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# EXISTENCE THEOREM OF CONE SADDLE-POINTS APPLYING A NONLINEAR SCALARIZATION

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**Abstract.** This paper is concerned with a nonlinear scalarization for vector-valued functions. We consider applying the scalarization to existence of cone saddle-points. Some properties of the scalarization about cone-continuity and cone-convexity are described and then as an application, an existence theorem for vector saddle-points is treated.

## 1. Introduction

This paper is concerned with applying a scalarization of vector-valued functions by nonconvex separation to existence theorems of cone saddle-points. The scalarization has been studied in [1,3]. In this paper, we compile some useful properties for the existence theorems. An application of those properties has been considered in Theorem 4 and [2].

The organization of this paper is given as follows. In Section 2 we consider some properties of an ordering cone in a normed space, and then we state some definitions concerned with continuity and convexity for vector-valued functions. In Section 3 we consider a scalarization with the ordering cone for vector-valued functions and study some properties of the scalarizing function. In Section 4 we show an existence theorem for a vector-valued saddle-point problem as an application by means of its scalarization.

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## 2. Preliminary and Terminology

In the beginning, we give some notations used throughout this paper. We denote the topological interior, closure and boundary of a set $S$ by $\operatorname{int} S$, $\operatorname{cl} S$ and $\operatorname{bd} S$, respectively, and the complementary set of $S$ by $S^{\mathrm{c}}$. In addition, we denotes the composite of two functions $f$ and $g$ by $g \circ f$.

Before dealing with the nonlinear scalarization for vector-valued functions, we consider some properties of ordering cones on normed spaces; a convex cone induces a partial ordering, and we call such one an ordering cone. Throughout of the paper, let $Z$ be a normed space over the real scalar field $R$, and let $C$ be a solid pointed convex cone in $Z$. Solidness means that the topological interior is nonempty, and then we have

$$\operatorname{int} C + (-\operatorname{int} C) = Z.$$

Moreover, for any $k \in \operatorname{int} C$ and $z \in Z$ there exists $t \in R$ such that

$$z \in (t \cdot k - C). \tag{1}$$

We see this property by the following facts:

(i) Every neighborhood $U$ of the origin of $Z$ is an absorbing set, i.e., for any $x \in Z$ there exists $t > 0$ such that $t \cdot x \in U$.

(ii) $C$ is a cone.

The property remains even if $C$ in (1) replaced by $\operatorname{int} C$ or $\operatorname{cl} C$. Pointedness means that

$$C \cap (-C) = \{0_Z\},$$

where $0_Z$ stands for the origin point of $Z$. Indeed, by the pointedness and solidness, if $C$ in (1) is replaced by $\operatorname{bd} C$ or $C^{\mathrm{c}}$, the property (1) is held; and especially the fact that for any $k \in \operatorname{int} C$ and $z \in Z$ there exists $t \in R$ such that

$$z \in (t \cdot k - \operatorname{bd} C), \tag{2}$$

has a meaning in Lemma 1.

Here, we note connections of Propositions, Lemmas, and Theorems in the paper briefly. Theorem 4 is led by Theorems 1, 2 and 3, and Corollary 1. Theorems 1 and 3 are led by Proposition 2 and Lemma 1. Theorem 2 and Corollary 1 are led by Lemma 1. Lemma 1 is led by Proposition 3 and Lemma 3. Lemma 3 is led by Propositions 3 and 4. Propositions 2 and 3 are led by Proposition 1.

