AUXILIARY PROBLEM METHOD FOR MIXED VARIATIONAL-LIKE INEQUALITIES

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Abstract. This paper deals with the convergence of the algorithm built on the auxiliary problem principle for solving a class of pseudomonotone type mixed variational-like inequalities. The convergence criteria of this method are presented under a joint pseudo-Dunn property or a joint strong pseudomonotonicity assumption on the operators involved in the mixed variational-like inequality problem. Moreover a posteriori error estimation for approximate solutions is also given.

1. INTRODUCTION

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of H. Let $T : K \to H, \eta : K \times K \to H$ be two mappings and let $f : K \to R$ be a real-valued function. We consider the mixed variational-like inequality problem which is to find $x^* \in K$ such that

$$(1) \qquad \langle T(x^*), \eta(y, x^*) \rangle + f(y) - f(x^*) \geq 0, \quad \forall y \in K.$$

Problem (1) was considered and studied by Ansari and Yao (Ref. 1) in 2001. They applied the auxiliary problem method to find the approximate solutions of problem (1) and proved the convergence of the approximate solutions to the exact solution of

problem (1) under the η-cocoercive assumption on T. On the other hand, a random version of problem (1) was also considered by Ding (Ref. 2) in 1997. When $f(x) = 0 \ \forall x \in K$, problem (1) is equivalent to finding $x^* \in K$ such that

$$(2) \qquad \langle T(x^*), \eta(y, x^*) \rangle \geq 0, \quad \forall y \in K.$$

Problem (2) was considered and studied previously by many authors; see, e.g., Refs. 3-8. Furthermore when $\eta(x, y) = x - y \ \forall x, y \in K$ and T has the pseudomonotonicity, problem (2) reduces to the following pseudomonotone variational inequality problem considered by Yao (Refs. 9-10) and Farouq (Ref. 11) which is to find $x^* \in K$ such that

$$\langle T(x^*), y - x^* \rangle \geq 0, \quad \forall y \in K.$$

In Yao (Refs. 9-10) some results on the existence of solutions to pseudomonotone variational inequalities can be found. Utilizing the technique developed by Cohen (Ref. 12), Farouq (Ref. 11) studied the convergence of the method based on the auxiliary problem principle under a pseudo-Dunn property or a strong pseudomonotonicity assumption on the operator T. Moreover she also gave a posteriori error estimation for the approximate solutions.

Motivated and inspired by Ansari and Yao (Ref. 1) and Farouq (Ref. 11), we investigate the convergence of the algorithm built on the auxiliary problem principle for solving the mixed variational-like inequality problem (1). We establish the convergence results on the approximate solutions to problem (1) involving general operators satisfying the joint pseudo-Dunn property and the jointly strong pseudomonotonicity property, respectively. In addition, we also give a posteriori error estimation for the approximate solutions to problem (1).

2. Preliminaries

In this section, we give various definitions and basic results which will be used in the sequel.

Definition 2.1. Let K be a nonempty subset of H. Let $T : K \to H$, $\eta : K \times K \to H$ be two mappings and let $f : K \to R$ be a real-valued function.

(1) T is η-monotone if

$$\langle T(x) - T(y), \eta(x, y) \rangle \geq 0, \quad \forall x, y \in K;$$

(2) T is η-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), \eta(x, y) \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in K;$$

(3) T has the η-Dunn property if there exists a constant $A > 0$ such that

$$\langle T(x) - T(y), \eta(x, y) \rangle \geq (1/A)\|T(x) - T(y)\|^2, \quad \forall x, y \in K;$$

(4) T is jointly pseudomonotone with respect to η and f if for each $x, y \in K$,

$$\langle T(x), \eta(y, x) \rangle + f(y) - f(x) \geq 0$$
$$\Rightarrow \langle T(y), \eta(y, x) \rangle + f(y) - f(x) \geq 0;$$

(5) T is jointly strongly pseudomonotone with respect to η and f if there exists a constant $e > 0$ such that for each $x, y \in K$,

$$\langle T(x), \eta(y, x) \rangle + f(y) - f(x) \geq 0$$
$$\Rightarrow \langle T(y), \eta(y, x) \rangle + f(y) - f(x) \geq e\|y - x\|^2;$$

