_\infty\, d\tau \\ &\le \|u_0\|_\infty + \|v_0\|_\infty \le K \end{aligned}$$

for all $(t,x)\in[0,\infty)\times[0,l]$. One may easily show by induction that

$$\left|\varphi_{n+1}(t,x)-\varphi_n(t,x)\right| \le K\tfrac{(kt)^n}{n!} \text{ and } \left|\phi_{n+1}(t,x)-\phi_n(t,x)\right| \le K\tfrac{(kt)^n}{n!}$$

for all $(t,x)\in[0,\infty)\times[0,l]$ and $n=0,1,2,3,....$ Therefore, for arbitrary fixed constant $T>0$,

$$\|\varphi_{n+1}-\varphi_n\| = \sup\left\{\left|\varphi_{n+1}(t,x)-\varphi_n(t,x)\right| : (t,x)\in[0,T]\times[0,l]\right\} \le K\tfrac{(kT)^n}{n!}$$

and

$$\|\phi_{n+1}-\phi_n\| = \sup\left\{\left|\phi_{n+1}(t,x)-\phi_n(t,x)\right| : (t,x)\in[0,T]\times[0,l]\right\} \le K\tfrac{(kT)^n}{n!}.$$

From the continuity of exponential function, for any $\varepsilon>0$, there is a constant $n_0\in N$ such that $e^{kT}-\sum_{i=0}^{n-1}\frac{(kT)^i}{i!}<\varepsilon K^{-1}$ for all $n\ge n_0$. Then for any $m>n\ge n_0$,

$$\begin{aligned} \|\varphi_m-\varphi_n\| &\le \|\varphi_m-\varphi_{m-1}\|+\|\varphi_{m-1}-\varphi_{m-2}\|+...+\|\varphi_{n+2}-\varphi_{n+1}\|+\|\varphi_{n+1}-\varphi_n\| \\ &\le K\left\{\frac{(kT)^{m-1}}{(m-1)!}+\frac{(kT)^{m-2}}{(m-2)!}+...+\frac{(kT)^{n+1}}{(n+1)!}+\frac{(kT)^n}{n!}\right\} \\ &\le K\left\{e^{kT}-\sum_{i=0}^{n-1}\frac{(kT)^i}{i!}\right\} \\ &< \varepsilon. \end{aligned}$$

Similarly, one may show $\|\phi_m-\phi_n\|<\varepsilon$ for any $m>n\ge n_0$. Hence, $\lim\limits_{n\to\infty}\varphi_n(t,x)$ and $\lim\limits_{n\to\infty}\phi_n(t,x)$ converge uniformly on $[0,T]\times[0,l]$. Denote

$$\lim_{n\to\infty}\varphi_n(t,x)=\widetilde{u}(t,x) \text{ and } \lim_{n\to\infty}\phi_n(t,x)=\widetilde{v}(t,x) \quad \text{on } [0,T]\times[0,l].$$

Since φ_n and ϕ_n are continuous on $[0,T]\times[0,l]$ for all $n=0,1,2,3,...$, the uniform convergence implies that $\widetilde{u}(t,x)$ and $\widetilde{v}(t,x)$ are continuous on $[0,T]\times[0,l]$. Since $T>0$ is arbitrary, $\widetilde{u}$, $\widetilde{v}\in C([0,\infty)\times[0,l])$. Furthermore, they can be expressed as

$$\widetilde{u}(t,x)=\begin{cases} e^{-at}u_0(x-\alpha t)+a\displaystyle\int_0^t e^{a(\tau-t)}\widetilde{v}(\tau,x+\alpha\tau-\alpha t)\,d\tau \\ \qquad \text{for all } (t,x)\in[0,\infty)\times[0,l] \text{ with } 0\le t<x\alpha^{-1}, \\ e^{-ax\alpha^{-1}}\eta+a\displaystyle\int_0^{x\alpha^{-1}} e^{a\left(\tau-x\alpha^{-1}\right)}\widetilde{v}\left(t-x\alpha^{-1}+\tau,\alpha\tau\right)d\tau \\ \qquad \text{for all } (t,x)\in[0,\infty)\times[0,l] \text{ with } x\alpha^{-1}\le t<\infty. \end{cases}$$

and

$$\widetilde{v}(t,x)=e^{-bt}v_0(x)+b\textstyle\int_0^t e^{b(\tau-t)}\widetilde{u}(\tau,x)\,d\tau \text{ for all } (t,x)\in[0,\infty)\times[0,l].$$

