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THE ABSTRACT GLIDING HUMP PROPERTIES AND APPLICATIONS

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Abstract. In this paper, by using the section mappings, we introduce the abstract strong gliding hump property and the 0-gliding hump property in dual pair $< E, F >$, and show that the gliding hump properties can substitute the AK-property of the dual spaces for the characterizions of the barrelledness of normed spaces.

1. INTRODUCTION

Let $< E, F >$ be a pair of (real or complex) vector spaces placed in duality by a bilinear mapping $<, >$: $E \times F \to K$. For every $n \in \mathbf{N}$, let $\rho_n : E \to E : x \to x^{[n]}$ and $\rho_n : F \to F$: $y \to y^{[n]}$ be linear mappings on E resp. F, continuous for $\sigma(E, F)$ resp. $\sigma(F, E)$, and suppose the following axioms are satisfied:

(S1) $< x, y^{[n]} >=< x^{[n]}, y^{[n]} >=< x^{[n]}, y >$ for every $n \in \mathbf{N}$ and all $x \in E, y \in F$;

(S2) $(x^{[n]})^{[m]} = x^{[n \wedge m]}$ whenever $x \in E, n, m \in \mathbf{N} : n \wedge m$ denotes min(n, m).

Then we shall refer to this construction as a system of sections on $< E, F >$ (see [1, P_{104}]).

Let a system of sections be fixed on $< E, F >$ and τ be any admissible locally convex topology on E. If for every $x \in E$, $\{x^{[n]}\}$ converges to x $(n \to \infty)$ with respect to τ, then (E, τ) is said to be an AK-space.

Example 1. For $1 \leq p < \infty$, let $l^p = \{(t_j) : \sum_j |t_j|^p < \infty\}$. Then $< l^p, l^q >$ is a dual pair, where $\frac{1}{p} + \frac{1}{q} = 1$. For each $n \in \mathbf{N}$, let $\rho_n : l^p \to l^p$ be

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$\rho_n(t_j) = (t_1, t_2, \cdots, t_n, 0, \cdots)$, then $\{\rho_n\}$ is a system of sections on $< l^p, l^q >$. Furthermore, $(l^p, ||.||_p)$ is an AK-space. But, $(l^\infty, ||.||_\infty)$ is not an AK-space.

We denote by $E^{[n]}, F^{[n]}, n \in \mathbf{N}$, the spaces of vectors $x^{[n]}, x \in E$, and $y^{[n]}, y \in F$, respectively, and refer to $x^{[n]}, y^{[n]}$ as the sections of x, y respectively.

Let a system of sections be fixed on $< E, F >$. We denote by $E^{<\beta>}$ the β-dual space of E which consisting of all sequences $(y_n)_{n=1}^\infty$ of vectors having $y_n \in F^{[n]}$, $y_n^{[m]} = y_m$ whenever $m < n$ and $\lim_{n\to\infty} < x, y_n >$ exists for every $x \in E$.

Example 2. Let λ be a scalar-valued sequence space and c_{00} be the scalar valued sequence space which are 0 eventually, the β-dual space of λ to be defined by: $\lambda^\beta = \{(u_j) : \sum_j u_j t_j$ is convergence for each $(t_j) \in \lambda\}$. Then $< \lambda, \lambda^\beta >$ is a dual pair with respect to the bilinear pairing $< \overline{t}, \overline{u} >= \sum_j u_j t_j$, where $\overline{t} = (t_j) \in \lambda, \overline{u} = (u_j) \in \lambda^\beta$. For each $n \in \mathbf{N}$, let $\rho_n : \lambda \to \lambda$ be $\rho_n(t_j) = (t_1, t_2, \cdots, t_n, 0, \cdots)$. Then $\{\rho_n\}$ is a system of sections on $< \lambda, \lambda^\beta >$, and $\lambda^{<\beta>}$ is just λ^β. That is, when the space E is a sequence space λ, the abstract β-dual space of E is just the usual β-dual space of λ.

