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REGULARITY AND BLOW-UP CONSTANTS OF SOLUTIONS FOR NONLINEAR DIFFERENTIAL EQUATION

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Abstract. In this paper we gain some results on the regularity and also the blow-up rates and constants of solutions to the equation $u'' - u^p = 0$ under some different situations. The blow-up rate and blow-up constant of $u^{(2n)}$ are $(p-2n+2)$ and $(\pm)(p-2n+2)\cdot\Pi_{i=0}^{n-1}(p-2i+2)(p-2i+1)E(0)^{p/2}$ respectively; blow-up rate and blow-up constant of $u^{(2n+1)}$ are $(p-2n+1)$ and $(p-2n+2)\ \Pi_{i=0}^{n-1}(p-2i+2)\cdot(p-2i+1)E(0)^{p-n}$ respectively, where $E(0) = u'(0)^2 - \frac{2}{p+1}u(0)^{p+1}$.

0. Introduction

In this paper, we deal with the estimate of blow-up rate and blow-up constant of $u^{(n)}$ and the regularity of solutions for the nonlinear ordinary differential equation

$$u'' - u^p = 0 \tag{0.1}$$

where $p > 1$.

Our motivation on the problem is based on the studying properties of solutions of the semi-linear wave equation $\Box u + f(u) = 0$ $[2,\ 3]$ with particular cases in zero space dimension and the blow-up phenomena of the solution to equation (0.1) $[4]$.

In this paper, if $p = \frac{r}{s}$, $r \in \mathbb{N}$, $s \in 2\mathbb{N}+1$, $(r,s) = 1$ (common factor) we say that p is odd (even respectively) if r is odd (even, respectively) .

For $p \in \mathbb{Q}$ and $p \geq 1$, the function u^p is locally Lipschitz, therefore by standard theory for ordinary differential equation there exists exactly one local classical solution to the equation (0.1) together with initial values $u(0) = u_0,\ u'(0) = u_1$.

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Notations and Fundamental Lemmata

For a given function u in this work we use the following abbreviations

$$a_u(t) = u(t)^2, \ E_u(t) = u'(t)^2 - \frac{2}{p+1} u(t)^{p+1}, \ J_u(t) = a_u(t)^{-\frac{p-1}{4}}.$$

Definition. A function $g : \mathbb{R} \to \mathbb{R}$ has a blow-up rate r means that g exists only in finite time, that is, there is a finite number T^* such that the following holds

$$\lim_{t \to T^*} g(t)^{-1} = 0$$

and there exists a non-zero $\beta \in \mathbb{R}$ with

$$\lim_{t \to T^*} (T^* - t)^r g(t) = \beta,$$

in this case β is called the blow-up constant of g.

One can find the detail in $[4]$ for the lemmas given as follows without rigorous argumentations.

Lemma 1. *Suppose that u is the solution of (0.1), then we have*

$$E(t)_u = E_u(0), \tag{0.2}$$

$$(p+3) u'(t)^2 = (p+1) E_u(0) + a_u''(t), \tag{0.3}$$

$$J_u''(t) = \frac{p^2 - 1}{4} E_u(0) J_u(t)^{\frac{p+3}{p-1}} \tag{0.4}$$

and

$$J_u'(t)^2 = J_u'(0)^2 - \frac{(p-1)^2}{4} E_u(0) J_u(0)^{\frac{2(p+1)}{p-1}} + \frac{(p-1)^2}{4} E_u(0) J_u(t)^{\frac{2(p+1)}{p-1}}. \tag{0.5}$$

Lemma 2. *Suppose that c_1 and c_2 are real constants and $u \in C^2(\mathbb{R})$ satisfies the inequality*

$$u'' + c_1 u' + c_2 u \leq 0, \quad u \geq 0,$$

$$u(0) = 0, \ u'(0) = 0,$$

then u must be null, that is, $u \equiv 0$.

Lemma 3. *If $g(t)$ and $h(t,r)$ are continuous with respect to their variables and the limit $\lim_{t\to T}\int_0^{g(t)} h(t,r)\,dr$ exists, then*

$$\lim_{t\to T}\int_0^{g(t)} h(t,r)\,dr = \int_0^{g(T)} h(T,r)\,dr.$$

Lemma 4. *If T is the life-span of u and u is the solution of the problem (0.1) with $E_u(0) < 0$ and $p > 1$ then T is finite, that is, u is only a local solution of (0.1). Further, for $a_u'(0) \geq 0$, we have the following estimates*

$$J_u'(t) = -\frac{p-1}{2}\sqrt{k_1 + E_u(0)\,J_u(t)^{k_2}} \leq J'(0) \quad \forall t \geq 0, \tag{0.6}$$

$$\int_{J_u(t)}^{J_u(0)} \frac{dr}{\sqrt{k_1 + E_u(0)\,r^{k_2}}} = \frac{p-1}{2}t \quad \forall t \geq 0 \tag{0.7}$$

and

$$T \leq T_1^*(u_0,u_1,p) = \frac{2}{p-1}\int_0^{J_u(0)} \frac{dr}{\sqrt{k_1 + E_u(0)\,r^{k_2}}}. \tag{0.8}$$

For $a_u'(0) < 0$, there is a constant $t_0(u_0,u_1,p)$ such that

$$\begin{cases} J_u'(t) = -\dfrac{p-1}{2}\sqrt{k_1 + E_u(0)\,J_u(t)^{k_2}} & \forall t \geq t_0(u_0,u_1,p), \\ J_u'(t) = \dfrac{p-1}{2}\sqrt{k_1 + E_u(0)\,J_u(t)^{k_2}} & \forall t \in [0, t_0(u_0,u_1,p)] \end{cases} \tag{0.9}$$

and

$$\begin{cases} \displaystyle\int_{J_u(t)}^{J_u(0)} \frac{dr}{\sqrt{k_1 + E_u(0)\,r^{k_2}}} = \frac{p-1}{2}\left(t - t_0(u_0,u_1,p)\right) & \forall t \geq t_0(u_0,u_1,p), \\ \displaystyle\int_{J_u(0)}^{J_u(t_0)} \frac{dr}{\sqrt{k_1 + E_u(0)\,r^{k_2}}} = \frac{p-1}{2}t_0(u_0,u_1,p). \end{cases} \tag{0.10}$$