(6) T has the joint pseudo-Dunn property with respect to η and f if there exists a constant $E > 0$ such that for each $x, y \in K$,

$$\langle T(x), \eta(y, x) \rangle + f(y) - f(x) \geq 0$$
$$\Rightarrow \langle T(y), \eta(y, x) \rangle + f(y) - f(x) \geq (1/E)\|T(y) - T(x)\|^2;$$

(7) T is jointly quasimonotone with respect to η and f if for each $x, y \in K$,

$$\langle T(x), \eta(y, x) \rangle + f(y) - f(x) > 0$$
$$\Rightarrow \langle T(y), \eta(y, x) \rangle + f(y) - f(x) \geq 0.$$

Remark 2.1. When $\eta(x, y) = x - y$ and $f(x) = 0$ for all $x, y \in K$, Definition 2.1 reduces to Definition 2.1 in Farouq (Ref. 11).

Lemma 2.1. *See Ref. 11. Let $\eta : K \times K \to H$ satisfy the condition:* $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in K.$

(i) *If $T : K \to H$ is jointly pseudomonotone with respect to η and f, then for any solutions x_1^* and x_2^* of problem (1),*

(3) $$\langle T(x_2^*), \eta(x_2^*, x_1^*) \rangle + f(x_2^*) - f(x_1^*) = 0.$$

(ii) *If T has the joint pseudo-Dunn property with respect to η and f, then the set*

$$S = \{T(x^*) : \langle T(x^*), \eta(x, x^*) \rangle + f(x) - f(x^*) \geq 0 \ \forall x \in K\}$$

is a singleton.

(*iii*) *If T is jointly strongly pseudomonotone with respect to η and f and problem (1) has a solution, then it is unique.*

Proof. Let x_1^* and x_2^* in K be two solutions of problem (1). Then

$$(4) \qquad \langle T(x_1^*), \eta(x_2^*, x_1^*) \rangle + f(x_2^*) - f(x_1^*) \geq 0,$$

$$(5) \qquad \langle T(x_2^*), \eta(x_1^*, x_2^*) \rangle + f(x_1^*) - f(x_2^*) \geq 0.$$

(i) If T is jointly pseudomonotone with respect to η and f, then (4) implies

$$\langle T(x_2^*), \eta(x_2^*, x_1^*) \rangle + f(x_2^*) - f(x_1^*) \geq 0.$$

Note that $\eta(x_1^*, x_2^*) = -\eta(x_2^*, x_1^*)$. Thus, from (5), we get (3).

(ii) If T has the joint pseudo-Dunn property with respect to η and f, then

$$0 = \langle T(x_2^*), \eta(x_2^*, x_1^*) \rangle + f(x_2^*) - f(x_1^*) \geq (1/E) \| T(x_2^*) - T(x_1^*) \|^2.$$

Therefore, $T(x_2^*) = T(x_1^*)$.

(iii) By using the same reasoning when T is jointly strongly pseudomonotone with respect to η and f, we get $x_2^* = x_1^*$. Therefore the solution of problem (1) is unique. ∎

Whenever $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in K$, we illustrate the following relationships between the η-monotonicity assumption and jointly generalized monotonicity assumptions:

$$\eta - \text{strong monotonicity} \quad \Rightarrow \quad \eta - \text{monotonicity} \quad \Leftarrow \quad \eta - \text{Dunn property}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow \qquad\qquad\qquad \Downarrow$$

$$\text{joint strong pseudomonotonicity} \Rightarrow \text{joint pseudomonotonicity} \Leftarrow \text{joint pseudo–Dunn property}$$