Lemma 1. [1, Prop. 3]. *Let E be a barrelled locally convex space with dual F and let a system of sections be fixed on $< E, F >$. If E is a (weakly) AK-space, then $E^{<\beta>} = F$.*

Lemma 2. [1, Prop. 2]. *Let E be a metrizable locally convex space with dual F and let a system of sections be fixed on $< E, F >$. If F is an AK-space with respect to the strong topology $\beta(F, E)$, then E is also an AK-space in its metrizable topology.*

Let a system of sections be fixed on $< E, F >$. The section mappings $\{\rho_n\}$ are said to have the uniform boundedness property if for every bounded subset B of $(E, \sigma(E, F))$, $\{\rho_n(x) : x \in B, n \in \mathbf{N}\}$ is a bounded subset of $(E, \sigma(E, F))$.

If $n, m \in \mathbf{N}$, $m > n$, denote $[n, m] = \{j : j \in \mathbf{N},\ n \leq j \leq m\}$ and $x^{[n,m]} = x^{[m]} - x^{[n]}$. A sequence of intervals $\{[n_k, m_k]\}$ is said to be increasing if $k_1 < k_2$ we have $m_{k_1} < n_{k_2}$. Generalizing Noll [2] we say that a sequence $\{z_k\}$ of non-zero vectors in E is a block sequence if there exists an increasing interval sequence $\{[n_k, m_k]\}$ in $\mathbf{N}$ and a sequence $\{x_k\} \subseteq E$ such that

$$z_k = x_k^{[m_k]} - x_k^{[n_k]}, k \in \mathbf{N}.$$

The section mappings $\{\rho_n\}$ are said to have the strong gliding hump property, if given any block sequence $\{z_k\}$ in E, which is weakly bounded in E, there exists a sequence of $\{k_i\}$ such that the series $\sum_{i=1}^\infty z_{k_i}$ is $\sigma(E, F)$-convergent to an element $z \in E$.

Let a system of sections be fixed on $< E, F >$ and τ be an admissible locally convex topology on E. The section mappings $\{\rho_n\}$ are said to have the 0-gliding hump property with respect to the topology τ, if $\{x_k\}$ converges to 0 with respect to τ and $\{[n_k, m_k]\}$ is an increasing sequence of intervals in $\mathbf{N}$, there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ and a subsequence $\{[n_{k_i}, m_{k_i}]\}$ of $\{[n_k, m_k]\}$ such that the series $\sum_{i=1}^{\infty} x_{k_i}^{[n_{k_i}, m_{k_i}]}$ is τ-converges to an element $z \in E$.

Many important classical sequence spaces have the strong gliding hump property or 0-gliding hump property (see [3, 4]). Now, we present two spaces, one has the strong gliding hump property and another has the 0-gliding hump property, but they are both not sequence spaces.

Let $(\Omega, \mathcal{U}, \mu)$ be a σ-finite measure space and $\{\Omega_n\}$ be an increasing sequence in Ω with union Ω such that $\mu(\Omega_n) < \infty$ for every $n \in \mathbf{N}$, where μ is a measure on $\mathcal{U}$. For $p \geq 1$, let $(L^p, ||.||_p) = (L^p(\Omega, \mathcal{U}, \mu), ||.||)$ denote the space of all equivalence classes of p-integrable functions and $(L^\infty, ||.||_\infty) = (L^\infty(\Omega, \mathcal{U}, \mu), ||.||_\infty)$ denote the space of all equivalence classes of essentially bounded functions.

Example 3. Let $E = L^\infty = L^\infty(\Omega, \mathcal{U}, \mu)$. Define a system of sections on $< E, E' >$ by setting $f^{[n]} = f\chi_n$, where χ denotes the characteristic function of Ω_n. Then $\{\rho_n\}$ has the uniform boundedness property and the strong gliding hump property, but $(L^\infty, ||.||_\infty)$ is not an AK-space.

Example 4. Let $E = L^p = L^p(\Omega, \mathcal{U}, \mu)$, $1 \leq p < \infty$. Define also a system of sections on $< L^p, L^q >$ by setting $f^{[n]} = f\chi_n$, where q satisfies that $\frac{1}{p} + \frac{1}{q} = 1$. The space $(L^p, ||.||_p)$ has the 0-gliding hump property and $(L^p, ||.||_p)$ is an AK-space.