Also we have

$$\begin{aligned} T &\leq T_2^*(u_0,u_1,p) \\ &= \frac{2}{p-1}\left(\int_0^{k} \frac{dr}{\sqrt{k_1 + E_u(0)\,r^{k_2}}} + \int_{J(0)}^{k} \frac{dr}{\sqrt{k_1 + E_u(0)\,r^{k_2}}}\right), \end{aligned} \tag{0.11}$$

where $k_1 := \dfrac{2}{p+1}, k_2 := \dfrac{2p+2}{p-1}$ *and* $k := \left(\dfrac{2}{p+1}\dfrac{-1}{E_u(0)}\right)^{\frac{p-1}{2p+2}}$.

Furthermore, if $E_u(0) = 0$ *and* $a_u'(0) > 0$, *then*

$$\begin{cases} J_u(t) = a_u(0)^{-\frac{p-1}{4}} - \dfrac{p-1}{4} a_u(0)^{-\frac{p-1}{4}-1} a_u'(0)\, t, \\ a_u(t) = a_u(0)^{\frac{p+3}{p-1}} \left(a_u(0) - \dfrac{p-1}{4} a_u'(0)\, t\right)^{-\frac{4}{p-1}} \end{cases} \tag{0.12}$$

for each $t \geq 0$, *and*

$$T \leq T_3^*(u_0, u_1, p) := \frac{4}{p-1} \frac{a_u(0)}{a_u'(0)}. \tag{0.13}$$

Lemma 5. *If* T *is the life-span of* u *and* u *is the solution of the problem* (0.1) *with* $E_u(0) > 0$, *then* T *is finite; that is,* u *is only a local solution of* (0.1). *If one of the following is valid*

(i) $a_u'(0)^2 > 4a_u(0) E_u(0)$ *or*

(ii) $a_u'(0)^2 = 4a_u(0) E_u(0)$ *and* $u_1 > 0$ *or*

(iii) $a_u'(0)^2 = 4a_u(0) E_u(0)$, $u_1 < 0$ *and* p *is odd.*

Further, in case of (i), *we have the estimate*

$$T \leq T_4^*(u_0, u_1, p) = \frac{2}{p-1} \int_0^{J_u(0)} \frac{dr}{\sqrt{k_1 + E_u(0)\, r^{k_2}}}, \tag{0.14}$$

and

$$a'(0) \geq 0. \tag{0.15}$$

In the case of (ii), *we have also*

$$T \leq T_5^*(u_0, u_1, p) = \frac{2}{p-1} \int_0^{\infty} \frac{dr}{\sqrt{k_1 + E_u(0)\, r^{k_2}}}. \tag{0.16}$$

In case of (iii), *we get*

$$T \leq T_6^*(u_0, u_1, p) = \frac{2}{p-1} \int_0^{\infty} \frac{dr}{\sqrt{k_1 + E_u(0)\, r^{k_2}}}. \tag{0.17}$$

Lemma 6. *Suppose that u is the solution of the problem (0.1) with one of the following property*

(i) $E_u(0) > 0$, $a_u'(0)^2 < 4a_u(0)E_u(0)$ *or*

(ii) $a_u'(0)^2 = 4a_u(0)E_u(0)$, $u_1 < 0$ *and p is odd.*

Then T_0 given by

$$T_0(u_0, u_1, p) = \int_{-u_0}^{-u(T_0)} \frac{dr}{\sqrt{E_u(0) - 2r^{p+1}/(p+1)}}, \tag{0.18}$$

where $-u(T_0) = ((p+1)E_u(0)/2)^{1/(p+1)}$ is the critical point of u, and u_0 must be non-positive.

Remark. Under condition $(i)u_0$ must be negative and p must be even.

If u is the solution of the problem (0.1) with $E_u(0) = 0$ and $a_u'(0) = 0$, then u must be null.

Lemma 8. *Suppose that u is the solution of the problem (0.1) with $E_u(0) > 0$ and one of the following holds*

(i) $a_u'(0)^2 < 4a_u(0)E_u(0)$.

(ii) $a_u'(0)^2 = 4a_u(0)E_u(0)$ *and* $u_1 < 0$, *p is even.*

Then u possesses a critical point $T_0(u_0, u_1, p)$ given by (0.18), provided condition (ii) *holds or condition* (i) *together with $a_u'(0) > 0$ holds; and under* (i), *there exists $z < \infty$ such that*

$$a(z) = 0.$$

For $a'(0) \leq 0$, we have the null point (zero) z_1 of a,

$$z_1(u_0, u_1, p) = \frac{\sqrt{p^2-1}}{\sqrt{2}} \int_0^{\sqrt{\frac{4a_u(0)}{(p^2-1)E_u(0)}}} \frac{dr}{\sqrt{2 - (p-1)k_3^2 r^{p+1}}},$$

and

$$T \leq T_7^*(u_0, u_1, p) := z_1(u_0, u_1, p) + T_5^*(u_0, u_1, p).$$

where $k_3 = \left(\frac{p^2-1}{4}E_u(0)\right)^{\frac{p-1}{4}}$.

