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SOME FIXED-POINT THEOREMS ON AN ALMOST G-CONVEX SUBSET OF A LOCALLY G-CONVEX SPACE AND ITS APPLICATIONS

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Abstract. In this paper, we first obtain the generalizations of the almost fixed point theorems on the almost G-convex sets and the Himmelberg fixed point theorems on a locally G-convex space. Next, we invoke non-convexity of constraint regions in place of convexity and we obtain the new fixed point theorems, " Let X be an almost G-convex subset of a locally G-convex space E. If $T \in \Gamma^* - KKM(X, X)$ is compact and closed, then T has a fixed point."

1. Introduction and Preliminaries

In 1929, Knaster, Kuratoaski and Mazurkiewicz [5] proved the well-known KKM theorem on n-simplex. In addition, in 1961, Ky Fan [3] generalized the KKM theorem in the infinite dimensional topological vector space. Later, the KKM theorem and related topics, for example, matching theorem, fixed point theorem, coincidence theorem, economic equilibrium problem, minmax inequalities and so on had been presented a grand occasions, (see, [2, 6-10]).

Along with the above development, many features of the concept of convex sets are extended to some general convexities; for example, convex spaces of Lassonde, C-spaces of Horvath, and others. These general convexities are all subsumed to the concept of G-convex spaces.

In this paper, we first obtain an elementary proof of generalizations of Himmelberg fixed point theorem on the G-convex spaces by applying the $\Gamma - KKM$ theorem. Next, we invoke non-convexity of constraint regions in place of convexity and we obtain the new fixed point theorems, " Let X be an almost G-convex subset

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of a locally G-convex space E. If $T \in \Gamma^* - KKM(X, X)$ is compact and closed, then T has a fixed point."

We now introduce the notations used in this paper and recall some basic facts.

For a nonempty set X, 2^X denotes the class of all nonempty subsets of X, and $\langle X \rangle$ denotes the class of all nonempty finite subsets of X.

Throughout this paper, for a set-valued function $T : X \to 2^Y$, the following notations are used:

(i) $T(x) = \{y \in Y | y \in T(x)\}$,

(ii) $T(A) = \cup_{x \in A} T(x)$,

(iii) $T^{-1}(y) = \{x \in X | y \in T(x)\}$, and

(iv) $T^{-1}(B) = \{x \in X | T(x) \cap B \neq \phi\}$,

Let X and Y be two topological spaces, and let $T : X \to 2^Y$ be a set-valued function. T is said to be upper semicontinuous if for each open subset G of Y, the set $\{x \in X : T(x) \subset G\}$ is open in X. It is well known that if Y is a compact Hausdorff space and $T(x)$ is closed for each $x \in X$, then T is upper semicontinuous if and only if the graph $\{(x, y) \in X \times Y : y \in T(x)\}$ of T is closed in $X \times Y$.

T is said to be compact if the image $T(X)$ of X under T is contained in a compact subset of Y. T is said to be closed if its graph $\mathcal{G}_T = \{(x, y) \in X \times Y | y \in Tx, \forall x \in X\}$ is a closed subset of $X \times Y$.

A subset D of a topological space X is said to be compactly closed (resp. open) in X if for any compact subset K of X the set $D \cap K$ is closed (resp. open) in K.Obviosly, D is compactly closed in X if and only if its complement $D^c = X \backslash D$ is compactly open in X.

We denote Δ_n the standard n-simplex with vectors $e_0, e_1, \ldots, e_n$, where e_i is the $(i+1)$-th unit vector in $\mathcal{R}^{n+1}$.

A generalized convex space [7] or a G-convex space $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X and a function $\Gamma : \langle D \rangle \to 2^X$ with nonempty values (in the sequel, we write $\Gamma(A)$ by ΓA for each $A \in \langle D \rangle$), such that

(i) for each $A, B \in \langle D \rangle$, $A \subset B$ implies $\Gamma A \subset \Gamma B$, and

(ii) for each $A \in \langle D \rangle$, with $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma A$ such that $J \in \langle A \rangle$, implies $\phi_A(\Delta_{|J|-1}) \subset \Gamma J$, where $\Delta_{|J|-1}$ denotes the faces of Δ_n corresponding to $J \in \langle A \rangle$.

