TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 3, pp. 699-712, March 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

INVERSES OF SOME NEW INEQUALITIES SIMILAR TO HILBERT'S INEQUALITIES

Zhao Changjian, Josip Pecarić and Leng Gangsong

Abstract. In the present paper we first establish inverse versions of some new inequalities similar to Hilbert's inequality involving series of nonnegative terms. Then, the integral analogues of our main results are also given. Our Theorems provide new estimates on these types of inequalities.

1. Introduction

In recent years several authors [1-9] have paid considerable attention to Hilbert's inequalities and Hilbert type inequalities and their various generalizations. In particular, in 1998, B. G. Pachpatte [1] proved four new inequalities similar to Hilbert's inequality involving series of nonnegative terms [10, P. 226] as follows (see the next section for definitions of notations to be used below):

Inequality A.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{m+n} \le \frac{1}{2}pq(kr)^{1/2}\Big(\sum_{m=1}^{k}(k-m+1)\big(a_m A_m^{p-1}\big)^2\Big)^{1/2} \times\Big(\sum_{n=1}^{r}(r-n+1)\big(b_n B_n^{q-1}\big)^2\Big)^{1/2}.$$

Received October 13, 2003; revised April 25, 2004.
Communicated by Sen-Yen Shaw.
2000 *Mathematics Subject Classification*: 26D15.
Key words and phrases: Hilbert's inequality, Hölder integral inequality, Jensen's inequality.
[1]Supported by National Natural Sciences Foundation of China (10271071).
[2]Supported by Zhejiang Provincial Natural Science Foundation of China (Y605065).
[3]Supported by Science Foundation of Education Department of Zhejiang Province (20050392).
[4]Supported by Academic Mainstay of Middle-age and Youth Foundation of Shandong Province of China.

Inequality B.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\phi(A_m)\psi(B_n)}{m+n}\le M(k,r)\Bigg(\sum_{m=1}^{k}\Big(k-m+1\Big)\Big(p_m\phi(\frac{a_m}{p_m})\Big)^2\Bigg)^{1/2} \times\Bigg(\sum_{n=1}^{r}\Big(r-n+1\Big)\Big(q_n\psi(\frac{b_n}{q_n})\Big)^2\Bigg)^{1/2}$$

where

$$M(k,r)=\frac{1}{2}\left(\sum_{m=1}^{k}\Big(\frac{\phi(P_m)}{P_m}\Big)^2\right)^{1/2}\left(\sum_{n=1}^{r}\Big(\frac{\psi(Q_n)}{Q_n}\Big)^2\right)^{1/2}.$$

Inequality C.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_mQ_n\phi(A_m)\psi(B_n)}{m+n}\le\frac{1}{2}(kr)^{1/2}\left(\sum_{m=1}^{k}(k-m+1)\Big(p_m\phi(a_m)\Big)^2\right)^{1/2} \times\left(\sum_{n=1}^{r}(r-n+1)\Big(q_n\psi(b_n)\Big)^2\right)^{1/2}.$$

Inequality D.

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{mn}{m+n}\phi(A_m)\psi(B_n)\le\frac{1}{2}(kr)^{1/2}\left(\sum_{m=1}^{k}(k-m+1)\Big(\phi(a_m)\Big)^2\right)^{1/2} \times\left(\sum_{n=1}^{r}(r-n+1)\Big(\psi(b_n)\Big)^2\right)^{1/2}.$$

Moreover, in [9] Pachpatte also establish integral analogues of the above inequalities as follows.

Inequality E.

$$\int_0^x\int_0^y\frac{F^h(s)G^l(t)}{s+t}dsdt\le\frac{1}{2}hl(xy)^{1/2}\left(\int_0^x\Big(x-s\Big)\Big(F^{h-1}(s)f(s)\Big)^2ds\right)^{1/2} \times\left(\int_0^y\Big(y-t\Big)\Big(G^{l-1}(t)g(t)\Big)^2dt\right)^{1/2}.$$

Inequality F.

$$\int_0^x\int_0^y\frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)}{s+t}dsdt\le L(x,y)\left(\int_0^x(x-s)\Big(p(s)\phi\Big(\frac{f(s)}{p(s)}\Big)\Big)^2ds\right)^{1/2}$$

$$\times\left(\int_0^y (y-t)\Big(q(t)\psi\Big(\frac{g(t)}{q(t)}\Big)\Big)^2 dt\right)^{1/2},$$

where

$$L(x,y)=\frac{1}{2}\left(\int_0^x \left(\frac{\phi\big(P(s)\big)}{P(s)}\right)^2 ds\right)^{1/2}\left(\int_0^y \left(\frac{\psi\big(Q(t)\big)}{Q(t)}\right)^2 dt\right)^{1/2}.$$

Inequality G.