**Proposition 1.** ([9]). *Let $Z$ be a normed space. If $C$ is a solid convex cone in $Z$, then*

$$\operatorname{cl} C + \operatorname{int} C = \operatorname{int} C.$$

**Proposition 2.** *Let $Z$ be a normed space and $C$ a solid convex cone in $Z$. If $z \notin (-\operatorname{int} C)$ then*

$$(z + \operatorname{cl} C) \cap (-\operatorname{int} C) = \emptyset.$$

*Moreover, If $z \notin (-\operatorname{cl} C)$ then*

$$(z + \operatorname{cl} C) \cap (-\operatorname{cl} C) = \emptyset.$$

*Proof.* This is clear from Proposition 1. ∎

**Proposition 3.** *Let $Z$ be a normed space, $C$ a solid convex cone in $Z$ with $C \neq Z$, and $k \in \operatorname{int} C$. Then, for $a, b \in R$ the following three conditions are equivalent each other:*

$(i)$ $a < b$,

$(ii)$ $(a\cdot k - \operatorname{cl} C) \subset (b\cdot k - \operatorname{int} C)$,

$(iii)$ $(a\cdot k - \operatorname{bd} C) \cap (b\cdot k - \operatorname{int} C) \neq \emptyset$.

*Proof.* This is clear from Proposition 1. ∎

**Remark 1.** Proposition 3 implies that if $a \neq b$ then $(a\cdot k - \operatorname{bd} C) \cap (b\cdot k - \operatorname{bd} C) = \emptyset$.

**Proposition 4.** *Let $Z$ be a normed space and $C$ a solid pointed convex cone in $Z$. Assume that $k \in \operatorname{int} C$ and that $t \in R$. Then*

(i) $z \notin t\cdot k - \operatorname{cl} C$ *if and only if there exists $\varepsilon > 0$ such that*

$$z \notin (t + \varepsilon)\cdot k - \operatorname{cl} C, \text{ and}$$

(ii) $z \in t\cdot k - \operatorname{int} C$ *if and only if there exists $\varepsilon > 0$ such that*

$$z \in (t - \varepsilon)\cdot k - \operatorname{int} C.$$

*Proof.* The proof is clear from the properties of closed set and open set, respectively. ∎

Next, we give some definitions about continuities and convexities concerned with respect to ordering cone $C$.

**Definition 1.** ([3, 8].) Let $X$ be a topological space, $Z$ a normed space with a partial ordering defined by a solid pointed convex cone $C$. A vector-valued function $f : X \to Z$ is said to be $C$-continuous at $x \in X$ if it satisfies one of the following three equivalent conditions:

(i) $f^{-1}(z+\operatorname{int} C)$ is open.

(ii) For any neighborhood $V \subset Z$ of $f(x)$, there exists a neighborhood $U \subset X$ of $x$ such that $f(u) \in V + C$ for all $u \in U$.

(iii) For any $k \in \operatorname{int} C$, there exists a neighborhood $U \subset X$ of $x$ such that $f(u) \in f(x) - k + \operatorname{int} C$ for all $u \in U$.

**Remark 2.** Whenever $Z = R$ and $C = R_+$, $C$-continuity and $(-C)$-continuity are the same as ordinary lower and upper semicontinuity, respectively. In [8, Definition 2.1 (pp.314-315)] corresponding to ordinary functionals, the above $C$-continuous is called $C$-lower semicontinuous, and $(-C)$-continuous is called $C$-upper semicontinuous.

**Definition 2.** ([7].) Let $K$ be a convex set in a real vector space $X$, $Z$ a normed space with a partial ordering defined by a solid pointed convex cone $C$. A vector-valued function $f : X \to Z$ is said to be $C$-convex on $K$ if

$$\lambda f(x_1) + (1-\lambda) f(x_2) \in f(\lambda x_1 + (1-\lambda)x_2) + C$$

for every $x_1, x_2 \in K$ and $\lambda \in [0,1]$.

**Definition 3.** ([7].) Let $K$ be a convex set in a real vector space $X$, $Z$ a normed space with a partial ordering defined by a solid pointed convex cone $C$. A vector-valued function $f : X \to Z$ is said to be $C$-properly quasiconvex on $K$ if either

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_1) - C,$$

or

$$f(\lambda x_1 + (1-\lambda)x_2) \in f(x_2) - C,$$

for every $x_1, x_2 \in K$ and $\lambda \in [0,1]$.