For $K \subset X$,

(i) K is G-convex if for each $A \in \langle D \rangle$, $A \subset K$ implies $\Gamma A \subset K$, and

(ii) The G-convex hull of K, denoted by $G\text{–}Co(K)$ is the set $\cap\{B \subset X | B$ is a G-convex subset of X containing $K\}$.

Let $(X, D; \Gamma)$ be a G-convex space. When $D = X$ we denote $(X, X; \Gamma)$ by X and X called a G-convex space.

A uniformity [4] for a set X is a non-void family $\mathcal{U}$ of subsets of $X \times X$ such that

(i) each member of $\mathcal{U}$ contains the diagonal Ω,

(ii) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$,

(iii) if $U \in \mathcal{U}$, then $V \circ V \subset U$ for some $V \in \mathcal{U}$,

(iv) if U and V are members of $\mathcal{U}$, then $U \cap V \in \mathcal{U}$, and

(v) if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a uniform space.

Let $(X, D; \Gamma)$ be a G-convex space which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$. Then a nonempty subset K of X is said to be almost G-convex if for any finite subset B of K and for any $V \in \mathcal{N}$, there is a mapping $h_{B,V} : B \to K$ such that $x \in V[h_{B,V}(x)]$ for all $x \in B$ and $G{-}co(h_{B,V}(B)) \subset K$. We call $h_{B,V}$ a G-convex-inducing mapping.

Remark 1.

(1.) In general, the G-convex-inducing mapping $h_{B,V}$ is not unique. If $U \in \mathcal{N}$ with $U \subset V$, then it is clear that any $h_{B,U} : B \to X$ can be regarded as an $h_{B,V} : B \to X$.

(2.) Let $M = R^2$ be the Euclidean topological space. Then the set $B = \{x = (x_1, x_2) \in M : x_1^{2/3} + x_2^{2/3} < 1\}$ is a G-convex set, but the set $B' = \{x = (x_1, x_2) \in M : 0 < x_1^{2/3} + x_2^{2/3} < 1\}$ is an almost G-convex set, not a G-convex set.

(3.) Our almost G-convex sets can be reduced to almost convex if X is a topological vector space.

Definition 1. [10] A G-convex space X is said to be a locally G-convex space if X is a uniform topological space with uniformity $\mathcal{U}$ which has an open base $\mathcal{N} = \{V_i | i \in I\}$ of symmetric encourages such that for each $V \in \mathcal{N}$, the set $V[x] = \{y \in X | (x, y) \in V\}$ is a G-convex set for each $x \in X$.

2. Almost Fixed-point Theorems for the $\Gamma{-}KKM$ Maps

First, we introduce a $\Gamma{-}KKM$ theorem for G-convex spaces. For a G-convex space $(X, D; \Gamma)$, a multimap $T : D \to 2^X$ is called a $\Gamma - KKM$ map if $\Gamma(A) \subset F(A)$ for each $A \in \langle D \rangle$. The following is known [9].

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space and $T : D \to 2^X$ a multimap with closed [resp., open] values. Suppose that T is a $\Gamma - KKM$ map. Then $\{T(z)\}_{z\in D}$ has the finite intersection property.*

Apply the above theorem 1, we shall prove the almost fixed point theorem as follows:

Theorem 2. *Let X be an almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$. Let $V \in \mathcal{N}$ such that $V[x]$ is G-convex for all $x \in X$. If $T : X \to 2^{\overline{X}}$ is closed and $T(X)$ is relatively compact, then T has a V-fixed point,that is; there is $x_V \in X$ such that $T(x_V) \cap V[x_V] \neq \phi$.*

Proof. Let $U \in \mathcal{N}$ such that $\overline{U} \circ \overline{U} \circ \overline{U} \subset V$. Since $T(X)$ is relatively compact, $X_1 = \overline{T(X)}$ is compact, there exists a finite subset $\{y_1, y_2, ..., y_n\}$ of $X_1 \subset \overline{X}$ such that $X_1 \subset \cup_{i=1}^n U[y_i] \subset \cup_{i=1}^n \overline{U}[y_i]$. For each $i \in \{1, 2, ..., n\}$, we can choose an $x_i \in X$ such that $y_i \in U[x_i]$. Then $X_1 \subset \cup_{i=1}^n U[y_i] \subset \cup_{i=1}^n \overline{U} \circ \overline{U}[x_i]$. Since X is almost G-convex and $B = \{x_1, x_2, ..., x_n\} \subset X$, there exists a G-convex-inducing $h_{B,U} : B \to X$ such that $x_i \in U[h_{B,U}(x_i)]$ for each $i = 1, 2, ..., n$ and $Z = G - co(h_{B,U}(B)) \subset X$. Now, we define $T' : h_{B,U}(B) \to 2^Z$ by