The space E is said to have the bounded uniform convergence property if for every $(z_k) \in E^{<\beta>}$ and every $\sigma(E, F)$-bounded subset B of E, the sequence $\{< x, z_k >\}$ converges uniformly with respect to $x \in B$.

It is clear that if $E^{<\beta>} = F$ and $\beta(F, E)$ is an AK-space, then E has the bounded uniform convergence property. Furthermore, we have

Lemma 3. *Let a system of sections be fixed on $< E, F >$ and the section mappings $\{\rho_n\}$ have the uniform boundedness property and the strong gliding hump property. Then E has the bounded uniform convergence property.*

Proof. If not, there is $\varepsilon > 0$, a bounded subset B of $\sigma(E, F)$ and $y = (y_k) \in E^{<\beta>}$ such that for every $k \in \mathbf{N}$, there is $x_k \in B$ and $n_k \in \mathbf{N}$, $k < n_k$ satisfying that $| < x_k, y^{[n_k]} > - < x_k, y > | \geq \varepsilon$. Note that $\lim_{n\to\infty} < x_k, y^{[n]} > = < x_k, y >$, so there is $m_k \in \mathbf{N}$, $n_k < m_k$ such that

$$| < x_k, y^{[m_k]} > - < x_k, y > | < \frac{\varepsilon}{2}.$$

Thus we have

$$| < x_k, y^{[m_k]} > - < x_k, y^{[n_k]} > | \geq \frac{\varepsilon}{2}.$$

Pick $x_{k+1} \in B$ and $n_{k+1} \in \mathbf{N}$ such that $m_k < n_{k+1}$ and $| < x_{k+1}, y^{[n_{k+1}]} > - < x_{k+1}, y > | \geq \varepsilon$. Similarly, we can obtain m_{k+1} such that $n_{k+1} < m_{k+1}$ and

$$| < x_{k+1}, y^{[m_{k+1}]} > - < x_{k+1}, y^{[n_{k+1}]} > | \geq \frac{\varepsilon}{2}.$$

Inductively, we obtain two sequences $\{n_k\}$ and $\{m_k\}$ in $\mathbf{N}$ such that $n_k < m_k < n_{k+1} < m_{k+1}$ and

$$| < x_k, y^{[m_k]} > - < x_k, y^{[n_k]} > | \geq \frac{\varepsilon}{2}, k \in \mathbf{N}.$$

It follows from the axiom (S1) that

$$| < x_k^{[m_k]}, y > - < x_k^{[n_k]}, y > | \geq \frac{\varepsilon}{2}, k \in \mathbf{N}.$$

Or equivalently,

$$(1) \qquad | < x_k^{[m_k]} - x_k^{[n_k]}, y > | \geq \frac{\varepsilon}{2}, k \in \mathbf{N}.$$

Note that the section mappings $\{\rho_n\}$ have the uniform boundedness property and the strong gliding hump property, so there are a subsequence $\{x_{k_i}^{[m_{k_i}]} - x_{k_i}^{[n_{k_i}]}\}$ of $\{x_k^{[m_k]} - x_k^{[n_k]}\}$ and $x \in E$ such that $\sum_i x_{k_i}^{[m_{k_i}]} - x_{k_i}^{[n_{k_i}]}$ converges to x with respect to $\sigma(E, F)$. Thus we have

$$\sum_i < x_{k_i}^{[m_{k_i}]} - x_{k_i}^{[n_{k_i}]}, y >=< x, y > .$$

So, $\lim_i < x_{k_i}^{[m_{k_i}]} - x_{k_i}^{[n_{k_i}]}, y >= 0$. This contradicts (1) and Lemma 3 is proved.

As we knew, the study of the barrelledness of locally convex spaces is an important topic in locally convex spaces theory ([5-10]). Noll and Stadler in [1] introduced the above section mappings $\{\rho_n\}$ and gave an abstract characterization of the barrelledness of the normed spaces by their β-dual spaces. Note that many such theorems asked that the dual spaces of the normed spaces must be AK-spaces, but, the normed space $(l^1, ||.||_1)$ is a Banach space, so it is also a barrelled space, but $(l^1, ||.||_1)' = (l^\infty, ||.||_\infty)$ is not an AK-space, thus, the barrelledness of $(l^1, ||.||_1)$ cannot be obtained by these known theorems. Now, we substitute the AK-property of the dual spaces of the normed spaces with the gliding hump property, then the barrelledness of normed spaces can also be characterized by their β-dual spaces.