With analogous arguments as in the proofs of Theorem 1 and Theorem 6, one can show that $(\widetilde{u},\widetilde{v})$ is the unique solution of the differential system (2.4) on $[0,\infty)\times[0,l]$.

On the other hand, since $\varphi_0(t,x)=u_0(x)\ge 0$ and $\phi_0(t,x)=v_0(x)\ge 0$ on $[0,\infty)\times[0,l]$, $\varphi_n(t,x)\ge 0$ and $\phi_n(t,x)\ge 0$ on $[0,\infty)\times[0,l]$. This implies that

$$\widetilde{u}(t,x)=\lim_{n\to\infty}\varphi_n(t,x)\ge 0 \text{ and } \widetilde{v}(t,x)=\lim_{n\to\infty}\phi_n(t,x)\ge 0 \quad \text{on } [0,\infty)\times[0,l].$$

Let $w_1(t,x)=u(t,x)-\widetilde{u}(t,x)$ and $w_2(t,x)=v(t,x)-\widetilde{v}(t,x)$ for all $(t,x)\in[0,\infty)\times[0,l]$. Since (u,v) is the solution of the differential equation (1.2) on the interval $[0,\infty)\times[0,l]$, (w_1,w_2) satisfy the following differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}w_1(t,x)=-\alpha\dfrac{\partial}{\partial x}w_1(t,x)+aw_2(t,x)-aw_1(t,x) \text{ for } t>0,\ x>0; \\ \dfrac{\partial}{\partial t}w_2(t,x)=bw_1(t,x)-bw_2(t,x)+\lambda\exp(v(t,x)) \text{ for } t>0,\ x>0; \\ w_1(t,0)=0 \text{ for } t\ge 0; \\ w_1(0,x)=0 \text{ and } w_2(0,x)=0 \text{ for } x>0. \end{cases}$$

With analogous arguments as in the previous proof, $w_1(t,x)\ge 0$ and $w_2(t,x)\ge 0$ on $[0,\infty)\times[0,l]$. Therefore, $u(t,x)\ge\widetilde{u}(t,x)\ge 0$ and $v(t,x)\ge\widetilde{v}(t,x)\ge 0$ for all $(t,x)\in[0,\infty)\times[0,l]$.

Since the existence of $v(t,x)$ is globally in t, one may choose t, large enough, such that $\alpha t>l$. This implies that

$$\begin{aligned} v_t(t,x)&=-bv(t,x)+be^{-ax\alpha^{-1}}\eta \\ &\quad+ab\int_0^{x\alpha^{-1}} e^{a\left(\tau-x\alpha^{-1}\right)}v\left(t-x\alpha^{-1}+\tau,\alpha\tau\right)d\tau+\lambda e^{v(t,x)} \end{aligned}$$

$$\geq -bv\,(t,x) + be^{-a\alpha^{-1}l}\eta + \lambda e^{v(t,x)} \quad \text{whenever } b \geq \lambda.$$

Since $\lambda e^y - by \geq b\left(1 - \ln b\lambda^{-1}\right)$ for all $y \geq 0$, we have $v_t(t,x) \geq b(1 - \ln b\lambda^{-1} + e^{-a\alpha^{-1}l}\eta) > 0$ for t large enough and $b \geq \lambda$.

On the other hand, if $b < \lambda$, then $\lambda e^y -$ by$\geq \lambda$ for all $y \geq 0$. Hence, we also have $v_t\,(t,x) \geq be^{-a\alpha^{-1}l}\eta + \lambda > 0$ for t large enough and $b < \lambda$.

This implies that $\lim_{t\to\infty} v\,(t,x) = \infty$. The proof of this theorem is complete now.

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