Furthermore, we also have

$$\lim_{t \to z_1} a_u(t)(z_1 - t)^{-2} = E_u(0), \tag{0.19}$$

$$\lim_{t\to z_1}(z_1-t)^{-1}a'(t)=-2E(0),\quad \lim_{t\to z_1}a_u''(t)=2E_u(0), \tag{0.20}$$

and $a_u(t)$ *blows up at* $T_7^*(u_0,u_1,p)$; *that is,* $\lim_{t\to T_7^*}1/a_u(t)=0$.
For $a_u'(0)>0$, *we have the null point* z_2 *of* a_u

$$z_2(u_0,u_1,p)=\frac{\sqrt{p^2-1}}{\sqrt{2}}\left(\int_0^{2^{\frac{1}{p+1}}(p-1)^{-\frac{1}{p+1}}k_3^{-\frac{2}{p+1}}}\frac{dr}{\sqrt{2-(p-1)k_3^2r^{p+1}}}+\int_{2a(0)^{1/2}(p^2-1)^{-1/2}E_u(0)^{-1/2}}^{2^{\frac{1}{p+1}}(p-1)^{-\frac{1}{p+1}}k_3^{-\frac{2}{p+1}}}\frac{dr}{\sqrt{2-(p-1)k_3^2r^{p+1}}}\right)$$

and

$$T\le T_8^*(u_0,u_1,p):=z_2(u_0,u_1,p)+T_6^*(u_0,u_1,p).$$

Furthermore, we also have

$$\lim_{t\to z_2}a_u(t)(z_2(u_0,u_1,p)-t)^{-2}=E_u(0), \tag{0.21}$$

$$\lim_{t\to z_2}(z_2-t)^{-1}a_u'(t)=-2E_u(0), \tag{0.22}$$

$$\lim_{t\to z_2}a_u''(t)=2E_u(0),$$

and $a_u(t)$ *blows up at* $T_8^*(u_0,u_1,p)$; *that is,* $\lim_{t\to T_8^*(u_0,u_1,p)}1/a_u(t)=0$.
Further, under the condition (ii), *we have the null point* $z_3(u_0,u_1,p)$ *of* a,

$$z_3(u_0,u_1,p)=2T_0(u_0,u_1,p),$$

$$T\le T_9^*(u_0,u_1,p)=z_3(u_0,u_1,p)+T_5^*(u_0,u_1,p)$$

and $a_u(t)$ *blows up at* $T_9^*(u_0,u_1,p)$. *Furthermore we have*

$$\lim_{t\to z_3}a_u(t)(z_3(u_0,u_1,p)-t)^{-2}=E_u(0), \tag{0.23}$$

$$\lim_{t\to z_3(u_0,u_1,p)}(z_3(u_0,u_1,p)-t)^{-1}a_u'(t)=-2E_u(0), \tag{0.24}$$

$$\lim_{t\to z_3} a_u''(t) = 2E_u(0). \tag{0.25}$$

In Section I, we consider the regularity of solution u of equation (1) for $p \in \mathbb{N}$ and gain the expansion of $u^{(n)}$ in terms of $u^{(k)}, k < n$; in section II, we consider the regularity of solution u as $p \in \mathbb{Q} - \mathbb{N}$. In the last section, we study the blow-up rates and blow-up constants of $u^{(n)}$ as t approach to life-span T^* and null point (zero) z under some situations.

1. Regularity of Solution to the Equation (0.1) with $p \in \mathbb{N}$

In this section we study the regularity of the solution u of the nonlinear equation (0.1) as $p \in \mathbb{N}$. First, we see that the well-defined function u^p is locally Lipschitz, hence we have the local existence and uniqueness of solution to the equation

$$\begin{cases} u'' = u^p, \\ u(0) = u_0, \ u'(0) = u_1. \end{cases} \tag{1.1}$$

Therefore, we rewrite $a_u(t) = a(t)$, $J_u(t) = J(t)$ and $E_u(t) = E(t)$ for convenience. Using (0.2) we have

$$u'(t)^2 = E(0) + \frac{2}{p+1} u(t)^{p+1}. \tag{1.2}$$

1.1 Regularity of Solution to the Equation (1.1) with $p \in \mathbb{N}$

Now we consider problem (1.1) with $p \in \mathbb{N}$, we have the following results:

Theorem 1. *If u is the solution of the problem (1.1) with the life-span T^* and $p \in \mathbb{N}$, then $u \in C^q(0, T^*)$ for any $q \in \mathbb{N}$ and*

$$u^{(2n)} = \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} E_{n\ i} u^{C_{n\ i}}, \tag{1.3}$$

$$\begin{aligned} u^{(2n+1)} &= \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} E_{n\ i} C_{n\ i} u^{C_{n\ i}-1} u' \\ &= \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} O_{n\ i} u^{C_{n\ i-1} u'} \end{aligned} \tag{1.4}$$

for positive integer n, *where* $\left[\frac{C_{n0}}{p+1}\right]$ *denotes the Gaussian integer number of* $\frac{C_{n0}}{p+1}$,

$$C_{n\ i} = (n-i)(p+1) - 2n + 1,$$

$$O_{n\ i} = E_{n\ i} C_{n\ i},\ E_{00} = 1$$

and

$$\begin{aligned} E_{n0} &= O_{(n-1)0}\left[\frac{2}{p+1}\left(C_{(n-1)0}-1\right)+1\right] \\ &= E_{(n-1)0}C_{(n-1)0}\left[\frac{2}{p+1}\left(C_{(n-1)0}-1\right)+1\right], \\ E_{n(n-1)} &= O_{(n-1)(n-2)}\left(C_{(n-1)(n-2)}-1\right)E(0) \\ &= E_{(n-1)(n-2)}C_{(n-1)(n-2)}\left(C_{(n-1)(n-2)}-1\right)E(0), \\ E_{nk} &= O_{(n-1)(k-1)}\left(C_{(n-1)(k-1)}-1\right)E(0) \\ &\quad + O_{(n-1)k}\left[\frac{2}{p+1}\left(C_{(n-1)k}-1\right)+1\right] \\ &= E_{(n-1)(k-1)}C_{(n-1)(k-1)}\left(C_{(n-1)(k-1)}-1\right)E(0) \\ &\quad + E_{(n-1)k}C_{(n-1)k}\left[\frac{2}{p+1}\left(C_{(n-1)k}-1\right)+1\right], \end{aligned}$$

for positive integer k *and* $0 < k < n$.