Our main theorem is:

Theorem 1. *Let* $(E, ||.||)$ *be a normed space with dual* F *and a system of section mappings* $\{\rho_n\}$ *be fixed on* $< E, F >$, E *be a* GAK*-space with respect to the norm topology. If* E *has the bounded uniform convergence property or the section mappings* $\{\rho_n\}$ *have the* 0*-gliding hump property with respect to the norm topology, then the following statements are equivalent:*

(1) $(E, ||.||)$ *is barrelled,*

(2) *Every* $(E^{[n]}, ||.||), n \in \mathbf{N}$, *is barrelled and* $E^{<\beta>} = F$.

Proof. (2) follows from (1) and Lemma 1 immediately.

If (2) is satisfied but $(E, ||.||)$ is not a barrelled space, then there is a pointwise bounded sequence $\{y_i\} \subseteq F$ such that $\sup\{||y_i|| : i \in \mathbf{N}\} = \infty$. i.e.,

$$\sup\{| < x, y_i > | : x \in B(E), i \in \mathbf{N}\} = \infty.$$

Here $B(E)$ is the unit ball of $(E, ||.||)$.

Note that for every $n \in \mathbf{N}$, $(E^{[n]}, ||.||)$ is a barrelled space and $\{y_i\} \subseteq F$ is pointwise bounded, so for every $n \in \mathbf{N}$,

$$\sup_i\{||y_i^{[n]}|| : i \in \mathbf{N}\} < \infty.$$

Case 1. E has the bounded uniform convergence property:

Let us define $\{i_n\}_{n=1}^{\infty}$ and $\{j_n\}_{n=1}^{\infty}$ as following:

Suppose that $i_1 = 1, i_2, \cdots, i_n$ and $j_1 = 1, j_2, \cdots, j_n$ have been defined. Pick $i_{n+1} > i_n$ and $x_{i_{n+1}} \in B(E)$ such that

$$||y_{i_{n+1}}|| \geq | < x_{i_{n+1}}, y_{i_{n+1}} > | > ||y_{i_{n+1}}|| - 1 > n2^n(1 + \sup_{q \leq j_n}\{||y_i^{[q]}|| : i \in \mathbf{N}\}).$$

Note that for every $x \in B(E), y \in F$, $\{< x, y^{[n]} >\}$ converges to $< x, y >$ uniformly with respect to $x \in B(E)$, so there is $j_{n+1} > j_n + 1$ such that

$$\sup\{| < x, y_{i_{n+1}} - y_{i_{n+1}}^{[j_{n+1}-1]} > | : x \in B(E)\} < \frac{1}{n}.$$

Thus, two strictly increasing sequences $i_1, i_2, \cdots$ and $j_1, j_2, \cdots$ have been well defined. If $j_n \leq k < j_{n+1}, n \in \mathbf{N}$, let

$$z_k = \frac{y_{i_{n+1}}^{[k-1,k]}}{| < x_{i_{n+1}}, y_{i_{n+1}} > |},$$

and

$$\omega_j = \sum_{k=1}^{j} z_k,$$

where $y_{i_2}^{[0]} = 0$. At first, we show that $(\omega_j) \notin E^{<\beta>} = F$, i.e., there is not a $\omega \in F$, such that $\omega^{[j]} = \omega_j, j \in \mathbf{N}$. If not, we can find a $\omega \in F$ satisfying the condition, then

$$
\begin{aligned}
&|<x_{i_{n+1}}, \omega^{[j_n, j_{n+1}-1]}>| = \frac{|<x_{i_{n+1}}, y_{i_{n+1}}^{[j_n, j_{n+1}-1]}>|}{|<x_{i_{n+1}}, y_{i_{n+1}}>|} \\
&\geq \frac{(|<x_{i_{n+1}}, y_{i_{n+1}}>| - |<x_{i_{n+1}}, y_{i_{n+1}}^{[j_n]}>| - |<x_{i_{n+1}}, y_{i_{n+1}} - y_{i_{n+1}}^{[j_{n+1}-1]}>|)}{|<x_{i_{n+1}}, y_{i_{n+1}}>|} \\
&> 1 - \frac{1}{n2^n} - \frac{1}{n^2 2^n}.
\end{aligned}
$$