$$T'(h_{B,U}(x_i)) = \{z \in Z : \ T(z) \subset X_1 \setminus \overline{U} \circ \overline{U}[x_i]\}, \text{ for each } x_i \in B, \ i = 1, 2, ..., n.$$

Since T is closed and $X_1 = \overline{TX}$ is compact, T is upper semi-continuous, and so we have that $T'(h_{B,U}(x_i))$ is open in Z for each $i = 1, 2, ..., n$. Moreover, $\cap_{i=1}^n T'(h_{B,U}(x_i)) = \{z \in Z : T(z) \subset X_1 \setminus \cup_{i=1}^n \overline{U} \circ \overline{U}[x_i]\} = \phi$, since $\overline{TX} \subset \cup_{i=1}^n \overline{U} \circ \overline{U}[x_i]$. By Theorem 1, T' is not a $\Gamma - KKM$ map, and then there exists a finite subset $S = \{h_{B,U}(x_{i_1}), h_{B,U}(x_{i_2}), ..., h_{B,U}(x_{i_k})\} \in \langle h_{B,U}(B)\rangle$ and an $x_V \in \Gamma(S) \subset X$ such that $x_V \notin T'(S)$. That is, $T(x_V) \not\subseteq X_1 \setminus \overline{U} \circ \overline{U}[x_{i_j}]$for all $j \in \{1, 2, ..., k\}$. Let $z \in T(x_V)$ and $z \notin X_1 \setminus \overline{U} \circ \overline{U}[x_{i_j}]$ for all $j \in \{1, 2, ..., k\}$. Then $z \in \overline{U} \circ \overline{U}[x_{i_j}] \subset \overline{U} \circ \overline{U} \circ \overline{U}[h_{B,U}(x_{i_j})] \subset V[h_{B,U}(x_{i_j})]$ for all $j \in \{1, 2, ..., k\}$, and hence $(h_{B,U}(x_{i_j}), z) \in V$ for all $j \in \{1, 2, ..., k\}$, $h_{B,U}(x_{i_j}) \in V[z]$ for all $j \in \{1, 2, ..., k\}$. Next, since $V[z]$ is G-convex and $x_V \in \Gamma(S)$, we have that $x_V \in V[z]$, and hence $(x_V, z) \in V$, $z \in V[x_V]$. Therefore, $T(x_V) \cap V[x_V] \neq \phi$. ∎

Corollary 1. *Let X be an almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$. Let $V \in \mathcal{N}$ such that $V[x]$ is G-convex for all $x \in X$. If $T : X \to 2^X$ is compact and closed, then T has a V-fixed point,that is; there is $x_V \in X$ such that $T(x_V) \cap V[x_V] \neq \phi$.*

Corollary 2. *Let X be an almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$. Let $V \in \mathcal{N}$ such that $V[x]$ is G-convex for all $x \in X$. If $T : X \to 2^{\overline{X}}$ is closed, $T(X)$ is* *totally bounded, then T has a V-fixed point,that is; there is $x_V \in X$ such that $T(x_V) \cap V[x_V] \neq \phi$.*

Proof. Since $T(X)$ is totally bounded and T is closed,we have that $T(X)$ is compact, and then we complete the proof. ■

From the almost fixed point theorem 2, we immediate get the fixed point theorem on an almost G-convex subset of the locally G-convex spaces.