Proof. Let v_n be the n-th derivative of u ; that is $v_n := u^{(n)}$, then $v_0^n = u^n$, $v_0 = u$, $v_1 = u'$, $v_2 = u''$, $v_1^2 = (u')^2$. To prove (1.3) we use mathematical induction. When $n = 1$, we have

$$v_2 = \sum_{i=0}^{\left[\frac{C_{10}}{p+1}\right]} E_{1\ i} u^{C_{1\ i}} = E_{10} u^{C_{10}} = v_0^p,$$

$$C_{00} = (0-0)(p+1) - 2\times 0 + 1 = 1,\ C_{10} = p$$

and

$$E_{10} = E_{00}C_{00}\left[\frac{2}{p+1}(C_{00}-1)+1\right] = 1.$$

Suppose $v_{2n} = \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} E_{n\ i} \cdot v_0^{C_{n\ i}},\ n \in \mathbb{N}$. Then

$$v_{2n+1} = \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} E_{n\ i} C_{n\ i} \cdot v_0^{C_{n\ i}-1} \cdot v_1$$

and

$$v_{2n+2} = \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} E_{n\ i} C_{n\ i} \left(v_0^{C_{n\ i}-1} \cdot v_2 + (C_{n\ i} - 1)\, v_0^{C_{n\ i}-2} \cdot v_1^2 \right).$$

By (1.2) we obtain

$$\begin{aligned}
v_{2n+2} = & \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} O_{n\ i} \cdot \left[\frac{2}{p+1} (C_{n\ i} - 1) + 1 \right] v_0^{C_{n\ i}+p-1} \\
& + \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} O_{n\ i} \cdot (C_{n\ i} - 1) \cdot E(0)\, v_0^{C_{n\ i}-2} \\
= & \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} O_{n\ i} \cdot \left[\frac{2}{p+1} (C_{n\ i} - 1) + 1 \right] v_0^{C_{(n+1)\ i}} \\
& + \sum_{i=0}^{\left[\frac{C_{n0}}{p+1}\right]} O_{n\ i} \cdot (C_{n\ i} - 1) \cdot E(0)\, v_0^{C_{(n+1)(i+1)}} \\
= & O_{n0} \cdot \left[\frac{2}{p+1} (C_{n0} - 1) + 1 \right] v_0^{C_{(n+1)0}} \\
& + O_{n0} \cdot (C_{n0} - 1) \cdot E(0)\, v_0^{C_{(n+1)\ 1}} \\
& + O_{n1} \cdot \left[\frac{2}{p+1} (C_{n1} - 1) + 1 \right] v_0^{C_{(n+1)\ 1}} \\
& + O_{n1} \cdot (C_{n1} - 1) \cdot E(0)\, v_0^{C_{(n+1)2}} \\
& + O_{n2} \cdot \left[\frac{2}{p+1} (C_{n2} - 1) + 1 \right] v_0^{C_{(n+1)2}} + \cdots \\
& + \ldots + O_{n\left[\frac{C_{n0}}{p+1}\right]} \cdot \left(C_{n\left[\frac{C_{n0}}{p+1}\right]} - 1 \right) \cdot E(0)\, v_0^{C_{(n+1)\left(\left[\frac{C_{n0}}{p+1}\right]+1\right)}}.
\end{aligned}$$

Hence

$$v_{2n+2} = \sum_{i=0}^{\left[\frac{C_{(n+1)0}}{p+1}\right]} E_{(n+1)\ i} \cdot v_0^{C_{(n+1)\ i}},$$

which completes the induction procedures and we obtain (1.3). Using (1.3), we get (1.4). ∎

1.2. The Properties of $u^{(n)}$

Drawing the graphs of the $u^{(n)}$ is not easy, so in this section we choose a spacial index $p = 2$.

We consider only on the properties of the solution u to the case that $E(0) = 0$ for the equation

$$(1.5) \qquad \begin{cases} u'' = u^2, \\ u(0) = 1, \quad u'(0) = \sqrt{\frac{2}{3}}. \end{cases}$$

The solution of equation (1.5) can be solved explicitly

$$u(t) = \frac{6}{\left(\sqrt{6} - t\right)^2}$$

and this affords the graphs of u, u', u'', $u^{(3)}$and $u^{(4)}$ below.

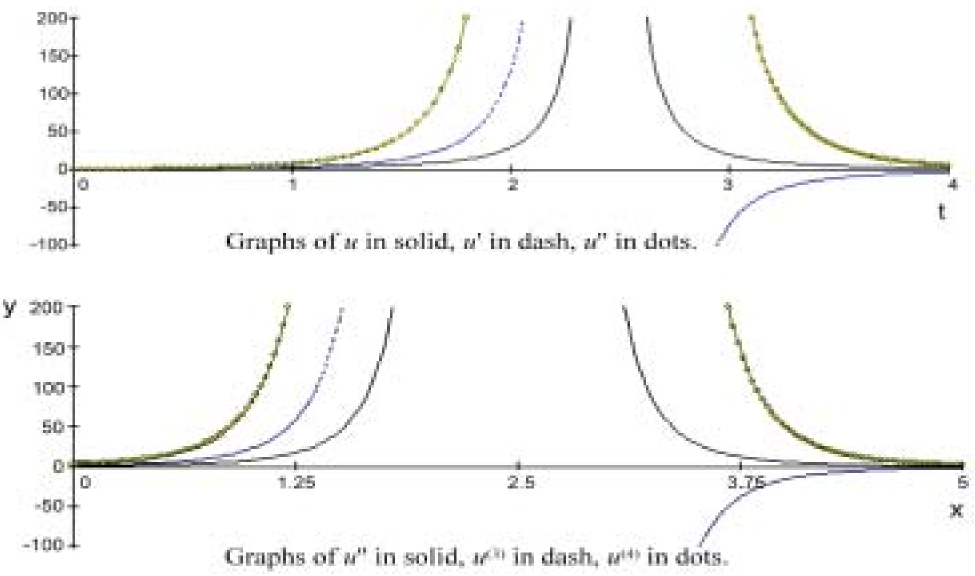

Fig. 1.5.