So whenever $n \to \infty$,

$$
|<x_{i_{n+1}}, \omega^{[j_n, j_{n+1}-1]}>| \geq \frac{1}{2}.
$$

Note that for every $y \in F$, $\{<x, y^{[n]}>\}$ converges to $<x, y>$ uniformly with respect to $x \in B(E)$, so $|<x_{i_{n+1}}, \omega^{[j_n, j_{n+1}-1]}>| \to 0$. This is a contradiction. Thus, $(\omega_j) \notin E^{<\beta>} = F$.

On the other hand, if $x \in B(E), j_1 \leq p_0 < q_0$, pick $k, l \in \mathbf{N}$ such that $j_l \leq p_0 < j_{l+1}, j_k \leq q_0 < j_{k+1}$, then

$$
\begin{aligned}
&|<x, \omega_{q_0} - \omega_{p_0-1}>| = \left|\sum_{j=p_0}^{q_0} <x, z_j>\right| \leq \sum_{m=l}^{k} \left|\sum_{j=j_m}^{j_{m+1}-1} <x, z_j>\right| \\
&+ \left|\sum_{j=j_l}^{p_0} <x, z_j>\right| + \left|\sum_{j=q_0}^{j_{k+1}-1} <x, z_j>\right|.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left|\sum_{j=j_m}^{j_{m+1}-1} <x, z_j>\right| = \frac{\left|<x, y_{i_{m+1}}^{[j_m, j_{m+1}-1]}>\right|}{\left|<x_{i_{m+1}}, y_{i_{m+1}}>\right|} \\
&\leq \frac{(\left|<x, y_{i_{m+1}}>\right| + \left|<x, y_{i_{m+1}}^{[j_m]}>\right| + \left|<x, y_{i_{m+1}} - y_{i_{m+1}}^{[j_{m+1}-1]}>\right|)}{\left|<x_{i_{m+1}}, y_{i_{m+1}}>\right|} \\
&\leq \frac{(\sup\{|<x, y_n>| : n \in \mathbf{N}\} + \left|<x, y_{i_{m+1}}^{[j_m]}>\right| + \frac{1}{m})}{\left|<x_{i_{m+1}}, y_{i_{m+1}}>\right|} \\
&\leq \frac{1}{2^m}(\sup\{|<x, y_n>| : n \in \mathbf{N}\} + 2).
\end{aligned}
$$

Thus we have

$$(2) \qquad \sum_{m=l}^{k} \Big| \sum_{j=j_m}^{j_{m+1}-1} < x, z_j > \Big| \leq \frac{1}{2^{l-1}} (\sup\{| < x, y_n > | : n \in \mathbf{N}\} + 2).$$

Since $||y_{i_{n+1}}|| \geq | < x_{i_{n+1}}, y_{i_{n+1}} > | > ||y_{i_{n+1}}|| - 1 > n2^n$, so

$$(3) \qquad \lim_n \frac{||y_{i_n}||}{| < x_{i_n}, y_{i_n} > |} = 1.$$

Note that $(E, ||.||)$ is an AK-space, we have

$$(4) \qquad \lim_{u,v} \sup_{y \in F, ||y|| \leq 1} | < x, y^{[u,v]} > | = \lim_{u,v} \sup_{y \in F, ||y|| \leq 1} | < x^{[u,v]}, y > | = 0.$$