$$\int_0^x\int_0^y \frac{P(s)Q(t)\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)}{s+t}dsdt$$
$$\leq \frac{1}{2}(xy)^{1/2}\left(\int_0^x \Big(x-s\Big)\Big(p(s)\phi\Big(f(s)\Big)\Big)^2 ds\right)^{1/2}$$
$$\left(\int_0^y \Big(y-t\Big)\Big(q(t)\psi\Big(g(t)\Big)\Big)^2 dt\right)^{1/2}.$$

Inequality H.

$$\int_0^x\int_0^y \frac{\phi\Big(F(s)\Big)\psi\Big(G(t)\Big)}{(st)^{-1}(s+t)}dsdt$$
$$\leq \frac{1}{2}(xy)^{1/2}\left(\int_0^x \Big(x-s\Big)\Big(\phi\Big(f(s)\Big)\Big)^2 ds\right)^{1/2}\left(\int_0^y \Big(y-t\Big)\Big(\psi\Big(g(t)\Big)\Big)^2 dt\right)^{1/2}.$$

The main purpose of this paper is to establish reverse versions of *Inequalities A-H.* Our main results provide new estimates on these types of inequalities.

2. Main Results

Theorem 1. *Let $0 < p_i \leq 1$ and $\{a_{i,m_i}\}$ be n positive sequences of real numbers defined for $m_i = 1, 2, \ldots, k_i$, where $k_i (i = 1, \ldots, n)$ are some natural numbers and let $A_{i,m_i} = \sum_{s_i=1}^{m_i} a_{i,s_i}$. Let $\frac{1}{\alpha_i} + \frac{1}{\beta_i} = 1, 0 < \beta_i < 1$ and $\sum_{i=1}^{n} \frac{1}{\alpha_i} = \frac{1}{\alpha}$. Then*

(1)
$$\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^n A_{i,m_i}^{p_i}}{\left(\alpha\sum_{i=1}^n\frac{1}{\alpha_i}m_i\right)^{1/\alpha}} \geq \prod_{i=1}^n p_i k_i^{1/\alpha_i}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)(a_{i,m_i}A_{i,m_i}^{p_i-1})^{\beta_i}\right)^{1/\beta_i}.$$

Proof. By using the following inequality(see Hardy *et al.* [10,P.39])

$$h_i a_{i,m_i}^{h_i-1}(a_{i,m_i}-b_{i,m_i}) \leq a_{i,m_i}^{h_i}-b_{i,m_i}^{h_i} \leq h_i b_{i,m_i}^{h_i-1}(a_{i,m_i}-b_{i,m_i}),$$

where $a_{i,m_i} > 0$, $b_{i,m_i} > 0$, and $0 \leq h_i \leq 1 (i = 1, 2, \ldots, n)$, we obtain that

(2)
$$A_{i,m_i}^{p_i} \geq p_i \sum_{s_i=1}^{m_i} a_{i,s_i}A_{i,s_i}^{p_i-1}.$$

Let us note that by the following means inequality

$$\prod_{i=1}^n m_i^{1/\alpha_i} \geq \left(\alpha\sum_{i=1}^n\frac{1}{\alpha_i}m_i\right)^{1/\alpha},$$

and from (2), and in view of inverse Hölder's inequality$[10, P.24]$, we have

(3)
$$\frac{\prod_{i=1}^n A_{i,m_i}^{p_i}}{\left(\alpha\sum_{i=1}^n\frac{1}{\alpha_i}m_i\right)^{1/\alpha}} \geq \prod_{i=1}^n p_i\left(\sum_{s_i=1}^{m_i}(a_{i,s_i}A_{i,s_i}^{p_i-1})^{\beta_i}\right)^{1/\beta_i}.$$

Taking the sum of both sides of (3) over m_i from 1 to $k_i (i = 1, 2, \ldots, n)$ first and then using again inverse Hölder's inequality, we obtain that

$$\begin{aligned}\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^n A_{i,m_i}^{p_i}}{\left(\alpha\sum_{i=1}^n\frac{1}{\alpha_i}m_i\right)^{1/\alpha}} &\geq \prod_{i=1}^n p_i\left(\sum_{m_i=1}^{k_i}\left(\sum_{s_i=1}^{m_i}(a_{i,s_i}A_{i,s_i}^{p_i-1})^{\beta_i}\right)^{1/\beta_i}\right)\\
&\geq \prod_{i=1}^n p_i k_i^{1/\alpha_i}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}(a_{i,s_i}A_{i,s_i}^{p_i-1})^{\beta_i}\right)^{1/\beta_i}\\
&= \prod_{i=1}^n p_i k_i^{1/\alpha_i}\left(\sum_{s_i=1}^{k_i}(k_i-s_i+1)(a_{i,s_i}A_{i,s_i}^{p_i-1})^{\beta_i}\right)^{1/\beta_i}\\
&= \prod_{i=1}^n p_i k_i^{1/\alpha_i}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)(a_{i,m_i}A_{i,m_i}^{p_i-1})^{\beta_i}\right)^{1/\beta_i}.\end{aligned}$$

This completes the proof.