**Definition 4.** ([7].) Let $K$ be a convex set in a real vector space $X$, $Z$ a normed space with a partial ordering defined by a solid pointed convex cone $C$. A vector-valued function $f : X \to Z$ is said to be $C$-naturally quasiconvex on $K$ if

$$f(\lambda x_1 + (1-\lambda)x_2) \in \operatorname{co}\{f(x_1), f(x_2)\} - C,$$

for every $x_1, x_2 \in K$ and $\lambda \in [0,1]$, where $\operatorname{co} S$ stands for the convex hull of the set $S$.

**Definition 5.** ([7].) Let $K$ be a convex set in a real vector space $X$, $Z$ a normed space with a partial ordering defined by a solid pointed convex cone $C$. A function $f : X \to Z$ is said to be $C$-quasiconvex on $K$ if it satisfies one of the following two equivalent conditions:

(i) for each $x_1, x_2 \in K$ and $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2) \in z - C, \text{ for all } z \in C(f(x_1), f(x_2)),$$

where $C(f(x_1), f(x_2))$ is the set of upper bounds of $f(x_1)$ and $f(x_2)$, i.e.,

(3) $$C(f(x_1), f(x_2)) := \{z \in Z : z \in f(x_1) + C \text{ and } z \in f(x_2) + C\};$$

(ii) for each $z \in Z$,

$$A(z) := \{x \in K : f(x) \in z - C\}$$

is convex or empty.

In Definitions 2-5, if $f$ is $(-C)$-convex, $(-C)$-properly quasiconvex, $(-C)$-naturally quasiconvex, $(-C)$-quasiconvex then we call $f$ $C$-concave, $C$-properly quasiconcave, $C$-naturally quasiconcave, $C$-quasiconcave, respectively.

## 3. Nonlinear Scalarization by Nonconvex Seperation for Vector-valued Maps

Let $Z$ be a normed space, $C$ a solid pointed convex cone in $Z$ and $k$ an interior point of $C$. We consider the scalarizing function $h$ from $Z$ to $R$ as follows:

(4) $$h(z;k) := \inf\{t \in R : z \in t{\cdot}k - C\}.$$

By the argument on (1) in Section 2, we see that for any $z \in Z$ and $k \in \operatorname{int} C$ there exists uniquely a corresponding real number to $h(z;k)$, and we know that $h(z;k)$ is subadditive and positive homogeneous. For convenience, it may be written as $h_k$ instead of $h(\cdot;k)$.

Next, we give some useful properties of the above scalarizing function.

**Lemma 1.** ([1, Theorem 2.1]) *Let $Z$ be a normed space, $C$ a solid pointed convex cone in $Z$, $k \in \operatorname{int} C$, and $h(\cdot;k)$ the scalarizing function defined by $(4)$. Then for any $z \in Z$ and $t \in R$ we have:*

$(i)$ $z \in t{\cdot}k - \operatorname{int} C$ *if and only if* $h(z;k) < t$,

$(ii)$ $z \in t{\cdot}k - \operatorname{bd} C$ *if and only if* $h(z;k) = t$, *and*

$(iii)$ $z \notin t{\cdot}k - \operatorname{cl} C$ *if and only if* $h(z;k) > t$.

**Corollary 1.** *Let $Z$ be a normed space with the partial ordering by solid pointed convex cone $C$, $k \in \operatorname{int} C$, and $h(\cdot;k)$ the scalarizing function defined by $(4)$. Then, we have:*

$(i)$ *if* $z_1 \in z_2 - C$ *then* $h(z_1;k) \le h(z_2;k)$,

$(ii)$ *if* $z_1 \in z_2 - \operatorname{int} C$ *then* $h(z_1; k) < h(z_2; k)$*, and*

$(iii)$ *if* $h(z_1; k) \leq h(z_2; k)$ *then* $z_2 \notin z_1 - \operatorname{int} C$.

