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NOTE ON THE IRRATIONALITY OF CERTAIN MULTIVARIATE q-FUNCTIONS

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To my friend Wolfram Jehne on the occasion of his 80th birthday

Abstract. Various irrationality results on certain infinite series and products representing q-functions were established in recent years by Borwein, Lubinsky, and Zhou [1], [7-10]. In all these papers, Padé approximants to appropriate functions were constructed to produce rational approximations that are too rapid to be consistent with rationality. The main purpose of this note is to show how an old and seemingly forgotten irrationality criterion of the present author [3], particularly suited for q-functions, can be used to deduce very easily much more general results.

1. Introduction and Main Result

Very recently, Zhou [9] published a paper whose main result reads as follows. Let $q, m \in \mathbb{N} := \{1, 2, ...\}$, $q > 1$, and define

$$F(x,y) := \sum_{i=0}^{\infty} q^{-mi} \prod_{j=0}^{i} (1 + q^{-mj}x + q^{-2mj}xy) \tag{1}$$

for each $(x, y) \in \mathbb{C}^2$. If $m \geq 2$ and $x, y \in \mathbb{Q}_+$ then at least one among the m numbers $F(q^{-\tau}x, q^{-\tau}y), \tau = 0, ..., m-1$, is irrational.

The same statement with $m = 1$ was claimed to be proven a few years earlier by Borwein and Zhou [1]. But, as Zhou [9] explained following the present author's review of [1] in MR 2001g:11114, "equation (2.19) in the proof of Theorem 2.2 in that paper is not correct. That critical error voids the proof...". Similar results, e.g., concerning the infinite product

$$\prod_{j=0}^{\infty} (1 + q^{-j}x + q^{-2j}xy), \tag{2}$$

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have been proved by Zhou and Lubinsky [10] and Zhou [7], but see also [8]. In all these papers, the general approach was to examine the Padé approximants to appropriate functions and to show that they provide rational approximations that are too rapid to be consistent with rationality.

It is the main purpose of the present note to show how much more general results than Zhou's above-mentioned one from [9] can easily be deduced from our old irrationality criterion [3], at least in the case $m > 2$. To cover also the case $m = 2$ we slightly extend our criterion proceeding along our former lines, i.e., using Newton interpolation series. As a matter of fact, our deduction here is much shorter than the proof in [9], since we have simply to check all conditions in the criterion (if necessary in its extended form).

To begin with our generalization of [9], let K denote either $\mathbb{Q}$ or an imaginary quadratic number field, and let O_K be the ring of integers in K. Furthermore, *in the whole paper, we suppose* $q \in O_K$ *with* $|q| > 1$.

Our first statement is the following.

Theorem 1. *Let* $\ell, m \in \mathbb{N}$ *satisfy*

$$m \geq \ell(\ell - 1) + 1, \tag{3}$$

and assume $r_1, ..., r_\ell \in K^\times := K \setminus \{0\}$ *such that*

$$1 + \sum_{\nu=1}^{\ell} q^{-k\nu} r_\nu \neq 0 \tag{4}$$

holds for each $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. *Then at least one among the* m *numbers*

$$\sum_{i=0}^{\infty} q^{-mi} \prod_{j=0}^{i} (1 + q^{-(mj+\tau)} r_1 + ... + q^{-(mj+\tau)\ell} r_\ell) \qquad (\tau = 0, ..., m-1)$$

is not in K.

Remark. We have not only Zhou's case $K = \mathbb{Q}, \ell = 2$ from [9] (by now only for $m \geq 3$, compare (3)). But even if $K = \mathbb{Q}$, some of the numbers $q, r_1, ..., r_\ell$ may be negative as long as condition (4) holds for each $k \in \mathbb{N}_0$.

The following one-dimensional case of Theorem 1 should be separately specified. Taking $m = 1$ we have

Corollary 1. *If* $r \in K^\times$ *satisfies* $r + q^j \neq 0$ *for each* $j \in \mathbb{N}_0$, *then the number*

$$\sum_{i=0}^{\infty} q^{-i} \prod_{j=0}^{i} (1 + q^{-j} r)$$

does not belong to K.