Remark 1. Taking $n = 2, \beta_i = \frac{1}{2}(i = 1, 2)$ to (1), (1) becomes

$$\sum_{m_1=1}^{k_1} \sum_{m_2=1}^{k_2} \frac{A_{1,m_1}^{p_1} A_{2,m_2}^{p_2}}{(m_1 + m_2)^{-2}} \geq 4p_1 p_2 (k_1 k_2)^{-1} \Big(\sum_{m_1=1}^{k_1} \big(k_1 - m_1 + 1\big)\big(a_{1,m_1} A_{1,m_1}^{p_1-1}\big)^{1/2}\Big)^2 \times \Big(\sum_{m_2=1}^{k_2} \big(k_2 - m_2 + 1\big)\big(a_{2,m_2} A_{2,m_2}^{p_2-1}\big)^{1/2}\Big)^2.$$

This is just an inverse of the inequality A stated in introduction.

Theorem 2. *Let* $\{a_{i,m_i}\}, A_{i,m_i}, k_i, \alpha_i$ and β_i *be as defined in Theorem 1. Let* $\{p_{i,m_i}\}$ *be n positive sequences for* $m_i = 1, 2, \ldots, k_i (i = 1, 2, \ldots, n)$. *Set* $P_{i,m_i} = \sum_{s_i=1}^{m_i} p_{i,s_i} (i = 1, 2, \ldots, n)$. *Let* $\phi_i (i = 1, 2, \ldots, n)$ *be n real-valued nonnegative, concave, and supermultiplicative functions defined on.* $R_+ = [0, +\infty)$. *Then*

$$\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} \phi_i(A_{i,m_i})}{\left(\alpha \sum_{i=1}^{n} \frac{1}{\alpha_i} m_i\right)^{1/\alpha}}$$

$$\geq M(k_1, k_2, \ldots, k_n) \prod_{i=1}^{n} \left(\sum_{m_i=1}^{k_i} (k_i - m_i + 1) \left(p_{i,m_i} \phi_i \left(\frac{a_{i,m_i}}{p_{i,m_i}} \right) \right)^{\beta_i} \right)^{1/\beta_i}, \tag{4}$$

where

$$M(k_1, k_2, \ldots, k_n) = \prod_{i=1}^{n} \left(\sum_{m_i=1}^{k_i} \left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \right)^{\alpha_i} \right)^{1/\alpha_i}.$$

Proof. From the hypotheses and by Jensen's inequality and inverse Hölder's inequality, we obtain that

$$\begin{aligned} \prod_{i=1}^{n} \phi_i(A_{i,m_i}) &= \prod_{i=1}^{n} \phi_i \left(\frac{P_{i,m_i} \sum_{s_i=1}^{m_i} p_{i,s_i} \left(\frac{a_{i,s_i}}{p_{i,s_i}} \right)}{\sum_{s_i=1}^{m_i} p_{i,s_i}} \right) \\ &\geq \prod_{i=1}^{n} \phi_i(P_{i,m_i}) \phi_i \left(\frac{\sum_{s_i=1}^{m_i} p_{i,s_i} \left(\frac{a_{i,s_i}}{p_{i,s_i}} \right)}{\sum_{s_i=1}^{m_i} p_{i,s_i}} \right) \\ &\geq \prod_{i=1}^{n} \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} \sum_{s_i=1}^{m_i} p_{i,s_i} \phi_i \left(\frac{a_{i,s_i}}{p_{i,s_i}} \right) \\ &\geq \prod_{i=1}^{n} \frac{\phi_i(P_{i,m_i})}{P_{i,m_i}} m_i^{1/\alpha_i} \left(\sum_{s_i=1}^{m_i} \left(p_{i,s_i} \phi_i \left(\frac{a_{i,s_i}}{p_{i,s_i}} \right) \right)^{\beta_i} \right)^{1/\beta_i}. \end{aligned} \tag{5}$$

Noticing the means inequality and dividing both sides of (5) by $\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}$, and then taking the sum over $m_i(i=1,2,\ldots,n)$ from 1 to k_i, and in view of inverse Hölder's inequality, we have

$$\begin{aligned}
&\sum_{m_1=1}^{k_1}\cdots\sum_{m_n=1}^{k_n}\frac{\prod_{i=1}^{n}\phi_i(A_{i,m_i})}{\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}m_i\right)^{1/\alpha}}\\
&\quad\ge\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\left(\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}\right)\\
&\quad\ge\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\left(\frac{\phi_i(P_{i,m_i})}{P_{i,m_i}}\right)^{\alpha_i}\right)^{1/\alpha_i}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}\\
&\quad=M(k_1,k_2,\ldots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}\sum_{s_i=1}^{m_i}\left(p_{i,s_i}\phi_i\left(\frac{a_{i,s_i}}{p_{i,s_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}\\
&\quad=M(k_1,k_2,\ldots,k_n)\prod_{i=1}^{n}\left(\sum_{m_i=1}^{k_i}(k_i-m_i+1)\left(p_{i,m_i}\phi_i\left(\frac{a_{i,m_i}}{p_{i,m_i}}\right)\right)^{\beta_i}\right)^{1/\beta_i}.
\end{aligned}$$

The proof is complete.