Thus,

$$\Big| \sum_{j=j_l}^{p_0} < x, z_j > \Big| = \frac{| < x, y_{i_{1+1}}^{[p_0]} - y_{i_{l+1}}^{[j_l]} > |}{| < x_{i_{l+1}}, y_{i_{l+1}} > |}$$

$$= \frac{||y_{i_{l+1}}||}{| < x_{i_{l+1}}, y_{i_{l+1}} > |} \times \frac{| < x^{[j_l, p_0]}, y_{i_{l+1}} > |}{||y_{i_{l+1}}||}.$$

$$\Big| \sum_{j=q_0}^{j_{k+1}-1} < x, z_j > \Big| = \frac{||y_{i_{k+1}}||}{| < x_{i_{k+1}}, y_{i_{k+1}} > |} \times \frac{| < x^{[q_0, j_{k+1}-1]}, y_{i_{k+1}} > |}{||y_{i_{k+1}}||}.$$

It follows from (2), (3) and (4) that whenever $p_0, q_0 \to \infty$,

$$| < x, \omega_{q_0} - \omega_{p_0 - 1} > | \to 0.$$

So $\{< x, \omega_k >\}$ is convergent. Thus, we have proved that $(\omega_k) \in E^{<\beta>}$. This is a contradiction. The case 1 is proved.

Case 2. $(E, ||.||)$ has the 0-gliding hump property:

Since $\sup\{| < x, y_i > | : x \in B(E), i \in \mathbf{N}\} = \infty$, so there are $r_1 \in \mathbf{N}$, $x_1 \in B(E)$ such that $| < x_1, y_{r_1} > | \geq 1 + \frac{1}{2}$. Note that $E^{<\beta>} = F$, so there exists $n_1 \in \mathbf{N}$ such that

$$| < x_1, y_{r_1}^{[n_1]} > | \geq 1.$$

Similarly, for $2 + \sup\{| < x, y_i^{[n_1]} > | : i \in \mathbf{N}, x \in B(E)\} + 1$, there are $r_2 > r_1, x_2 \in B(E)$ such that

$$| < x_2, y_{r_2} > | \geq 2 + \sup\{| < x, y_i^{[n_1]} > | : i \in \mathbf{N}, x \in B(E)\} + 1.$$

So, there is a $n_2 \in \mathbf{N}$, satisfying

$$| < x_2, y_{r_2}^{[n_1+1,n_2]} > | \geq 2.$$

Continuing this construction we can obtain $n_1 < n_2 < \cdots, r_1 < r_2 < \cdots$, and $\{x_k\} \subseteq B(E)$ such that

$$| < x_k, y_{r_k}^{[n_{k-1}+1,n_k]} > | \geq k, k \in \mathbf{N}, k \geq 2.$$

That is

$$\frac{1}{\sqrt{k}} | < x_k, \frac{y_{r_k}^{[n_{k-1}+1,n_k]}}{\sqrt{k}} > | \geq 1, k \in \mathbf{N}, k \geq 2.$$

Equivalently,

$$\frac{1}{\sqrt{k}} | < \frac{x_k^{[n_{k-1}+1,n_k]}}{\sqrt{k}}, y_{r_k} > | \geq 1, k \in \mathbf{N}, k \geq 2. \tag{5}$$

Consider the infinite matrix $(\frac{1}{\sqrt{k}} < \frac{x_i^{[n_{i-1}+1,n_i]}}{\sqrt{i}}, y_{r_k} >)$. It is clear that for every $i \in \mathbf{N}$,

$$\lim_{k\to\infty} \frac{1}{\sqrt{k}} < \frac{x_i^{[n_{i-1}+1,n_i]}}{\sqrt{i}}, y_{r_k} >= 0.$$

For every $k \in \mathbf{N}$, it follows from $\{\frac{x_i}{\sqrt{i}}\}$ converging to 0 and $(E, ||.||)$ having the 0-gliding hump property that every subsequence $\{[n_{i_p-1}+1, n_{i_p}]\}$ of $\{[n_{i-1}+1, n_i]\}$ has a subsequence $\{[n_{i_{p_m}-1}+1, n_{i_{p_m}}]\}$ of $\{[n_{i_p-1}+1, n_{i_p}]\}$ and $x_0 \in E$ such that in the norm topology,

$$\sum_{m=1}^{\infty} \frac{x_{i_{p_m}}^{[n_{i_{p_m}-1}+1,n_{i_{p_m}}]}}{\sqrt{i_{p_m}}} = x_0.$$