Remark. Here we clearly see that the non-vanishing of $r+q^j$ for each j is a necessary condition for the truth of the assertion (under the other conditions on q and r).

To get Zhou's full result in [9], i.e., for $m \geq 2 = \ell$ we prove

Theorem 1'. *Condition (3) in Theorem 1 can be replaced by*

(3') $$m \geq \ell \quad and \quad m(\ell+m)^2 > \ell^2(\ell(m-1)+m^2),$$

or by

(3") $$\ell \geq m \quad and \quad (\ell+m)^2 > \ell((\ell-1)m+\ell^2).$$

Remarks.

(1) Theorem 1' implies Theorem 1 since both inequalities in (3') are consequences of condition (3). For the first, this is evident. Assuming $m(\ell+m)^2 \leq \ell^2(\ell(m-1)+m^2)$ leads to $m(\ell+m) < \ell(m-1)+m^2$, by (3), and this last inequality is certainly false.

(2) It should be also noted that $\ell \geq 2$ and $m \geq \ell(\ell-1)$ together imply (3'). Hence Theorem 1' contains Zhou's main result.

(3) In Theorem 1.1 of [9], the author claims, as an application of the main result with $m=2$, that at least one of the series

$$\sum_{i=0}^{\infty} q^{-i} \prod_{j=0}^{i}(1+q^{-j}r+q^{-2j}s) \quad \text{and} \quad \sum_{i=0}^{\infty} q^{-2i} \prod_{j=0}^{i}(1+q^{-2j}r+q^{-4j}s)$$

is irrational for $q \in \mathbb{N} \setminus \{1\}$, and $r,s \in \mathbb{Q}_+$. Unfortunately, this assertion remains open since the identity on p. 451, on which the proof depends essentially, is incorrect.

(4) The case $\ell = 2, m = 1$, unsuccessfully studied in [1], is *not* covered by Theorem 1' since the second inequality in (3") fails in this case.

Whereas we will deduce Theorems 1 and 1' from the irrationality criterion in [3] and from its refined version in our subsequent Lemma 1', respectively, we take this opportunity to mention another application of Lemma 1' to get a result on infinite products generalizing (2). Since the arguments in this case are very similar to those for Theorem 1', we will leave the proof to the reader as an exercise after giving a short hint at the end of Section 4.

Theorem 2. *Let $\ell, m \in \mathbb{N}$ satisfy (3') or (3"). Suppose further $r_1, ..., r_\ell \in K^\times$ such that inequality (4) holds for each $k \in \mathbb{N}_0$. Then at least one of the infinite products*

$$\prod_{j=0}^{\infty}(1 + q^{-(mj+\tau)}r_1 + ... + q^{-(mj+\tau)\ell}r_\ell) \qquad (\tau = 0, ..., m-1)$$

is not contained in K.

Remark. For $\ell = 1$ we may take $m = 1$. Assuming $r \in K^\times$ with $r + q^j \neq 0$ for each $j \in \mathbb{N}_0$ we can conclude $\prod_{j=0}^{\infty}(1 + q^{-j}r) \notin K$. This is essentially Lototsky's [4] classical result from 1943, but compare also [2] for a quantitative version.

2. Two Lemmas

First we quote our irrationality criterion [2], Satz 1, as

Lemma 1. *Let*

$$f(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu \in K[[z]] \tag{5}$$

be an entire transcendental function satisfying the two following conditions:

- (i) *For every $n \in \mathbb{N}_0$, there exists $b_n \in O_K \setminus \{0\}$ such that $b_n c_\nu \in O_K$ for $\nu = 0, ..., n$, and $|b_n| \leq |q|^{\lambda n^2 + o(n^2)}$ with some fixed real $\lambda \geq 0$.*
- (ii) *There are $a \in K^\times$, and a sequence $(T_k)_{k=0,1,...}$ in $O_K \setminus \{0\}$ satisfying $|T_k| \leq |q|^{\mu k^2 + o(k^2)}$ with some fixed real $\mu \geq 0$, such that, for every $k \in \mathbb{N}_0$, $T_k f(aq^{-\kappa}) \in O_K$ holds for $\kappa = 0, ..., k$. Then the inequality*

$$\rho^*(f) := \limsup_{r \to \infty} \frac{\log |f|_r}{\log^2 r} \geq (\frac{1}{\lambda} + \frac{1}{\mu})\frac{1}{4 \log |q|} \tag{6}$$

holds, where $|f|_r := \max_{|z|=r} |f(z)|$.