Remark 2. Taking $n=2,\beta_i=\frac{1}{2}(i=1,2)$ to (4), (4) becomes

$$\begin{aligned}
&\sum_{m_1=1}^{k_1}\sum_{m_2=1}^{k_2}\frac{\phi_1(A_{1,m_1})\phi_2(A_{2,m_2})}{(m_1+m_2)^{-2}}\\
&\quad\ge M(k_1,k_2)\Big(\sum_{m_1=1}^{k_1}\Big(k_1-m_1+1\Big)\Big(p_{1,m_1}\phi_1\big(\frac{a_{1,m_1}}{p_{1,m_1}}\big)\Big)^{1/2}\Big)^2\\
&\quad\times\Big(\sum_{m_2=1}^{k_2}\Big(k_2-m_2+1\Big)\Big(p_{2,m_2}\phi_2\big(\frac{a_{2,m_2}}{p_{2,m_2}}\big)\Big)^{1/2}\Big)^2
\end{aligned}$$

where

$$M(k_1,k_2)=4\left(\sum_{m_1=1}^{k_1}\Big(\frac{\phi_1\big(P_{1,m_1}\big)}{P_{1,m_1}}\Big)^{-1}\right)^{-1}\left(\sum_{m_2=1}^{k_2}\Big(\frac{\phi_2\big(P_{2,m_2}\big)}{P_{2,m_2}}\Big)^{-1}\right)^{-1}.$$

This is just an inverse of the inequality B stated in introduction.
Similarly, the following theorem also can be established.

Theorem 3. *Let $P_{i,m_i}, \{a_{i,m_i}\}, \{p_{i,m_i}\}, k_i, \alpha_i$ and β_i be is in Theorem 2, and define $A_{i,m_i} = (1/P_{i,m_i}) \sum_{s_i=1}^{m_i} p_{i,s_i} a_{i,s_i}$, for $m_i = 1, 2, \ldots, k_i$. Let $\phi_i(i = 1, 2, \ldots, n)$ be n real-valued, nonnegative, and concave functions defined on R_+. Then*

$$\sum_{m_1=1}^{k_1} \cdots \sum_{m_n=1}^{k_n} \frac{\prod_{i=1}^{n} P_{i,m_i} \phi_i(A_{i,m_i})}{\left(\alpha \sum_{i=1}^{n} \frac{1}{\alpha_i} m_i\right)^{1/\alpha}} \geq \prod_{i=1}^{n} k_i^{1/\alpha_i} \left(\sum_{m_i=1}^{k_i} (k_i - m_i + 1)(p_{i,m_i} \phi_i(a_{i,m_i}))^{\beta_i} \right)^{1/\beta_i}. \tag{6}$$

The proof of Theorem 3 can be completed by following the same steps as in the proof of Theorem 2 with suitable changes. Here, we omit the details.

Remark 3. Taking $n = 2, \beta_i = \frac{1}{2}(i = 1, 2)$ to (6), (6) becomes

$$\sum_{m_1=1}^{k_1} \sum_{m_2=1}^{k_2} \frac{P_{1,m_1} P_{2,m_2} \phi_1(A_{1,m_1}) \phi_2(A_{2,m_2})}{(m_1 + m_2)^{-2}} \geq 4(kr)^{-1} \left(\sum_{m_1=1}^{k_1} (k_1 - m_1 + 1) \Big(p_{1,m_1} \phi_1(a_{1,m_1}) \Big)^{1/2} \right)^2 \times \left(\sum_{m_2=1}^{k_2} (k_2 - m_2 + 1) \Big(p_{2,m_2} \phi_2(a_{2,m_2}) \Big)^{1/2} \right)^2.$$

This is just an inverse of the inequality C stated in introduction.