$$\Downarrow$$

$$\text{joint quasimonotonicity}$$
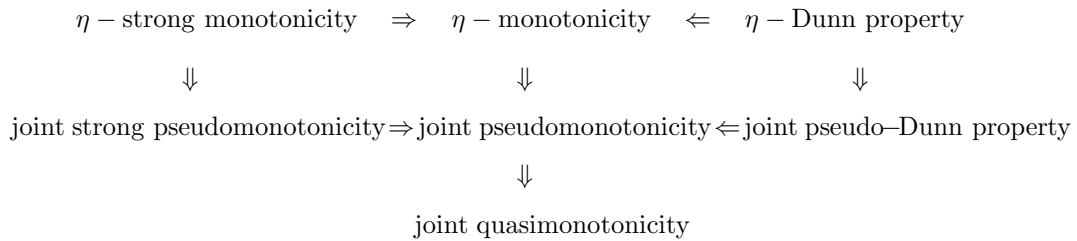
Lemma 2.2. *See Ref. 11. If $T : K \rightarrow H$ is jointly strongly pseudomonotone with constant e and Lipschitz continuous with constant L, then it has the joint pseudo-Dunn property with constant L^2/e.*

Proof. Suppose that

$$\langle T(x), \eta(y, x) \rangle + f(y) - f(x) \geq 0, \quad \forall x, y \in K.$$

Then it follows from the joint strong pseudomonotonicity and Lipschitz continuity of T that

$$(e/L^2)\|T(y) - T(x)\|^2 \leq (e/L^2)L^2\|y - x\|^2 = e\|y - x\|^2$$
$$\leq \langle T(y), \eta(y, x)\rangle + f(y) - f(x). \qquad \blacksquare$$

Definition 2.2. See Ref. 1. A mapping $\eta : K \times K \to H$ is called Lipschitz continuous if there exists a constant $\lambda > 0$ such that $\|\eta(x, y)\| \leq \lambda\|x - y\| \ \forall x, y \in K$.

Definition 2.3. See Ref. 1. A differentiable function $h : K \to R$ on a convex subset K is called

(i) η-convex if

$$h(y) - h(x) \geq \langle h'(x), \eta(y, x)\rangle, \quad \forall x, y \in K,$$

where $h'(x)$ is the Frechet derivative of h at x;

(ii) η-strongly convex if there exists a constant $\mu > 0$ such that

$$h(y) - h(x) - \langle h'(x), \eta(y, x)\rangle \geq (\mu/2)\|x - y\|^2, \quad \forall x, y \in K.$$

Proposition 2.1. *See Ref. 1. Let h be a differentiable η-strongly convex functional with constant μ on a convex subset K of H and let $\eta : K \times K \to H$ be a mapping such that $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in K$. Then h' is η-strongly monotone with constant μ, i.e.,*

$$\langle h'(x) - h'(y), \eta(x, y)\rangle \geq \mu\|x - y\|^2 \ \forall x, y \in K.$$

Proof. Since h is a differentiable η-strongly convex functional with constant μ, we have

$$h(y) - h(x) - \langle h'(x), \eta(y, x)\rangle \geq (\mu/2)\|x - y\|^2,$$
$$h(x) - h(y) - \langle h'(y), \eta(x, y)\rangle \geq (\mu/2)\|y - x\|^2.$$

Note that $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in K$. Thus adding these two inequalities, we derive the conclusion. $\blacksquare$

A real-valued function $F : K \to R$ is called sequentially continuous at x_0 if $F(x_k) \to F(x_0)$ for all sequences $x_k \to x_0$. F is called sequentially continuous on K if it is sequentially continuous at each of its points.

Lemma 2.3. *See Ref. 1. Let $\eta(\cdot, y) : K \to H$ and h' be sequentially continuous from the weak topology to the weak topology and from the weak topology*

to the strong topology, respectively, where y is any fixed point in K. Then the function $g : K \to R$, defined as $g(x) = \langle h'(x), \eta(y, x) \rangle$ for each fixed $y \in K$, is weakly sequentially continuous on K.

For each $D \subseteq H$, we denote by $\mathrm{co}(D)$ the convex hull of D. A point-to-set mapping $G : H \to 2^H$ is called a KKM mapping if for every finite subset $\{x_1, x_2, ..., x_k\}$ of H,

$$\mathrm{co}(\{x_1, x_2, ..., x_k\}) \subseteq \bigcup_{i=1}^{k} G(x_i).$$

Lemma 2.4. *See Ref. 13. Let K be an arbitrary nonempty subset in a Hausdorff topological vector space E and let $G : K \to 2^E$ be a KKM mapping. If $G(x)$ is closed for all $x \in K$ and is compact for at least one $x \in K$, then $\bigcap_{x \in K} G(x) \neq \emptyset$.*

3. ITERATIVE ALGORITHM AND CONVERGENCE ANALYSIS

We introduce the following basic algorithm framework for problem (1).