With the help of graphing with maple we find that the n-th derivative $u^{(n)}$ is smooth and that the blow-up rate of $u^{(n)}$ is increasing in n. Here we do not give rigorous proof, we will illustrate this in section III.

2. Regularity of Solution to the Equation (0.1) with $p \in \mathbb{Q} - \mathbb{N}$

According to the preceding section we obtain the solution $u \in C^q(0,T)$ of (0.1) with $p \in \mathbb{N}$ for any $q \in \mathbb{N}$. In this section we consider the equation of (0.1) with $p \in \mathbb{Q} - \mathbb{N}$.

Except the null points (zeros) of u, $u^{(q)}$ are also differentiable for any $q \in \mathbb{N}$. We have

Theorem 2. *If u is the solution of the problem (0.1) with $p \in \mathbb{Q} - \mathbb{N}$, $p \geq 1$ and the followings do not hold*

(i) $a'(0)^2 < 4a(0)E(0)$, $E(0) > 0$,

(ii) $a'(0)^2 = 4a(0)E(0)$, $E(0) > 0$ *and* $u_1 < 0$, p *is even,*

then $u \in C^q(0,T)$ for any $q \in \mathbb{N}$. Further, we have

$$u^{(2n)} = \sum_{i=0}^{n-1} E_{n\ i} u^{C_{n\ i}} \tag{2.1}$$

and

$$\begin{aligned} u^{(2n+1)} &= \sum_{i=0}^{n-1} E_{n\ i} C_{n\ i} u^{C_{n\ i}-1} u' \\ &= \sum_{i=0}^{n-1} O_{n\ i} u^{C_{n\ i}-1} u'. \end{aligned} \tag{2.2}$$

Proof. Same as the procedures given in the proof of Theorem 1, to prove (2.1) and (2.2) by mathematical induction. If t_0 is the null (zero) point of u, then

$$\lim_{t \to t_0} u^{c_{n\ i}}(t_0)^{-1} = 0$$

for $i > \frac{n(p-1)+1}{p+1} = \frac{C_{n0}}{p+1}$ since that $C_{n\ i} < 0$, for $i > \frac{C_{n0}}{p+1}$. By lemma 8 we know that u possesses the null point (zero) only in the case (i) or (ii). Hence, we obtain the assertions by Theorem 1.

Similarly, by the same arguments above, we have also a result as following:

Theorem 3. *If u is the solution of the problem (0.1) with $p \in \mathbb{Q} - \mathbb{N}$, $p \geq 1$ and one of the followings holds*

(i) $a'(0)^2 < 4a(0)E(0), E(0) > 0$

(ii) $a'(0)^2 = 4a(0)E(0), E(0) > 0 and u_1 < 0$, p *is even.*

Then $u \in C^{[p]+2}(0,T)$, *where* $[p]$ *mean that Gaussian integer number of* p. *Further, we have*

$$u^{(2n)} = \sum_{i=0}^{n-1} E_{n\ i} u^{C_{n\ i}}, \quad \text{for } n \leq \left[\frac{p}{2}\right] + 1 \tag{2.3}$$

and

$$u^{(2n+1)} = \sum_{i=0}^{n-1} E_{n\ i} C_{n\ i} u^{C_{n\ i}-1} u'$$

$$= \sum_{i=0}^{n-1} O_{n\ i} u^{C_{n\ i}-1} u', \quad \text{for } n \leq \left[\frac{p}{2}\right] + 1. \tag{2.4}$$

Proof. Same as the proof of Theorem 1, we obtain also the identities (2.3) and (2.4).

By lemma 8, we know that u possesses the null point (zero) in the case (i) or (ii). (Figure 2.1) If t_0 is the null point of u then $\lim_{t \to t_0} u^{-c_{n\ i}}(t) = 0$ for $C_{n\ i} < 0$. Hence, in the case of (i) or (ii), we should find the range of n with $C_{n\ i} \geq 0$ as $i = n-1$, and then $u^{(2n)}$ exists only in such situation.

Here

$$C_{n\ i} = (p+1)(n-i) - 2n + 1.$$

Let

$$C_{n(n-1)} = (p+1)(n-(n-1)) - 2n + 1 \geq 0,$$

then we get $n \leq \frac{p}{2} + 1$. Since n is an integer, we have $n \leq \left[\frac{p}{2}\right] + 1$.

Now $u^{(2n)}$ exists for $n \leq \left[\frac{p}{2}\right] + 1$ in the case (i) or (ii); thus we obtain that $u \in C^{[p]+2}(0.T)$.

Example 2.1. To draw the graphs of $u^{(n)}$ for $p \in \mathbb{Q} - \mathbb{N}$ is not easy, so we choose a special index $p = \frac{7}{3}$.

We consider on the properties of the solution u to the case that $E(0) > 0$ for the equation

$$\begin{cases} u'' = u^{\frac{7}{3}}, \\ u(0) = -1, \ u'(0) = 1. \end{cases} \tag{2.5}$$