Remark 4. If $p_{i,m_i} = 1$, then $P_{i,m_i} = m_i (i = 1, 2)$. Taking these results to Remark 3 , we obtain that

$$\sum_{m_1=1}^{k_1} \sum_{m_2=1}^{k_2} \frac{m_1 m_2}{(m_1 + m_2)^{-2}} \phi_1(A_{1,m_1}) \phi_2(A_{2,m_2}) \geq 4(kr)^{-1} \left(\sum_{m_1=1}^{k_1} (k_1 - m_1 + 1) \Big(\phi_1(a_{1,m_1}) \Big)^{1/2} \right)^2 \times \left(\sum_{m_2=1}^{k_2} (k_2 - m_2 + 1) \Big(\phi_2(a_{2,m_2}) \Big)^{1/2} \right)^2.$$

This is just an inverse of the inequality D stated in introduction.

3. Integral Analogues

Theorem 4. *Let* $h_i \geq 1$ *and let* $f_i(\sigma_i) \in C^1[(0, x_i), (0, \infty)], i = 1, \ldots, n$, *where* x_i *are positive real numbers, and define* $F_i(s_i) = \int_0^{s_i} f_i(\sigma_i) d\sigma_i$, for $s_i \in (0, x_i)$. *Then for* $\frac{1}{\alpha_i} + \frac{1}{\beta_i} = 1$,$0 < \beta_i < 1$ *and* $\sum_{i=1}^n \frac{1}{\alpha_i} = \frac{1}{\alpha}$,

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{\left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i\right)^{1/\alpha}} ds_1 \cdots ds_n \geq \prod_{i=1}^n x_i^{1/\alpha_i} h_i \left(\int_0^{x_i} (x_i - s_i) \left(F_i^{h_i-1}(s_i) f_i(s_i)\right)^{\beta_i} ds_i\right)^{1/\beta_i}. \tag{7}$$

Proof. From the hypotheses and in view of inverse Hölder integral inequality [11], it is easy to observe that

$$\begin{aligned}\prod_{i=1}^n F_i^{h_i}(s_i) &= \prod_{i=1}^n h_i \int_0^{s_i} F_i^{h_i-1}(\sigma_i) f_i(\sigma_i) d\sigma_i \\ &\geq \prod_{i=1}^n h_i s_i^{1/\alpha_i} \left(\int_0^{s_i} \left(F_i^{h_i-1} f_i(\sigma_i)\right)^{\beta_i} d\sigma_i\right)^{1/\beta_i}.\end{aligned}$$

In view of the means inequality, we obtain that

$$\frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{\left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i\right)^{1/\alpha}} \geq \prod_{i=1}^n h_i \left(\int_0^{s_i} \left(F_i^{h_i-1} f_i(\sigma_i)\right)^{\beta_i} d\sigma_i\right)^{1/\beta_i}. \tag{8}$$

Integrating both sides of (8) over s_i from 0 to $x_i (i = 1, 2, \ldots, n)$ and using the special case of inverse Hölder integral inequality, we observe that

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n F_i^{h_i}(s_i)}{\left(\alpha \sum_{i=1}^n \frac{1}{\alpha_i} s_i\right)^{1/\alpha}} ds_1 \cdots ds_n \geq \prod_{i=1}^n h_i \int_0^{x_i} \left(\int_0^{s_i} \left(F_i^{h_i-1} f_i(\sigma_i)\right)^{\beta_i} d\sigma_i\right)^{1/\beta_i} ds_i$$

$$\geq \prod_{i=1}^{n} h_i x_i^{1/\alpha_i} \left(\int_0^{x_i} \left(\int_0^{s_i} \left(F_i^{h_i-1}(\sigma_i) f_i(\sigma_i) \right)^{\beta_i} d\sigma_i \right) ds_i \right)^{1/\beta_i}$$

$$= \prod_{i=1}^{n} x_i^{1/\alpha_i} h_i \left(\int_0^{x_i} (x_i - s_i) \left(F_i^{h_i-1}(s_i) f_i(s_i) \right)^{\beta_i} ds_i \right)^{1/\beta_i}.$$

The proof is complete.

Remark 5. Taking $n = 2, \beta_i = \frac{1}{2}$ to (7), (7) changes to

$$\int_0^{x_1} \int_0^{x_2} \frac{F_1^{h_1}(s_1) F_2^{h_2}(s_2)}{(s_1 + s_2)^{-2}} ds_1 ds_2$$

$$\geq 4 h_1 h_2 (x_1 x_2)^{-1} \left(\int_0^{x_1} \Big(x_1 - s_1 \Big) \Big(F_1^{h_1-1}(s_1) f_1(s_1) \Big)^{1/2} ds_1 \right)^2$$

$$\times \left(\int_0^{x_2} \Big(x_2 - s_2 \Big) \Big(F_2^{h_2-1}(s_2) f_2(s_2) \Big)^{1/2} ds_2 \right)^2.$$

This is just an inverse inequality similar to the Inequality E stated in introduction.