Theorem 3. *Let X be an almost G-convex subset of a locally G-convex space E. If $T : X \to 2^X$ is compact and closed, then T has a fixed point in X.*

Proof. Since E is a locally G-convex space, there exists a uniform structure $\mathcal{U}$, let $\mathcal{N} = \{V_i | i \in I\}$ be an open symmetric base family for the uniform structure $\mathcal{U}$ such that for any $U \in \mathcal{N}$ the set $U[x] = \{y \in X | (x, y) \in U\}$ is open G-convex for each $x \in X$. Then for each $V_i \in \mathcal{N}$, there exists $x_{V_i} \in X$ such that $V_i[x_{V_i}] \cap T(x_{V_i}) \neq \phi$, by using corollary 1. Let $y_{V_i} \in V_i[x_{V_i}] \cap T(x_{V_i})$, then $(x_{V_i}, y_{V_i}) \in \mathcal{G}_T$ and $(x_{V_i}, y_{V_i}) \in V_i$. Since $\overline{TX}$ is compact in X, without loss of generality, we may assume that $\{y_{V_i}\}_{i \in I}$ converges to y_0, that is there exists $V_0 \in \mathcal{N}$ such that $(y_{V_j}, y_0) \in V_j$ for all $V_j \in \mathcal{N}$ with $V_j \subset V_0$. Let $V_U \in \mathcal{N}$ with $V_U \circ V_U \subset V_j \subset V_0$, then we have $(x_{V_U}, y_{V_U}) \in V_U$ and $(y_{V_U}, y_0) \in V_U$, so $(x_{V_U}, y_{V_U}) \circ (y_{V_U}, y_0) = (x_{V_U}, y_0) \in V_U \circ V_U \subset V_j$, that is $x_{V_U} \to y_0$. By T is closed, then $(y_0, y_0) \in \mathcal{G}_T$, and so $y_0 \in T(y_0)$. We complete the proof. ■

Corollary 3. *Let X be an almost G-convex subset of a locally G-convex space E. If $T : X \to 2^X$ is closed, and TX is totally bounded in X, then T has a fixed point in X.*

3. Fixed-point Theorems for Maps in $\Gamma^* - KKM(X, Y)$

Next, we introduce the new definition of the $\Gamma^* - KKM$ property on an almost G-convex subset of a G-convex space.

Definition 2. Let X be an almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$, Y a topological space. If $T, F : X \to 2^Y$ are two set-valued functions satisfying for any $A \in \langle X \rangle$ and for any $V \in \mathcal{N}$, there is a G-convex-inducing mapping $h_{A,V} : A \to X$ such that

$$T(\Gamma(h_{A,V}(A))) \subset F(A), \qquad (*)$$

then F is called a generalized $\Gamma^* - KKM$ mapping with respect to T. If the set-valued function $T : X \to 2^Y$ satisfies the requirement that for any generalized

$\Gamma^* - KKM$ mapping F with respect to T the family $\{\overline{F(x)} | x \in X\}$ has the finite intersection property, then T is said to have the $\Gamma^* - KKM$ property. The class $\Gamma^* - KKM(X,Y)$ is defined to be the set $\{T : X \to 2^Y | T$ has the $\Gamma^* - KKM$ property$\}$.

Lemma 1. *Let X be an almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open symmetric base family $\mathcal{N}$, Y and Z be two topological spaces. If $T \in \Gamma^* - KKM(X,Y)$, then $fT \in \Gamma^* - KKM(X,Z)$ for all $f \in \mathcal{C}(Y,Z)$.*

From the above Lemma 1, we get the main results of this section.

Theorem 4. *Let X be an almost G-convex subset of a locally G-convex space E. If $T \in \Gamma^* - KKM(X,X)$ is compact and closed, then T has a fixed point.*

Proof. Since E is a locally G-convex space, there exists a uniform structure $\mathcal{U}$, and let $\mathcal{N} = \{V_i | i \in I\}$ be an open symmetric base family for the uniform structure $\mathcal{U}$ such that for any $U \in \mathcal{N}$, the set $U[x] = \{y \in X | (x,y) \in U\}$ is open G-convex for each $x \in X$. Now we claim that for any $V \in \mathcal{N}$, there exists $x_V \in X$ such that $V[x_V] \cap T(x_V) \neq \phi$. Suppose it is not the case, then there is an $V \in \mathcal{N}$ such that $V[x_V] \cap T(x_V) = \phi$ for all $x_V \in X$. Let $V' \in \mathcal{N}$ such that $V' \circ V' \subset V$. Let $K = \overline{TX}$. Then K is compact and closed in X, since T is compact and X is Hausdorff. Define $F : X \to 2^X$ by

$$F(x) = K \backslash V'[x] \qquad \text{for each } x \in X.$$

Then we have

(1) $F(x)$ is nonempty compact and closed, and

(2) F is a generalized $\Gamma^* - KKM$ mapping with respect to T.