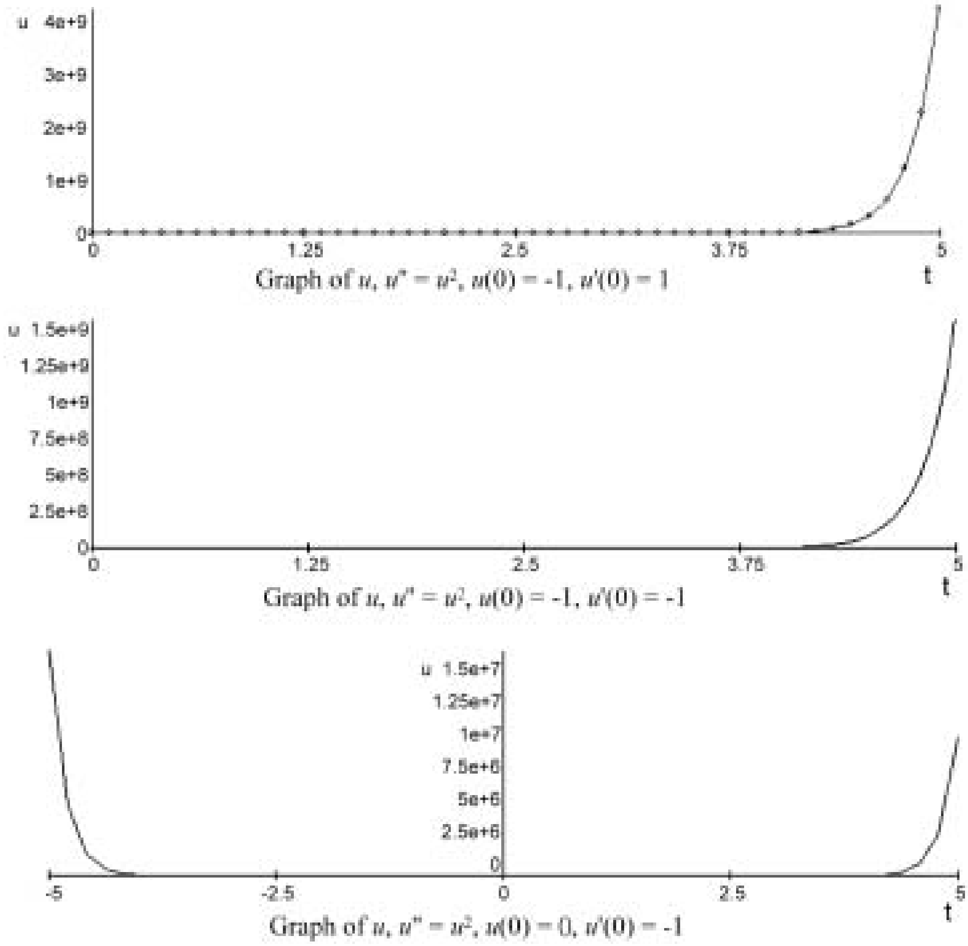

Fig. 2.1.

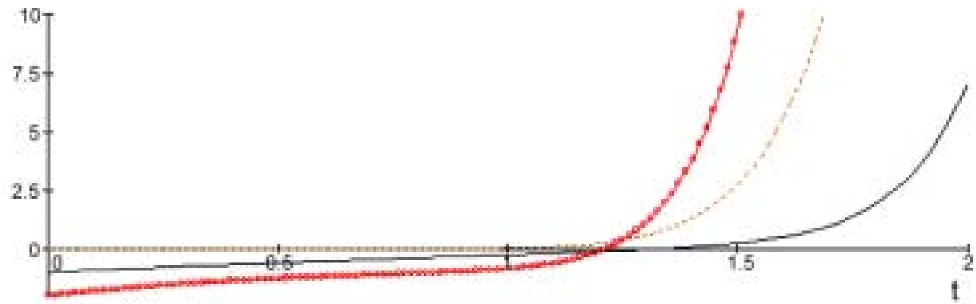

Fig. 2.2. Graphs of u in solid, u' in dash, u'' in dots.

Because the solution of equation (2.5) can not be solved explicitly, we solve this ode numerically and obtain the graphs of u, u', u'', $u^{(3)}$ $u^{(4)}$ and $u^{(5)}$ below by Maple.

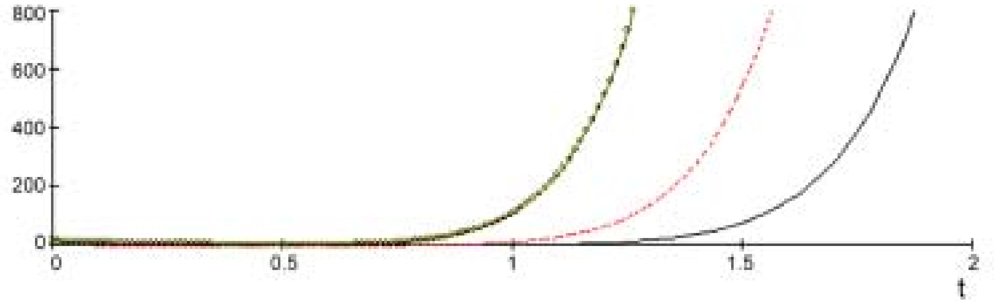

Fig. 2.3. Graphs of $u^{(3)}$ in solid, $u^{(4)}$ in dash, $u^{(5)}$ in dots.

By Theorem 3, we know that $u \in C^4(0,T)$. With the help of graph with maple, we find the $t_0 \sim 1.4$ of the null point of u (Figure 2.2) and the $u^{(5)}$ is close in infinite as t approach to 1.4 (Figure 2.3). Hence we know that $u^{(5)}(t)$ does not exist for $t = t_0$ by the graph. The blow-up rate of $u^{(n)}$ is increasing in n. It will be illustrate in the next section .

3. The Blow-up Rate and Blow-up Constant

Finding out the blow-up rate and blow-up constant of $u^{(n)}$ of the equation (0.1) given as follows is our main result, we have the following results:

Theorem 4. *If u is the solution of the problem (0.1) with one of the following properties that*

(i) $E(0) < 0$ *or*

(ii) $E(0) = 0,\ a'(0) > 0$ *or*

(iii) $E(0) > 0,\ a'(0)^2 > 4a(0)E(0)$ *or*

(iv) $E(0) > 0,\ a'(0)^2 = 4a(0)E(0),\ u_1 > 0$ *or*

(v) $E(0) > 0,\ a'(0)^2 = 4a(0)E(0),\ u_1 < 0$ *and p is odd.*

Then the blow-up rate of $u^{(2n)}$ is $\frac{2}{p-1} + 2n$, and the blow-up constant of $u^{(2n)}$ is $\left|E_{n0}\left(\frac{\sqrt{2(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n}\right|$; *that is, for $n \in \mathbb{N}$, $m \in \{1,2,3,4,5,6\}$*

$$\begin{aligned}(3.1)\qquad &\lim_{t\to T_m^*} u^{(2n)}(T_m^* - t)^{\frac{2}{p-1}+2n} \\ &= (\pm 1)^{C_{n0}} E_{n0}\left(\frac{\sqrt{2(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n} := K_{2n}\end{aligned}$$