Remark. If $\lambda\mu = 0$ the right-hand side of (6) has to be interpreted as $+\infty$.

The following lemma is needed to precisely calculate the left-hand side of (6) in a wide class of cases.

Lemma 2. *Suppose that the entire transcendental function f satisfies a functional equation*

$$f(Qz) = P_0(z)f(z) + P_1(z) \tag{7}$$

with fixed $Q \in \mathbb{C}, |Q| > 1$, and $P_0, P_1 \in \mathbb{C}[z]$. Then

$$\log|f|_r = \frac{\deg P_0}{2\log|Q|}\log^2 r + O(\log r)$$

and hence

$$\rho^*(f) = \frac{\deg P_0}{2\log|Q|}. \tag{8}$$

Remark. The above sharper assertion can be found in [5], Lemma 2, whereas (8) is already contained in [6]. Of course, (7) implies that P_0 is non-constant, i.e., $\deg P_0 \geq 1$ since f itself is not a polynomial.

3. Proof of Theorem 1

For fixed $\underline{x} := (x_1, ..., x_\ell) \in \mathbb{C}^\ell$, the infinite series

$$f(z) := \sum_{i=0}^{\infty} q^{-mi} \prod_{j=0}^{i} (1 + q^{-mj}x_1 z + ... + q^{-\ell mj} x_1 \cdot ... \cdot x_\ell z^\ell) \tag{9}$$

defines an entire function of z, where $f(1) = F(x_1, x_2)$ in terms of (1). It is easily checked that f satisfies the functional equation

$$f(q^m z) = (1 + q^m x_1 z + ... + q^{\ell m} x_1 \cdot ... \cdot x_\ell z^\ell)(q^{-m} f(z) + 1), \tag{10}$$

or, with $Q := q^m$,

$$f(Qz) = (1 + Qx_1 z + ... + Q^\ell x_1 \cdot ... \cdot x_\ell z^\ell)(Q^{-1} f(z) + 1). \tag{11}$$

Assuming $x_1 \cdot ... \cdot x_\ell \neq 0$ every entire solution f of this functional equation is transcendental and $\log|f|_r = (\ell/2\log|Q|)\log^2 r + O(\log r)$ follows from Lemma 2. Hence we note

Proposition 1. *For $x_1, ..., x_\ell \in \mathbb{C}^\times$ the entire transcendental function f in (9) satisfies*

$$\rho^*(f) = \frac{\ell}{2m\log|q|}.$$

If f has Taylor series (5) about the origin, then (11) leads to the following recurrence formula for the c's[1]

$$c_n(Q^{n+1}-1) = \sum_{i=1}^{\min(\ell,n-1)} Q^i x_1 \cdot \ldots \cdot x_i c_{n-i} + \gamma_n \frac{Q^{n+2}}{Q-1} x_1 \cdot \ldots \cdot x_n \quad (n \in \mathbb{N}) \tag{12}$$

with $\gamma_n := 1$ for $n = 1, \ldots, \ell$, and $\gamma_n := 0$ for $n > \ell$, and additionally $c_0 = Q/(Q-1)$. From (12) we easily deduce **Proposition 2.** *For each* $n \in \mathbb{N}_0$, *we have*

$$c_n \prod_{\nu=1}^{n+1} (Q^\nu - 1) \in \mathbb{Z}[Q, x_1, \ldots, x_\ell], \tag{13}$$

and this polynomial is homogeneous of degree n *in* $x_1, \ldots, x_\ell$.

Proof. Since $c_0(Q-1) = Q$, (13) holds for $n = 0$, where no x's occur on the right-hand side. Let $n > 0$ and assume the truth of the assertion for all subscripts less than n. Multiplying (12) by $\prod_{\nu=1}^{n}(Q^\nu - 1)$ gives the assertion for n.