Algorithm 3.1. See Ref. 1. Let $\{\rho_n\}_{n=0}^{\infty}$ be a sequence of positive parameters and let x_0 be any initial guess in K. For each given iterate x_n, consider the auxiliary problem that consists of finding x_{n+1} such that

$$(6) \quad \langle \rho_n T(x_n) + h'(x_{n+1}) - h'(x_n), \eta(y, x_{n+1}) \rangle + \rho_n(f(y) - f(x_{n+1})) \geq 0, \quad \forall y \in K$$

where $h'(x)$ is the Frechet derivative of a functional $h : K \to R$ at x.

Theorem 3.1. *Suppose that problem (1) has a solution. Let $f : K \to R$ be a lower semicontinuous and convex functional and let $T : K \to H$ has the joint pseudo-Dunn property with respect to η and f with constant E. Assume that there hold the following conditions:*

(i) *$\eta : K \times K \to H$ is Lipschitz continuous with constant λ such that*

    (a) *$\eta(x, y) + \eta(y, x) = 0, \; \forall x, y \in K$,*

    (b) *$\eta(x, y) = \eta(x, z) + \eta(z, y), \; \forall x, y, z \in K$,*

    (c) *$\eta(\cdot, \cdot)$ is affine in the first variable,*

    (d) *for each fixed $y \in K$, $\eta(y, \cdot) : K \to H$ is sequentially continuous form the weak topology to the weak topology;*

(ii) *$h : K \to R$ is η-strongly convex with constant μ and, its derivative h' is both Lipschitz continuous with constant B and sequentially continuous from the weak topology to the strong topology;*

(iii) *for each fixed $n \geq 0$ and $z \in K$, the set*

$$\{x \in K : \langle \rho_n T(z) + h'(x) - h'(z), \eta(y, x) \rangle + \rho_n [f(y) - f(x)] \geq 0\}$$

is bounded for at least one $y \in K$.

Then there exists a unique solution $x_{n+1} \in K$ to (6) for each given iterate x_n. If

(7) $\rho_{n+1} \leq \rho_n$ and $\alpha < \rho_n < 2\mu/\lambda(E + \beta)$ $\forall n \geq 0$ for some $\alpha, \beta > 0$,

then $\{x_n\}$ is bounded, $\{\|x_{n+1} - x_n\|\}$ converges to zero, and $\{T(x_n)\}$ converges strongly to $T(x^*)$. In addition, if T is sequentially continuous from the weak topology to the strong topology, then $\{x_n\}$ converges weakly to a solution of problem (1).

Proof.   Existence of Solutions of Problem (6). For the sake of simplicity, we write (6) as follows: Find $\bar{x} \in K$ such that

$$\langle \rho_n T(x_n) + h'(\bar{x}) - h'(x_n), \eta(y, \bar{x}) \rangle + \rho_n [f(y) - f(\bar{x})] \geq 0, \quad \forall y \in K.$$

For each fixed $n \geq 0$ and each $y \in K$, we define

$$G(y) = \{x \in K : \langle \rho_n T(x_n) + h'(x) - h'(x_n), \eta(y, x) \rangle + \rho_n [f(y) - f(x)] \geq 0\}.$$

Note that since $y \in G(y)$, $G(y)$ is nonempty for each $y \in K$. Now we claim that G is a KKM mapping. Indeed suppose that there exists a finite subset $\{y_1, y_2, ..., y_k\}$ of K and $\alpha_i \geq 0$ $\forall i = 1, 2, ..., k$ with $\sum_{i=1}^{k} \alpha_i = 1$ such that $\hat{x} = \sum_{i=1}^{k} \alpha_i y_i \notin G(y_i)$ $\forall i = 1, 2, ..., k$. Then by virtue of assumptions (a), (c) in (i), we have

$$0 = \langle \rho_n T(x_n) + h'(\hat{x}) - h'(x_n), \eta(\hat{x}, \hat{x}) \rangle + \rho_n [f(\hat{x}) - f(\hat{x})]$$
$$\leq \sum_{i=1}^{k} \alpha_i [\langle \rho_n T(x_n) + h'(\hat{x}) - h'(x_n), \eta(y_i, \hat{x}) \rangle + \rho_n (f(y_i) - f(\hat{x}))] < 0$$

which is a contradiction. Hence G is a KKM mapping.