*Proof.* From Lemma 1, the proof follows immediately. ■

**Theorem 1.** *Let* $X$ *be a topological space,* $Z$ *a normed space with the partial ordering by a solid pointed convex cone* $C$ *in* $Z$ *and* $k \in \operatorname{int} C$*. Let* $h_k$ *be the scalarizing function on* $Z$ *defined by* $(4)$ *and* $f$ *a vector-valued function from* $X$ *to* $Z$*.*

$(i)$ *If* $f$ *is* $C$*-continuous at* $x \in X$*, then* $(h_k \circ f)$ *is lower semicontinuous at* $x \in X$*.*

$(ii)$ *If* $f$ *is* $(-C)$*-continuous at* $x \in X$*, then* $(h_k \circ f)$ *is upper semicontinuous at* $x \in X$*.*

**Remark 3.** If cone $C$ is closed, then Theorem 3.1 and Corollary 3.1 in [5] for single-valued cases are reduced to (i) and (ii) of Theorem 1, respectively.

**Corollary 2.** *Let* $X$ *be a topological space,* $Z$ *a normed space with the partial ordering by a solid pointed convex cone* $C$ *in* $Z$ *and* $k \in \operatorname{int} C$*. Let* $h_k$ *be the scalarizing function on* $Z$ *defined by* $(4)$ *and* $f$ *a vector-valued function from* $X$ *to* $Z$*. If* $f$ *is* $C$*-continuous and* $(-C)$*-continuous at* $x \in X$*, then* $(h_k \circ f)$ *is continuous at* $x \in X$*.*

**Remark 4.** In the case that $C$ has a bounded closed convex base, the $C$-continuity and $(-C)$-continuity of the vector-valued function $f$ guarantee the continuity of $f$, nevertheless $h_k \circ f$ is continuous; see [3, Theorem 5.3 and Remark 5.4 (pp. 22-23)].

**Theorem 2.** (see [3, Proposition 6.3 (p. 30)]). *Let* $K$ *be a convex set in a real vector space* $X$*,* $Z$ *a normed space, a solid pointed convex cone* $C$ *in* $Z$ *and* $k \in \operatorname{int} C$*. Let* $h_k$ *be the scalarizing function on* $Z$ *defined by* (4) *and* $f$ *a vector-valued function from* $X$ *to* $Z$*. Then,* $f$ *is* $C$*-quasiconvex on* $K$ *if and only if* $(h_k \circ f)$ *is quasiconvex on* $K$*.*

**Corollary 3.** *Let* $K$ *be a convex set in a real vector space* $X$*,* $Z$ *a normed space and a solid pointed convex cone* $C$ *in* $Z$ *and* $k \in \operatorname{int} C$*. Let* $h_k$ *be the scalarizing function on* $Z$ *defined by* $(4)$ *and* $f$ *a vector-valued function from* $X$ *to* $Z$*. If* $f$ *is* $C$*-naturally quasiconvex on* $K$*, then* $(h_k \circ f)$ *is quasiconvex on* $K$*. Moreover, if* $f$ *is* $C$*-properly quasiconvex on* $K$*, then* $(h_k \circ f)$ *is quasiconvex on* $K$*.*

*Proof.* If $f$ is $C$-properly quasiconvex on $K$ then $f$ is also $C$-naturally quasiconvex on $K$, and if $f$ is $C$-naturally quasiconvex on $K$ then $f$ is also $C$-quasiconvex on $K$; see [7, Theorem 2.1]. ■

**Theorem 3.** *Let $K$ be a convex set in a real vector space $X$, $Z$ a normed space, a solid pointed convex cone $C$ in $Z$ and $k \in \operatorname{int} C$. Let $h_k$ be the scalarizing function on $Z$ defined by $(4)$ and $f$ a vector-valued function from $X$ to $Z$. If $f$ is $(-C)$-properly quasiconvex on $K$, then $(h_k \circ f)$ is quasiconcave on $K$.*

*Proof.* Let $\operatorname{Lev}_{\geq}((h_k \circ f); \alpha)$ be the upper level set of $(h_k \circ f)$ at a scalar $\alpha$, i.e.,

$$\operatorname{Lev}_{\geq}((h_k \circ f); \alpha) := \{x \in K : (h_k \circ f)(x) \geq \alpha\}.$$