On the other hand, for $\beta_i = \frac{n-1}{n} (i = 1, \ldots, n)$, (7) becomes

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} F_i^{h_i}(s_i)}{(s_1 + \cdots + s_n)^{-n/(n-1)}} ds_1 \cdots ds_n \geq n^{n/(n-1)} \prod_{i=1}^{n} x_i^{-1/(n-1)} h_i \tag{9}$$

$$\left(\int_0^{x_i} (x_i - s_i) \left(F_i^{h_i-1}(s_i) f_i(s_i) \right)^{(n-1)/n} ds_i \right)^{n/(n-1)}.$$

Theorem 5. *Let $f_i(\sigma_i)$,$F_i(s_i)$, α_i and β_i be as in Theorem 4. Let $p_i(\sigma_i)$ be n positive functions defined for $\sigma_i \in (0, x_i) (i = 1, 2, \cdots, n)$ and define $P_i(s_i) = \int_0^{s_i} p_i(\sigma_i) d\sigma_i$, where x_i are positive real numbers. Let $\phi_i (i = 1, 2, \cdots, n)$ be n real-valued nonnegative, concave, and supermultiplicative functions defined on R_+. Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} \phi_i(F_i(s_i))}{\left(\alpha \sum_{i=1}^{n} \frac{1}{\alpha_i} s_i \right)^{1/\alpha}} ds_1 \cdots ds_n \geq L(x_1, \cdots, x_n) \prod_{i=1}^{n} \tag{10}$$

$$\left(\int_0^{x_i} (x_i - s_i) \left(p_i(s_i) \phi_i\Big(\frac{f_i(s_i)}{p_i(s_i)}\Big) \right)^{\beta_i} ds_i \right)^{1/\beta_i},$$

where

$$L(x_1,\cdots,x_n)=\prod_{i=1}^{n}\left(\int_0^{x_i}\left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)}\right)^{\alpha_i}ds_i\right)^{1/\alpha_i}.$$

Proof. By using Jensen integral inquality and inverse Hölder integral inequality and noticing that $\phi_i(i=1,2,\cdots,n)$ are n real-valued supermultiplicative functions, it is easy to observe that

$$\phi_i\left(F_i(s_i)\right)=\phi_i\left(\frac{P_i(s_i)\int_0^{s_i}p_i(\sigma_i)\frac{f_i(\sigma_i)}{p_i(\sigma_i)}d\sigma_i}{\int_0^{s_i}p_i(\sigma_i)d\sigma_i}\right)$$

(11)
$$\begin{aligned}&\geq\phi_i(P_i(s_i))\phi_i\left(\frac{\int_0^{s_i}p_i(\sigma_i)\frac{f_i(\sigma_i)}{p_i(\sigma_i)}d\sigma_i}{\int_0^{s_i}p_i(\sigma_i)d\sigma_i}\right)\\&\geq\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)}\int_0^{s_i}p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)d\sigma_i\\&\geq\left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)}\right)s_i^{1/\alpha_i}\left(\int_0^{s_i}\left(p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right)^{\beta_i}d\sigma_i\right)^{1/\beta_i}.\end{aligned}$$

In view of the means inequality and integrating two sides of (11) over s_i from 0 to $x_i(i=1,2,\ldots,n)$ and noticing Hölder integral inequality, we observe that

$$\begin{aligned}&\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^{n}\phi_i(F_i(s_i))}{\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}s_i\right)^{1/\alpha}}ds_1\cdots ds_n\\&\geq\prod_{i=1}^{n}\int_0^{x_i}\left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)}\right)\left(\int_0^{s_i}\left(p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right)^{\beta_i}d\sigma_i\right)^{1/\beta_i}ds_i\\&\geq\prod_{i=1}^{n}\left(\int_0^{x_i}\left(\frac{\phi_i(P_i(s_i))}{P_i(s_i)}\right)^{\alpha_i}ds_i\right)^{1/\alpha_i}\left(\int_0^{x_i}\int_0^{s_i}\left(p_i(\sigma_i)\phi_i\left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)}\right)\right)^{\beta_i}d\sigma_i ds_i\right)^{1/\beta_i}\\&=L(x_1,\ldots,x_n)\prod_{i=1}^{n}\left(\int_0^{x_i}(x_i-s_i)\left(p_i(s_i)\phi_i(\frac{f_i(s_i)}{p_i(s_i)})\right)^{\beta_i}ds_i\right)^{1/\beta_i}.\end{aligned}$$

This completes the proof of Theorem 2.