From Proposition 2 we immediately conclude

Proposition 3. *If* $x_1, \ldots, x_\ell \in K^\times$ *have a common denominator* $\xi \in O_K \setminus \{0\}$, *then*

$$c_n \xi^n \prod_{\nu=1}^{n+1} (Q^\nu - 1) \in O_K.$$

Since $Q = q^m$, *we may take* $b_n := \xi^n \prod_{\nu=1}^{n+1}(q^{m\nu} - 1)$ *for* $n = 0, 1, \ldots$ *in condition* (i) *of Lemma 1, hence* $\lambda := m/2$.

Henceforth we suppose $x_1, \ldots, x_\ell \in K^\times$ as in Proposition 3, and moreover

$$\sum_{\nu=0}^{\ell} q^{-k\nu} x_1 \cdot \ldots \cdot x_\nu \neq 0 \tag{14}$$

for every $k \in \mathbb{N}_0$. *Assuming* $f(q^{-\tau}) \in K$ *for* $\tau = 0, \ldots, m-1$ we find from (10)

$$f(q^{-(k-m)}) = (q^{-m} f(q^{-k}) + 1) \sum_{\nu=0}^{\ell} q^{-(k-m)\nu} x_1 \cdot \ldots \cdot x_\nu \tag{15}$$

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[1] As usual, empty sums or products have to be interpreted as 0 or 1, respectively.

for each $k \geq m$. By (14), this leads inductively to $f(q^{-k}) \in K$ for *every* $k \in \mathbb{N}_0$.

Now we want to apply Lemma 1 with $a = 1$. For this we have to find a sequence (T_k) as described in that lemma. To perform this last step, we write $k = hm + \tau$ with $h := [k/m] \in \mathbb{N}_0, \tau \in \{0, ..., m-1\}$ to prepare the following consideration. If $h \geq 1$ and $t_{h-1,\tau} \in O_K \setminus \{0\}$ has the property $t_{h-1,\tau} f(q^{-(h-1)m-\tau}) \in O_K$, then, by (14) and (15),

$$t_{h,\tau} := t_{h-1,\tau} \xi^\ell \sum_{\nu=0}^{\ell} q^{(\ell-\nu)((h-1)m+\tau)} x_1 \cdot ... \cdot x_\nu \in O_K \setminus \{0\}$$

is a denominator of $f(q^{-hm-\tau})$, where ξ is as in Proposition 3. Hence, if $\Xi \in O_K \setminus \{0\}$ is a denominator for all $f(q^{-\tau})$ with $\tau = 0, ..., m-1$, then

$$\Xi \prod_{j=0}^{h-1} \left(\xi^\ell \sum_{\nu=0}^{\ell} q^{(\ell-\nu)(jm+\tau)} x_1 \cdot ... \cdot x_\nu \right)$$

is a denominator for $f(q^{-\kappa})$ with $\kappa \leq k$ and $\kappa \equiv \tau \pmod m$. Taking each residue class modulo m into account it is evident that

$$T_k := \Xi \prod_{\tau=0}^{m-1} \prod_{j=0}^{[k/m]-1} \left(\xi^\ell \sum_{\nu=0}^{\ell} q^{(\ell-\nu)(jm+\tau)} x_1 \cdot ... \cdot x_\nu \right)$$

is a sufficient choice. This leads to the estimate $|T_k| \leq |q|^{(\ell/2)k^2+O(k)}$, and we may apply Lemma 1 with $\mu := \ell/2$. Combined with Proposition 1 and (3), inequality (6) of that lemma says $\ell/m \geq 1/m + 1/\ell$, or equivalently $\ell(\ell-1) \geq m$ contradicting (3). Hence our above assumption (after (14)) cannot be true.

We finally remark that, given $r_1, ..., r_\ell$ according to Theorem 1, we define $x_1 := r_1$ and $x_\nu := r_\nu / r_{\nu-1}$ for $\nu = 2, ..., \ell$ (if $\ell > 1$). Hence $x_1 \cdot ... \cdot x_\nu = r_\nu$ for $\nu = 1, ..., \ell$ and conditions (4) and (14) are equivalent. Then we work with these $x_1, ..., x_\ell$ from the beginning of the present section.