In view of conditions (i) (d), (ii), and Lemma 2.3, we can readily see that $G(y)$ is a weakly closed subset of K for each $y \in K$. Moreover, from condition (iii) we know that $G(y)$ is weakly compact for at least one point $y \in K$. Hence, by Lemma 2.4, we have $\bigcap_{y \in K} G(y) \neq \emptyset$, which clearly implies that there exists at least one solution to (6).

Uniqueness of Solutions of Problem (6). Let x_1, x_2 be two solutions of (6). Then

(8) $\langle \rho_n T(x_n) + h'(x_1) - h'(x_n), \eta(y, x_1) \rangle + \rho_n [f(y) - f(x_1)] \geq 0,$

(9) $\langle \rho_n T(x_n) + h'(x_2) - h'(x_n), \eta(y, x_2) \rangle + \rho_n[f(y) - f(x_2)] \geq 0.$

for all $y \in K$. Taking $y = x_2$ in (8) and $y = x_1$ in (9), and adding these inequalities, we get

$$\rho_n \langle T(x_n), \eta(x_2, x_1) \rangle + \langle h'(x_1) - h'(x_n), \eta(x_2, x_1) \rangle + \rho_n[f(x_2) - f(x_1)]$$
$$+ \rho_n \langle T(x_n), \eta(x_1, x_2) \rangle + \langle h'(x_2) - h'(x_n), \eta(x_1, x_2) \rangle + \rho_n[f(x_1) - f(x_2)] \geq 0.$$

By using condition (i) (a), we have $\langle h'(x_1) - h'(x_2), \eta(x_1, x_2) \rangle \leq 0$. From the η-strong convexity of h and Proposition 2.1, we get $\mu \|x_1 - x_2\|^2 \leq 0$ and therefore, $x_1 = x_2$ since $\mu > 0$. Hence the solution of (6) is unique.

Let x^* be any fixed solution of problem (1). Since x_{n+1} is the unique solution to (6), we get

(10) $\langle h'(x_{n+1}) - h'(x_n), \eta(y, x_{n+1}) \rangle + \rho_n \langle T(x_n), \eta(y, x_{n+1}) \rangle$
$$+\rho_n[f(y) - f(x_{n+1})] \geq 0, \quad \forall y \in K.$$

We consider the function Λ defined by

$$\Lambda(x, \rho) = \Phi(x) + \Omega(x, \rho),$$

where

(11) $\Phi(x) = h(x^*) - h(x) - \langle h'(x), \eta(x^*, x) \rangle$ and $\Omega(x, \rho)$
$$= \rho[\langle T(x^*), \eta(x, x^*) \rangle + f(x) - f(x^*)].$$

From the η-strong convexity of h, we obtain

(12) $$\Phi(x_n) \geq (\mu/2)\|x_n - x^*\|^2 \geq 0.$$

Since $\Omega(x_n, \rho_n)$ is nonnegative, we have

(13) $$\Lambda(x_n, \rho_n) \geq (\mu/2)\|x_n - x^*\|^2 \geq 0.$$

Let us study the variation of Λ for one stage of Algorithm 3.1,

$$\Gamma_n^{n+1} = \Lambda(x_{n+1}, \rho_{n+1}) - \Lambda(x_n, \rho_n).$$

Then we have

$$\Gamma_n^{n+1} = s_1 + s_2 + s_3$$

where

$$s_1 = h(x_n) - h(x_{n+1}) - \langle h'(x_n), \eta(x_n, x_{n+1}) \rangle,$$
$$s_2 = \langle h'(x_n) - h'(x_{n+1}), \eta(x^*, x_{n+1}) \rangle,$$

and

$$s_3 = \rho_{n+1}\langle T(x^*), \eta(x_{n+1}, x^*)\rangle - \rho_n\langle T(x^*), \eta(x_n, x^*)\rangle$$
$$+\rho_{n+1}(f(x_{n+1}) - f(x^*) - \rho_n(f(x_n) - f(x^*)).$$