(1) is obvious. To prove (2), we use the contradiction. Let $A = \{x_1, x_2, ..., x_n\} \in \langle X \rangle$. Suppose that F is not a generalized $\Gamma^* - KKM$ mapping with respect to T. Then there is a $U \in \mathcal{N}$ such that for any G-convex-inducing mapping $h_{A,U} : A \to X$ one has $T(\Gamma(h_{A,U}(A))) \nsubseteq F(A)$. Let $W = V' \cap U$. Then $T(\Gamma(h_{A,W}(A)) \nsubseteq F(A)$. Choose $u \in \Gamma(h_{A,W}(A))$ and $z \in T(u)$ such that $z \notin \cup_{i=1}^n F(x_i)$, so $z \in V'[x_i]$ for all $i = 1, 2, ..., n$. Since X is almost G-convex, we have $z \in V'[x_i] \subset V' \circ W[h_{A,W}(x_i)] \subset V[h_{A,W}(x_i)]$. So $(z, h_{A,W}(x_i)) \in V$, $(h_{A,W}(x_i), z) \in V$ and $h_{A,W}(x_i) \in V[z]$ for all $i = 1, 2, ..., n$, since V is symmetric. Since $V[z]$ is G-convex and $u \in \Gamma(h_{A,W}(A)) \subset V[z]$, we have $(u, z) \in V$ and $z \in V[u]$. So $V[u] \cap T(u) \neq \phi$, a contradiction. So F is a generalized $\Gamma^* - KKM$ mapping with respect to T. Since $T \in \Gamma^* - KKM(X,X)$ and K is compact, so $\cap_{x \in X} Fx \neq \phi$. Let $\xi \in \cap_{x \in X} Fx = \cap_{x \in X} K \backslash V'[x]$. Then $\xi \notin V'[\xi]$, a contradiction. Therefore, we have proved that for each $V_i \in \mathcal{N}$, there is an $x_{V_i} \in X$ such that $V_i[x_{V_i}] \cap T(x_{V_i}) \neq \phi$. Let $y_{V_i} \in V_i[x_{V_i}] \cap T(x_{V_i})$,

then $(x_{V_i}, y_{V_i}) \in \mathcal{G}_T$ and $(x_{V_i}, y_{V_i}) \in V_i$. Since T is compact, without loss of generality, we may assume that $\{y_{V_i}\}_{i \in I}$ converges to y_0, that is there exists $V_0 \in \mathcal{N}$ such that $(y_{V_j}, y_0) \in V_j$ for all $V_j \in \mathcal{N}$ with $V_j \subset V_0$. Let $V_U \in \mathcal{N}$ with $V_U \circ V_U \subset V_j \subset V_0$, then we have $(x_{V_U}, y_{V_U}) \in V_U$ and $(y_{V_U}, y_0) \in V_U$, so $(x_{V_U}, y_{V_U}) \circ (y_{V_U}, y_0) = (x_{V_U}, y_0) \in V_U \circ V_U \subset V_j$, that is $x_{V_U} \to y_0$. By T is closed, then $(y_0, y_0) \in \mathcal{G}_T$, so $y_0 \in T(y_0)$. We have completed the important proof. ■

We introduce the following concepts to establish the remainder results of this section.

Definition 3. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two uniform spaces, and let $A : X \to 2^Y$ be a set-valued function. A is upper semi-continuous at $x \in X$ if and only if for any $V \in \mathcal{V}$ there exists a $U \in \mathcal{U}$ such that $A(U[x]) \subset V[A(x)]$. A is said to be upper semi-continuous on X if A is upper semi-continuous at x for each x in X. Moreover, A is said to be upper semi-continuous if for any $V \in \mathcal{V}$, there exists a $U \in \mathcal{U}$ such that $A \circ U \subset V$.

Definition 4. Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ be two uniform spaces, $U \in \mathcal{U}$, $V \in \mathcal{V}$, and let $A : X \to 2^Y$ be a set-valued function. A single-valued function $s : X \to Y$ is said to be a $U - V$-selection of A if $s(x) \in V[A(U[x])]$ for any $x \in X$.