Let $\lambda \in [0, 1]$ and $x_1, x_2 \in \operatorname{Lev}_{\geq}((h_k \circ f); \alpha)$, by Lemma 1,

$$f(x_1), f(x_2) \notin (\alpha \cdot k - \operatorname{int} C).$$

By Proposition 2, we have

$$(f(x_i) + C) \cap (\alpha \cdot k - \operatorname{int} C) = \emptyset \text{ for } i = 1, 2.$$

Thus, by the $(-C)$-properly quasiconvexity of $f$, we have

$$f(\lambda x_1 + (1 - \lambda)x_2) \notin (\alpha \cdot k - \operatorname{int} C),$$

which implies that $\lambda x_1 + (1 - \lambda)x_2 \in \operatorname{Lev}_{\geq}((h_k \circ f); \alpha)$, by Lemma 1. ■

## 4. Applications

In this section, we consider the vector-valued saddle-point problem, and we show an existence theorem of weak $C$-saddle-points as an application of the scalarization.

Let $X$ and $Y$ be nonempty subsets in two normed spaces, respectively, and $Z$ a normed space with a partial ordering by induced a solid pointed convex cone in $Z$. Suppose that $F$ is a vector-valued function from $X \times Y$ to $Z$, then the vector-valued saddle-point problem is to find a pair $x \in X$ and $y \in Y$ such that

$$(P) \quad \begin{cases} F(x, y) - F(u, y) \notin \operatorname{int} C & \text{for all } u \in X, \\ F(x, v) - F(x, y) \notin \operatorname{int} C & \text{for all } v \in Y. \end{cases}$$

A point $(x, y) \in X \times Y$ is said to be a weak $C$-saddle-point of function $F$ on $X \times Y$, if it is a solution of the problem.

**Theorem 4.** *Let $X$ and $Y$ be nonempty compact convex sets in two normed spaces, respectively, and $Z$ a normed space with a partial ordering induced by a solid pointed convex cone $C$ in $Z$. If a vector-valued function $F : X \times Y \to Z$ satisfies that*

$(i)$ $x \mapsto F(x,y)$ *is* $C$*-continuous and* $C$*-quasiconvex on* $X$ *for every* $y \in Y$*,*

$(ii)$ $y \mapsto F(x,y)$ *is* $(-C)$*-continuous and* $(-C)$*-properly quasiconvex on* $Y$ *for every* $x \in X$*,*

*then* $F$ *has at least one weak* $C$*-saddle point.*

*Proof.* Since $C$ is solid we can take $k \in \operatorname{int} C$, and so we can define the scalarizing function $h_k$ in (4). We see that, by Theorems 1 and 2, the map $x \mapsto (h_k \circ F)(x,y)$ is lower semicontinuous and quasiconvex on $X$, and we see that, by Theorems 1 and 3, the map $y \mapsto (h_k \circ F)(x,y)$ is upper semicontinuous and quasiconcave on $Y$. By Sion's minimax theorem [6], $(h_k \circ F)$ has an ordinary saddle point and by Corollary 1, $F$ has at least one weak $C$-saddle-point. ■

**Theorem 5.** *Let* $X$ *be a compact convex set of a normed space, and* $Z$ *a normed space with a partial ordering defined by a solid pointed convex cone* $C$*. If* $f : X \to Z$ *is* $C$*-quasiconvex on* $X$*, then* $\operatorname{argmin} h \circ f(x)$ *is a convex set in* $X$*, and*

$$(\text{argmin } h \circ f(x)) \subset \{x \in X : f(u) - f(x) \notin -\operatorname{int} C \ \text{ for all } u \in X\},$$

*where* $\operatorname{argmin} h \circ f(x) := \{x \in X : h \circ f(x) = \min_{u \in X} h \circ f(u)\}$.

**Remark 5.** Theorem 5 is a useful result. We can consider something like a convex envelope for vector-valued functions by using Theorem 5 with Theorems 1, 2, and 3. Its detail and application have been studied in [2].

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