By using the η-strong convexity of h, we have $s_1 \leq -(\mu/2)\|x_{n+1} - x_n\|^2$. Utilizing (10) with $y = x^*$ yields

$$s_2 \leq \rho_n\langle T(x_n), \eta(x^*, x_{n+1})\rangle + \rho_n(f(x^*) - f(x_{n+1}))$$
$$= \rho_n\langle T(x_n), \eta(x^*.x_n)\rangle + \rho_n\langle T(x_n), \eta(x_n, x_{n+1})\rangle$$
$$+\rho_n(f(x^*) - f(x_{n+1})).$$

By using (1) with $y = x_n$ and the joint pseudo-Dunn assumption on T, we get

$$\langle T(x_n), \eta(x_n, x^*)\rangle + f(x_n) - f(x^*) \geq (1/E)\|T(x_n) - T(x^*)\|^2.$$

Thus

$$s_2 \leq -(\rho_n/E)\|T(x_n) - T(x^*)\|^2 + \rho_n\langle T(x_n), \eta(x_n, x_{n+1})\rangle$$
$$+\rho_n(f(x_n) - f(x_{n+1})).$$

Since $\rho_{n+1} \leq \rho_n\ \forall n \geq 0$, we obtain

$$s_2 + s_3 \leq -(\rho_n/E)\|T(x_n) - T(x^*)\|^2 + \rho_n\langle T(x_n) - T(x^*), \eta(x_n, x_{n+1})\rangle.$$

Therefore,

$$\Gamma_n^{n+1} \leq -(\mu/2)\|x_{n+1} - x_n\|^2 - (\rho_n/E)\|T(x_n) - T(x^*)\|^2$$
$$+\lambda\rho_n\|T(x_n) - T(x^*)\|\|x_{n+1} - x_n\|.$$

Thus by using the inequality

$$\rho_n\|T(x_n) - T(x^*)\|\|x_{n+1} - x_n\| \leq (\rho_n^2/2\omega)\|T(x_n) - T(x^*)\|^2 + (\omega/2)\|x_{n+1} - x_n\|^2,$$

where ω is a positive number chosen so that $\omega < \mu/\lambda$, we get

$$\Gamma_n^{n+1} \leq -(\mu/2 - \lambda\omega/2)\|x_{n+1} - x_n\|^2 - \rho_n(1/E - \lambda\rho_n/2\omega)\|T(x_n) - T(x^*)\|^2.$$

For $\alpha < \rho_n < 2\omega/\lambda(E + \beta)$ where $\alpha > 0$ and $\beta > 0$, we have

$$\Gamma_n^{n+1} \leq -(\mu/2 - \lambda\omega/2)\|x_{n+1} - x_n\|^2 - (\alpha\beta/E(E + \beta))\|T(x_n) - T(x^*)\|^2,$$

and for $\omega < \mu/\lambda$, Γ_n^{n+1} is negative unless $x_{n+1} = x_n$ and $T(x_n) = T(x^*)$. Then, according to (10), x_n is a solution to problem (1).

Note that the sequence $\{\Lambda(x_n, \rho_n)\}$ is strictly decreasing. But since it is positive, it must converge and the difference between two consecutive terms tends to zero, that is, $\Gamma_n^{n+1} \to 0$ as $n \to \infty$. Therefore, $\|x_{n+1} - x_n\|$ and $\|T(x_n) - T(x^*)\|$ converge to zero. Moreover, since the sequence $\{\Lambda(x_n, \rho_n)\}$ converges, it is bounded, and so does $\{x_n\}$ according to (13).