Definition 5. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two uniform spaces. A set-valued function $A : X \to 2^Y$ is said to be approachable if it has a continuous $U - V$-selection for any $U \in \mathcal{U}$ and $V \in \mathcal{V}$, that is, for any $U \in \mathcal{U}$ and $V \in \mathcal{V}$ there exists a continuous $U - V$-selection $s : X \to Y$ of A such that $s(x) \in V[A(U[x])]$ for all $x \in X$.

Let X and Y be uniform spaces. The classes of approachable maps are defined as follows.

(i) $\mathcal{A}_0(X, Y) = \{A : X \to 2^Y | A \text{ is approachable}\}$,

(ii) $\mathcal{A}(X, Y) = \{A \in \mathcal{A}_0 | A \text{ is } u.s.c. \text{ with compact-values}\}$, and

(iii) $\mathcal{A}_c(X, Y) = \{A = A_m \circ A_{m-1} \circ \cdots \circ A_1 | A_i \in \mathcal{A}(X, Y) \text{ for } i = 1, 2, \ldots, m\}$.

Applying Theorem 3.13 [1], we have the following:

Theorem 5. *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be uniform spaces and X be compact. If $\mathcal{G}r$ is a closed subset of $X \times Y$, then the following statements are equivalent:*

(i) *$\mathcal{G}_f \cap \mathcal{G}r \neq \phi$, for each $f \in \mathcal{C}(X, Y)$,*

(ii) *$\mathcal{G}_A \cap \mathcal{G}r \neq \phi$, for each $A \in \mathcal{A}_c(X, Y)$.*

We now apply the above theorem 5 to establish the following proposition 1.

Proposition 1. *Let X be an almost G-convex subset of a locally G-convex space E and Y a compact subset of a uniform space. Suppose that $T \in \Gamma^* - KKM(X,Y)$is closed. Then for any $H \in \mathcal{A}_c(Y,X)$, TH has a fixed point in Y.*

4. Applications

We now establish the following $\Gamma^* - KKM$-type theorems, which is equivalent to the matching theorem following it.

Theorem 6. *Let X be an almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open base family $\mathcal{N}$, and Y a Hausdorff space. If $T, F : X \to 2^Y$ are two set-valued functions satisfies the following:*

(i) *$T \in \Gamma^* - KKM(X,Y)$ such that $\overline{T(\Gamma(X))}$ is compact in Y,*

(ii) *for any $x \in X$, Fx is compactly closed in Y,*

(iii) *F is a generalized $\Gamma^* - KKM$ mapping with respect to T.*

Then $\overline{T(\Gamma(X))} \cap (\cap\{Fx : x \in X\}) \neq \phi$.

As a consequence of the above theorem 6, we obtain the following generalization of the Ky Fan matching theorem.

Theorem 7. *Let X be an almost G-convex subset of a G-convex space E which has a uniformity $\mathcal{U}$ and $\mathcal{U}$ has an open base family $\mathcal{N}$, and Y a Hausdorff space. Suppose that $T, H : X \to 2^Y$ are two set-valued functions satisfies the following:*

(i) *$T \in \Gamma^* - KKM(X,Y)$ such that $\overline{T(\Gamma(X))}$ is compact in Y,*

(ii) *for any $x \in X$, $H(x)$ is compactly open in Y,*

(iii) *$\overline{T(\Gamma(X))} \subset H(X)$.*

Then there is $M \in \langle X \rangle$ such that $T(\Gamma(X))) \cap (\cap\{H(x) : x \in M\}) \neq \phi$.

From Theorem 4 we have the following quasi-equilibrium theorem.

Theorem 8. *Let X be an almost G-convex subset of a locally G-convex space, and $f : X \times X \to \Re$ an upper semicontinuous function. Suppose $H : X \to 2^X$ is compact and closed, and suppose that*

(i) *the function M defined on X by $M(x) = max_{y \in H(x)} f(x,y)$ for $x \in X$, is lower semicontinuous, and*

(ii) *the multifunction $T : X \to 2^X$ defined by $T(x) = \{y \in H(x) : f(x,y) = M(x)\}$, is in $\Gamma^* - KKM(X,X)$.*

Then there is an $x_0 \in X$ *such that* $x_0 \in H(x_0)$ *and* $f(x_0, x_0) = M(x_0)$.

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