The blow-up rate of $u^{(2n+1)}$ *is* $\frac{2}{p-1}+2n+1$, *and the blow-up constant of* $u^{(2n+1)}$ *is* $\left|E_{n0}C_{n0}\sqrt{\frac{2}{p+1}}\left(\frac{\sqrt{2(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n+1}\right|$*; that is, for* $n \in \mathbb{N}$, $m \in \{1,2,3,4,5,6\}$

$$\begin{aligned}&\lim_{t\to T_m^*} u^{(2n+1)}\left(T_m^*-t\right)^{\frac{2}{p-1}+2n}\\ &=(\pm)^{C_{n0}}E_{n0}C_{n0}\sqrt{\tfrac{2}{p+1}}\left(\frac{\sqrt{2(P+1)}}{p-1}\right)^{\frac{2}{p-1}+2n+1} := K_{2n+1}\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}C_{n0}&=(p-1)\,n+1,\\ E_{n0}&=\Pi_{i=0}^{n-1}\left[\frac{2\,(p-1)^2\,i^2+(p-1)\,i}{p+1}+(p-1)\,i+1\right].\end{aligned}$$

Proof. Under condition (i), $E(0)<0,\ a'(0)\geq 0$ by (0.7) and (0.8), we obtain that

$$\int_0^{J(t)} \frac{1}{T_1^*-t}\frac{dr}{\sqrt{k_1+E(0)\,r^{k_2}}}=\frac{p-1}{2} \quad \forall t\geq 0. \tag{3.3}$$

Using lemma 3 and (3.3) we have

$$\lim_{t\to T_1^*}\frac{1}{\sqrt{k_1}}\frac{J(t)}{T_1^*-t}=\frac{p-1}{2}; \text{ (see appendix A.1)}$$

in other words,

$$\lim_{t\to T_1^*} a(t)\left(T_1^*-t\right)^{\frac{4}{p-1}}=\left(\frac{2}{(p-1)\sqrt{k_1}}\right)^{\frac{4}{p-1}} \tag{3.4}$$

and then

$$\lim_{t\to T_1^*} u(t)\left(T_1^*-t\right)^{\frac{2}{p-1}}=\pm\left(\frac{2}{(p-1)\sqrt{k_1}}\right)^{\frac{2}{p-1}}. \tag{3.5}$$

Here $C_{n\ i}=p+(n-1-i)\,(p+1)-2\,(n-1)$, hence we have $C_{n\ i}>C_{n\ j}$ as $i<j$.

By (2.1) and (3.5), we obtain

$$\begin{aligned}&\lim_{t\to T_1^*} u^{(2n)}\left(T_1^*-t\right)^{\frac{2}{p-1}\times C_{n0}}\\ &=(\pm 1)^{C_{n0}}E_{n0}\left(\frac{2}{(p-1)\sqrt{k_1}}\right)^{\frac{2}{p-1}\times C_{n0}}.\end{aligned}$$

Since $\frac{2}{p-1} \times C_{n0} = \frac{2}{p-1} + 2n$ and $k_1 = \frac{2}{p+1}$, so we get (3.1) for $m = 1$.

By (0.6), we find that

$$\lim_{t \to T_1^*} J'(t) = -\frac{p-1}{\sqrt{2p+2}} \tag{3.6}$$

and

$$\frac{2\sqrt{2}}{\sqrt{p+1}} = \lim_{t \to T_1^*} \left(a(t) (T_1^* - t)^{\frac{4}{p-1}} \right)^{-\frac{p-1}{4}-1} \cdot \lim_{t \to T_1^*} a'(t) (T_1^* - t)^{\frac{4}{p-1} \times \frac{p+3}{4}}.$$

Together (3.4) and (2.2) we obtain that

$$\lim_{t \to T_1^*} u'(t) (T_1^* - t)^{\frac{2}{p-1}+1} = \pm\sqrt{k_1} \left(\frac{2}{(p-1)\sqrt{k_1}} \right)^{\frac{2}{p-1}+1} \tag{3.7}$$

and

$$\begin{aligned}
&\lim_{t \to T_1^*} u^{(2n+1)} (T_1^* - t)^{\frac{2}{p-1}C_{n0}+1} \\
&= \lim_{t \to T_1^*} \sum_{i=0}^{n-1} E_{n\ i} C_{n\ i} u^{C_{n\ i}-1} \cdot u' \cdot (T_1^* - t)^{\frac{2}{p-1}C_{n0}+1} \\
&= \lim_{t \to T_1^*} E_{n0} C_{n0} u^{C_{n0}-1} \cdot u' \cdot (T_1^* - t)^{\frac{2}{p-1}C_{n0}+1} \\
&= \lim_{t \to T_1^*} E_{n0} C_{n0} u^{C_{n0}-1} \cdot (T_1^* - t)^{\frac{2}{p-1}C_{n0}-1} \cdot u' \cdot (T_1^* - t)^{\frac{2}{p-1}+1} \\
&= \lim_{t \to T_1^*} (\pm)^{C_{n0}} E_{n0} C_{n0} \sqrt{k_1} \left(\frac{2}{(p-1)\sqrt{k_1}} \right)^{\frac{2}{p-1}C_{n0}+1};
\end{aligned}$$

thus (3.2) for $m = 1$ is proved.