Remark 6. Taking $n = 2, \beta_i = \frac{1}{2}$ to (10), (10) changes to

$$\int_0^{x_1}\int_0^{x_2}\frac{\phi_1\Big(F_1(s_1)\Big)\phi_2\Big(F_2(s_2)\Big)}{(s_1+s_2)^{-2}}ds_1ds_2$$

(12)
$$\geq L(x_1,x_2)\left(\int_0^{x_1}(x_1-s_1)\left(p_1(s_1)\phi_1\Big(\frac{f_1(s_1)}{p_1(s_1)}\Big)\right)^{1/2}ds_1\right)^2$$

$$\times\left(\int_0^{x_2}(x_2-s_2)\left(p_2(s_2)\phi_2\Big(\frac{f_2(s_2)}{p_2(s_2)}\Big)\right)^{1/2}ds_2\right)^2,$$

where

$$L(x_1,x_2)=4\left(\int_0^{x_1}\left(\frac{\phi_1\Big(P_1(s_1)\Big)}{P_1(s_1)}\right)^{-1}ds_1\right)^{-1}\left(\int_0^{x_2}\left(\frac{\phi_2\Big(P_2(s_2)\Big)}{P_2(s_2)}\right)^{-1}ds_2\right)^{-1}.$$

This is just an inverse inequality similar to the following Inequality F stated in introduction.

On the other hand, for $\beta_i = \frac{n-1}{n}(i = 1,\ldots,n)$, (10) changes to

(13)
$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^{n}\phi_i(F_i(s_i))}{(s_1+\cdots+s_n)^{-n/(n-1)}}ds_1\cdots ds_n$$

$$\geq \bar{L}(x_1,\cdots,x_n)\prod_{i=1}^{n}\left(\int_0^{x_i}(x_i-s_i)\left(p_i(s_i)\phi_i\left(\frac{f_i(s_i)}{p_i(s_i)}\right)\right)^{(n-1)/n}ds_i\right)^{n/(n-1)},$$

where

$$\bar{L}(x_1,\cdots,x_n) = n^{n/(n-1)}\prod_{i=1}^{n}\left(\int_0^{x_i}\left(\frac{\phi_i\left(P_i(s_i)\right)}{P_i(s_i)}\right)^{-(n-1)}ds_i\right)^{-1/(n-1)}.$$

Theorem 6 *Let* $f_i(\sigma_i), p_i(\sigma_i), P_i(\sigma_i), \alpha_i$ *and* β_i *be as in Theorem 5 and define* $F_i(s_i) = \frac{1}{P_i(s_i)}\int_0^{s_i}p_i(\sigma_i)f_i(\sigma_i)d\sigma_i$ for $\sigma_i, s_i \in (0, x_i)$ *where* x_i *are positive real numbers. Let* $\phi_i(i = 1, 2, \ldots, n)$ *be n real-valued, nonnegative, and concave functions on* R_+*. Then*

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^{n}P_i(s_i)\phi_i(F_i(s_i))}{\left(\alpha\sum_{i=1}^{n}\frac{1}{\alpha_i}s_i\right)^{1/\alpha}}ds_1\cdots ds_n$$

$$\geq \prod_{i=1}^{n} x_i^{1/\alpha_i} \left(\int_0^{x_i} (x_i - s_i) \left(p_i(s_i)\phi_i(f_i(s_i)) \right)^{\beta_i} ds_i \right)^{1/\beta_i}. \tag{14}$$

Proof. From the hypotheses and by using Jensen integral inequality and the inverse Hölder integral inequlity, we have

$$\begin{aligned}\phi_i\left(F_i(s_i)\right) &= \phi_i\left(\frac{1}{P_i(s_i)}\int_0^{s_i} p_i(\sigma_i) f_i(\sigma_i) d\sigma_i\right) \geq \frac{1}{P_i(s_i)}\int_0^{s_i} p_i(\sigma_i)\phi_i\left(f_i(\sigma_i)\right) d\sigma_i \\ &\geq \frac{1}{P_i(s_i)} s_i^{1/\alpha_i} \left(\int_0^{s_i} \left(p_i(\sigma_i)\phi_i\left(f_i(\sigma_i)\right)\right)^{\beta_i} d\sigma_i\right)^{1/\beta_i}.\end{aligned}$$

Hence

$$\begin{aligned}&\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} P_i(s_i)\phi_i(F_i(s_i))}{\left(\alpha \sum_{i=1}^{n} \frac{1}{\alpha_i} s_i\right)^{1/\alpha}} ds_1 \cdots ds_n \\ &\geq \prod_{i=1}^{n} \int_0^{x_i} \left(\int_0^{s_i} \left(p_i(\sigma_i)\phi_i(f_i(\sigma_i))\right)^{\beta_i} d\sigma_i \right)^{1/\beta_i} ds_i \\ &\geq \prod_{i=1}^{n} x_i^{1/\alpha_i} \left(\int_0^{x_i} \int_0^{s_i} \left(p_i(\sigma_i)\phi_i(f_i(\sigma_i))\right)^{\beta_i} d\sigma_i ds_i \right)^{1/\beta_i} \\ &= \prod_{i=1}^{n} x_i^{1/\alpha_i} \left(\int_0^{x_i} (x_i - s_i) \left(p_i(s_i)\phi_i(f_i(s_i))\right)^{\beta_i} ds_i \right)^{1/\beta_i}.\end{aligned}$$