4. Proof of Theorem 1'

To prove this strenghtened version of Theorem 1 we use just the fact that more precise arithmetical informations on the power series coefficients of f as contained, e.g., in Proposition 3, lead to a sharper version of Lemma 1, namely

Lemma 1'. *Let the hypotheses of Lemma 1 be satisfied but with condition* (i) *replaced by the following one.*

(i') *There are* $\xi \in O_K \setminus \{0\}, m \in \mathbb{N}, t \in \mathbb{N}_0$ *such that*

$$c_n \xi^n \prod_{\nu=1}^{n+t} (q^{m\nu} - 1) \in O_K$$

holds for every $n \in \mathbb{N}_0$.

Then the inequality

$$\rho^*(f) \geq \frac{(m+2\mu)^2}{(2\mu m + m_2^2 - m_1)m_1} \cdot \frac{1}{2\log|q|} \tag{6'}$$

is valid, where $m_1 := \min(2\mu, m)$ *and* $m_2 := \max(2\mu, m)$.

Accepting, for a moment, the truth of Lemma 1', our assumption $f(q^{-\tau}) \in K$ for $\tau = 0, ..., m-1$ leads again to $\mu = \ell/2$ as at the end of the last section. Hence we have $m_1 = \min(\ell, m)$ and $m_2 = \max(\ell, m)$, and Proposition 1 and (6') yield $\ell^2(\ell(m-1) + m^2) \geq m(\ell + m)^2$ if $m \geq \ell$, and $\ell((\ell-1)m + \ell^2) \geq (\ell + m)^2$ if $\ell \geq m$. This proves Theorem 1'.

To demonstrate finally (6') we resume our proof of Lemma 1 as given in [3], pp.177-179. The decisive fact is that the former denominator $B_{k+j(k)-1} \in O_K \setminus \{0\}$ of $A^*_{k+j(k)-1}$ defined in [3], (9) can now be replaced by the substantially smaller denominator

$$B^*_{k+j(k)-1} := s^{k+j(k)-1} T_{k-1} \xi^{j(k)-1} \cdot \prod_{\nu=1}^{j(k)+t-1} (q^{m\nu} - 1) \cdot \prod_{\nu=j(k)+t}^{k-1} (q^\nu - 1)$$

the last product being 1 if it is empty. Here we have $j(k) := [\frac{\mu+\varepsilon}{\lambda+\varepsilon} k]$ with arbitrary real $\varepsilon > 0$ as in [3],(2), λ being $m/2$ in our present situation. Hence estimate (10) in [3] can now be replaced by

$$\begin{aligned} &\left|B^*_{k+j(k)-1}\right| \\ &\leq \exp\left\{\left(\mu + \varepsilon + \frac{m}{2}\Big(\frac{\mu+\varepsilon}{\lambda+\varepsilon}\Big)^2 + \frac{\delta}{2}\Big(1 - \Big(\frac{\mu+\varepsilon}{\lambda+\varepsilon}\Big)^2\Big)\right) k^2 \log|q| + O(k)\right\} \end{aligned} \tag{16}$$

with $\delta := 0$ for $\mu \geq \lambda$ (i.e., $j(k) \geq k$) and $\delta := 1$ for $\mu < \lambda$. From this point on, the reasoning is as in [3]. Combination of formulae (6) and (8) in [3] leads to our old upper estimate for $|A^*_{k+j(k)-1}|$. This and our above inequality (16) lead to

$$\frac{2(\lambda+\mu)^2}{2\lambda^2\mu + m\mu^2 + \delta(\lambda^2 - \mu^2) - \lambda^2} \leq 4\rho^*(f)\log|q|$$

if we let ε tend to zero. Taking $\lambda = m/2$ into account this last inequality is equivalent with (6').

Remark. To prove Theorem 2 the reader may use the infinite product

$$\prod_{j=0}^{\infty}(1+q^{-mj}x_1z+\ldots+q^{-\ell mj}x_1\cdot\ldots\cdot x_\ell z^\ell)$$

as new function $f(z)$.

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