For $E(0) < 0,\ a'(0) < 0$, by (0.10) we have

$$\int_0^{J(t)} \frac{dr}{(T_2^* - t)\sqrt{k_1 + E(0) r^{k_2}}} = \frac{p-1}{2} \quad \forall t \geq t_0. \tag{3.8}$$

Using lemma 3, (3.8) and (2.1), therefore we gain the estimate (3.1) for $m = 2$, and by (0.9) we get the estimate (3.2) for $m = 2$. (see appendix A.2)

Under (ii), $E(0) = 0,\ a'(0) > 0$, inducing (0.12), we have

$$a(t) = a(0)^{\frac{p+3}{p-1}} \left(\frac{p-1}{4} a'(0) (T_3^* - t) \right)^{-\frac{4}{p-1}} \quad \forall t \geq 0. \tag{3.9}$$

In view of (3.9) and (2.1), we get the estimate (3.1) for $m = 3$.

Using (0.12), we also obtain

$$J'(t) = J'(0) \quad \forall t \geq 0$$

and

$$\lim_{t \to T_1^*} a(t)^{-\frac{p-1}{4}-1} a'(t) = -\frac{p-1}{4} a(0)^{-\frac{p-1}{4}-1} a'(0).$$

By (3.9) and (2.2), the estimate (3.2) for $m = 3$ is completely proved.

Under (iii) or (iv) or (v), the proofs of estimates (3.1) and (3.2) for $m = 4, 5, 6$ are similar to the above arguments, we omit the argumentations.

Theorem 5. *Suppose that u is the solution of the problem (0.1) with $E(0) > 0$ and one of the following properties holds*

(i) $a'(0)^2 < 4a(0)E(0)$ *and* $a'(0) \leq 0$.

(ii) $a'(0)^2 < 4a(0)E(0)$ *and* $a'(0) > 0$.

(iii) $a'(0)^2 = 4a(0)E(0)$ *and* $u_1 < 0$, p *is even. Then we have*

$$\lim_{t \to z_1} u^{(2n)}(t)(z_m - t)^{-C_{n(n-1)}} = (\pm)^{C_{n(n-1)}} E_{n(n-1)} E(0)^{\frac{C_{n(n-1)}}{2}} \tag{3.10}$$

and

$$\lim_{t \to z_1} u^{(2n+1)}(t)(z_m - t)^{-C_{n(n-1)}+1} = E_{n(n-1)} C_{n(n-1)} E(0)^{C_{n(n-1)}-1} \tag{3.11}$$

for $n \in \mathbb{N}$, $m \in \{1, 2, 3\}$, where z_m is the null point (zero) of u and

$$C_{n(n-1)} = p - 2n + 2,$$

$$E_{n(n-1)} = \Pi_{i=0}^{n-1}(p - 2i + 2)(p - 2i + 1) E(0)^{n-1}.$$

Proof. Under (i) using (0.19) and (0.20) we get

$$\lim_{t \to z_1} u(t)(z_1 - t)^{-1} = \pm E(0)^{\frac{1}{2}} \tag{3.12}$$

and

$$\lim_{t \to z_1} u'(t)(z_1 - t)^{-1} = \mp E(0)^{\frac{1}{2}}. \tag{3.13}$$

By (2.1) and (3.12) we obtain that

$$\begin{aligned} &\lim_{t \to z_1} u^{(2n)}(z_1 - t)^{-C_{n(n-1)}} \\ &= \lim_{t \to z_1} \sum_{i=0}^{n-1} E_{n\ i} u^{C_{n\ i}} (z_1 - t)^{-C_{n(n-1)}} \\ &= \lim_{t \to z_1} E_{n(n-1)} u^{C_{n(n-1)}} (z_1 - t)^{-C_{n(n-1)}} \\ &= (\pm 1)^{C_{n(n-1)}} E_{n(n-1)} E(0)^{\frac{C_{n(n-1)}}{2}}. \end{aligned}$$

Therefore, (3.10) for $m=1$ is proved.

From (2.2), (3.12) and (3.13), we obtain

$$\begin{aligned}&\lim_{t\to z_1} u^{(2n+1)}(z_1-t)^{-C_{n(n-1)}+1}\\ &=\lim_{t\to z_1}\sum_{i=0}^{n-1}E_{ni}C_{ni}u^{C_{ni}-1}u'(z_1-t)^{-C_{n(n-1)}+1}\\ &=\lim_{t\to z_1}E_{n(n-1)}C_{n(n-1)}u^{C_{n(n-1)}-1}u'(z_1-t)^{-C_{n(n-1)}+1}\\ &=E_{n(n-1)}C_{n(n-1)}E(0)^{C_{n_{(n-1)}}}.\end{aligned}$$

Thus, (3.11) for $m=1$ is obtained.

Under the (ii) or (iii), the proofs of estimations (3.10) and (3.11) for $m=2,3$ are similar to the above arguments, we do not bother them again.

Appendix Proof of Theorem 4

A.1 Lemma

Lemma A1. *If* $\int_0^{J(t)}\frac{1}{T^*-t}\frac{dr}{\sqrt{k_1+E(0)r^{k_2}}}=\frac{p-1}{2}$ *for each* $t\geq 0$, *then*

$$\lim_{t\to T^*}\frac{1}{\sqrt{k_1}}\frac{J(t)}{T^*-t}=\frac{p-1}{2}.$$

Proof. Let $r=(T^*-t)s$, then using lemma 3, we conclude

$$\begin{aligned}&\lim_{t\to T^*}\int_0^{\frac{J(t)}{(T^*-t)}}\frac{ds}{\sqrt{k_1+E(0)(T^*-t)^{k_2}s^{k_2}}}\\ &=\frac{1}{\sqrt{k_1}}\int_0^{\lim_{t\to T^*}\frac{J(t)}{(T^*-t)}}ds=\lim_{t\to T^*}\frac{1}{\sqrt{k_1}}\frac{J(t)}{T^*-t}.\end{aligned}$$

A.2. Lemma

Lemma A2. *If u is the solution of the problem* (0.1) *with* $E(0)<0$ *and* $a'(0)<0$, *then* (3.1) *and* (3.2) *hold for* $m=2$.

Proof. By lemma A1.

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