Remark 7. Taking $n = 2, \beta_i = \frac{1}{2}$ to (14), (14) changes to

$$\begin{aligned}&\int_0^{x_1} \int_0^{x_2} \frac{P_1(s_1)P_2(s_2)\phi_1\Big(F_1(s_1)\Big)\phi_2\Big(F_2(s_2)\Big)}{(s_1+s_2)^{-2}} ds_1 ds_2 \\ &\geq 4(x_1x_2)^{-1}\left(\int_0^{x_1} \Big(x_1 - s_1\Big)\Big(p_1(s_1)\phi_1\Big(f_1(s_1)\Big)\Big)^{1/2} ds_1 \right)^2 \\ &\times\left(\int_0^{x_2} \Big(x_2 - s_2\Big)\Big(p_2(s_2)\phi_2\Big(f_2(s_2)\Big)\Big)^{1/2} ds_2 \right)^2.\end{aligned} \tag{15}$$

This is just an inverse inequality similar to the Inequality G stated in introduction.

Moreover, in (15), if $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore (15) changes to

$$\int_0^{x_1}\int_0^{x_2} \frac{\phi_1\Big(F_1(s_1)\Big)\phi_2\Big(F_2(s_2)\Big)}{(s_1+s_2)^{-2}}ds_1ds_2$$

$$\geq 4(x_1x_2)^{-1}\left(\int_0^{x_1}\Big(x_1-s_1\Big)\Big(\phi_1\Big(f_1(s_1)\Big)\Big)^{1/2}ds_1\right)^2$$

$$\left(\int_0^{x_2}\Big(x_2-s_2\Big)\Big(\phi_2\Big(f_2(s_2)\Big)\Big)^{1/2}ds_2\right)^2.$$

This is just an inverse inequality similar to the Inequality H stated in introduction.

On the other hand, for $\beta_i = \frac{n-1}{n}(i = 1, \ldots, n)$, (14) changes to

$$\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^{n} P_i(s_i)\phi_i(F_i(s_i))}{(s_1+\cdots+s_n)^{-n/(n-1)}}ds_1\cdots ds_n$$

$$\geq n^{n/(n-1)}\prod_{i=1}^{n} x_i^{-1/(n-1)}\left(\int_0^{x_i}(x_i-s_i)(p_i(s_i)\phi_i(f_i(s_i)))^{(n-1)/n}\,ds_i\right)^{n/(n-1)}.$$

References

1. B. G. Pachpatte, On some new inequalities similar to Hilbert's inequality, *J. Math. Anal. Appl.*, **226** (1998), 166-179.

2. G. D. Handley, J. J. Koliha and J. E. Pecarić, New Hilbert-Pachpatte Type Integral Inequalities, *J. Math. Anal. Appl.*, **257** (2001), 238-250.

3. Gao Minzhe and Yang Bicheng, On the extended Hilbert's inequality, *Proc. Amer. Math. Soc.*, **126** (1998), 751-759.

4. Kuang Jichang, On new extensions of Hilbert's integral inequality, *J. Math. Anal. Appl.*, **235** (1999), 608-614.

5. Yang Bicheng, On new generalizations of Hilbert't inequality, *J. Math. Anal. Appl.*, **248** (2000), 29-40.

6. Zhao Changjian, On Inverses of disperse and continuous Pachpatte's inequalities, *Acta Math. Sin.*, **46** (2003), 1111-1116.

7. Zhao chang-jian, Generalizations on two new Hilbert type inequalities, *J. Math.*, **20** (2000), 413-416.

8. Zhao Changjian and L. Debnath, Some New Inverse Type Hilbert Integral Inequalities, *J. Math. Anal. Appl.*, **262** (2001), 411-418.

9. G. D. Handley, J. J. Koliha and J. E. Pecarić, A Hilbert type inequality, *Tamkang J. Math.*, **31** (2000), 311-315.

10. G. H. Hardy, J. E. *Littlewood and Pólya*, Inequalities, Cambridge Univ. Press, Cambridge, U.K., 1934.

11. E. F. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag, Berlin-Göttingen, Heidelberg, 1961.

Zhao Changjian[1] Josip Pecarić[2] and Leng Gangsong[3]

[1]Department of Information and Mathematics Sciences,
College of Science,
China Jiliang University,
Hangzhou 310018, P. R. China
E-mail: chjzhao@163.com

[2]Faculty of Textile Technology,
University of Zagreb,
Pierottijeva 6, 10000 Zagreb,
Croatia

[3]Department of Mathematics,
Shanghai University,
Shanghai 200444, P. R. China
E-mail: lenggangsong@163.com