So we have

$$\lim_{k\to\infty} \frac{1}{\sqrt{k}} < \sum_{m=1} \frac{x_{i_{p_m}}^{[n_{i_{p_m}-1}+1,n_{i_{p_m}}]}}{\sqrt{i_{p_m}}}, y_{r_k} >= \lim_{k\to\infty} \frac{1}{\sqrt{k}} < x_0, y_{r_k} >= 0.$$

By the Antosik-Mikusinski basic matrix theorem ([11]) that

$$\lim_{k\to\infty} \frac{1}{\sqrt{k}} < \frac{x_k^{[n_{k-1}+1,n_k]}}{\sqrt{k}}, y_{r_k} >= 0.$$

This contradicts (5) and so case 2 is also true. The theorem is proved.

It follows from Lemma 3 and Theorem 1 that:

Corollary 1. *Let E be a normed space with dual F and let a system of sections be fixed on $< E, F >$ and E be an AK-space with respect to the norm topology. If the section mappings $\{\rho_n\}$ have the uniform boundedness property and the strong gliding hump property, then the following statements are equivalent:*

(1) E *is barrelled,*

(2) *Every* $E^{[n]}, n \in \mathbf{N}$, *is barrelled and* $E^{<\beta>} = F$.

Note that if $E^{<\beta>} = F$ and $\beta(F, E)$ is an AK-space, then E has the bounded uniform convergence property. In addition, it follows from Lemma 2 that if E' is an AK-space, then E is also an AK-space. Thus, we can obtain the main result of $[1]$, *that is*

Corollary 2. [1]. *Let E be a normed space with dual F and let a system of sections be fixed on $< E, F >$, $E' = F$ be an AK-space with respect to the dual norm topology. Then the following statements are equivalent:*

(1) E *is barrelled,*

(2) *Every* $E^{[n]}, n \in \mathbf{N}$, *is barrelled and* $E^{<\beta>} = F$.

Finally, we use Theorem 1 to prove the barrelledness of the dense subspace $(l^p, ||.||)$ $(0 < p < 1)$ *of* $(l^1, ||.||_1)$.

Following the terminology of references [1], [6] and [7], a Banach space E with a system of sections has the Wilansky property, provided a dense subspace D of E is barrelled if and only if the β-duals of D and E coincide. The practical use of this property is the following:

Suppose we want to show that two spaces E and F coincide, (where F is a dense subspace of E, say). If the Banach space E has the Wilansky property, and if F is itself a Banach space with a finer topology, then it suffices to show that the β-duals of E and F coincide. Because then F will be barrelled as a subspace of E, and the identity $I : F \to E$ will be continuous for these two topologies by the closed graph theorem, so the two topologies will coincide, and since F is dense, this will imply $E = F$.

In a concrete situation, the use of this might be that we want to show that two properties E and F are equivalent. While it may be hard to show this directly, it could be much easier to show that the β-duals of E and F coincide. So we hope that E has the Wilansky property.

But, note that the Banach space $(l^1, ||.||_1)$ does not have the Wilansky property in the sense above, so it is not possible to identify barrelledness of the subspace $(l^p, ||.||_1)$ $(0 < p < 1)$ by means of their β-dual space. In addition, note that the dual space $(l^\infty, ||.||_\infty)$ of $(l^p, ||.||_1)$ $(0 < p < 1)$ is not an AK-space, so the barrelledness of $(l^p, ||.||_1)$ $(0 < p < 1)$ can not also be obtained by the Corollary 2. On the other hand, it is very easily to prove that $(l^p, ||.||_1)$ $(0 < p < 1)$ has the

0-gliding hump property, so it follows from Theorem 1 that $(l^p, ||.||_1)$ $(0 < p < 1)$ is a barrelled subspace of $(l^1, ||.||_1)$. That is

Corollary 3. *Let* $0 < p < 1$ *and* $l^p = \{(t_j) : \sum_j |t_j|^p < \infty\}$. *Then* $(l^p, ||.||_1)$ *is a proper dense barrelled subspace of* $(l^1, ||.||_1)$.

Corollary 3 showed that our Theorem 1 extended substantially the main result in [1].

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