Let $\bar{x}$ be a weak cluster point of the sequence $\{x_n\}$, and let $\{x_{n_i}\}$ be a sub-sequence converging weakly to $\bar{x}$. By using (10), since h' is Lipschitz continuous with constant B and $\rho_n > \alpha$, it is known that for each fixed $x \in K$,

(14)
$$
\begin{aligned}
&\langle T(x_n), \eta(x, x_{n+1})\rangle + f(x) - f(x_{n+1}) \\
&\geq -(1/\rho_n)\langle h'(x_{n+1}) - h'(x_n), \eta(x, x_{n+1})\rangle \\
&\geq -(\lambda B/\alpha)\|x_{n+1} - x_n\|\|x - x_{n+1}\|.
\end{aligned}
$$

Note that the functional $f : K \to R$ is convex and lower semicontinuous. Thus, it is weakly lower semicontinuous on K. This implies that $-f : K \to R$ is weakly upper semicontinuous on K. On the other hand, note also that T is sequentially continuous form the weak topology to the strong topology. Thus, we conclude that $\|T(x_{n_i}) - T(\bar{x})\| \to 0$ as $i \to \infty$. Since $\|x_{n_i+1} - x_{n_i}\| \to 0$ as $i \to \infty$ and $\|T(x_{n_i}) - T(x^*)\| \to 0$ as $i \to \infty$, it follows that $T(\bar{x}) = T(x^*)$, $w - \lim_{i\to\infty} \eta(x, x_{n_i+1}) = \eta(x, \bar{x})$. Hence

$$
\begin{aligned}
&\limsup_{i\to\infty} |\langle T(x_{n_i}), \eta(x, x_{n_i+1})\rangle - \langle T(\bar{x}), \eta(x, \bar{x})\rangle| \\
&\leq \limsup_{i\to\infty} |\langle T(x_{n_i}) - T(\bar{x}), \eta(x, x_{n_i+1})\rangle| + \limsup_{i\to\infty} |\langle T(\bar{x}), \eta(x, x_{n_i+1}) - \eta(x, \bar{x})\rangle| \\
&\leq \limsup_{i\to\infty} \|T(x_{n_i}) - T(\bar{x})\|\|\eta(x, x_{n_i+1})\| + \limsup_{i\to\infty} |\langle T(\bar{x}), \eta(x, x_{n_i+1}) - \eta(x, \bar{x})\rangle| \\
&= 0.
\end{aligned}
$$

This shows that

$$
\lim_{i\to\infty} \langle T(x_{n_i}), \eta(x, x_{n_i+1})\rangle = \langle T(\bar{x}), \eta(x, \bar{x})\rangle.
$$

Now taking the superior limit for the subsequence $\{n_i\}$ in inequality (14) yields

(15)
$$
\langle T(\bar{x}), \eta(x, \bar{x})\rangle + f(x) - f(\bar{x}) \geq 0.
$$

From (15), it follows that $\bar{x}$ is a solution of problem (1).

Finally, we claim that $\{x_n\}$ converges weakly to a solution of problem (1). Indeed, it is sufficient to prove that $\{x_n\}$ has the unique weak cluster point. Let $\bar{x}$ and $\hat{x}$ be two weak cluster points of $\{x_n\}$. Then, both weak cluster points can be used as x^* to define the Lyapunov function Λ. This yields two possible Lyapunov

functions, denoted $\bar{\Lambda}$ and $\hat{\Lambda}$, respectively. It was proved that $\Lambda(x_n, \rho_n)$ has a limit that may depend on the solution x^* used to define Λ. Let the corresponding limits be denoted by $\bar{l}$ and $\hat{l}$, respectively. Consider the subsequences $\{n_i\}$ and $\{m_j\}$ such that $\{x_{n_i}\}$ and $\{x_{m_j}\}$ converge weakly to $\bar{x}$ and $\hat{x}$, respectively. Then by using (3) and the fact that $T(\bar{x}) = T(\hat{x})$, we get

$$\hat{\Lambda}(x_{n_i}, \rho_{n_i}) = \bar{\Lambda}(x_{n_i}, \rho_{n_i}) + R(x_{n_i})$$

where

$$R(x_{n_i}) = h(\hat{x}) - h(\bar{x}) - \langle h'(x_{n_i}), \eta(\hat{x}, \bar{x}) \rangle + \rho_{n_i}[\langle T(\hat{x}), \eta(\bar{x}, \hat{x}) \rangle + f(\bar{x}) - f(\hat{x})].$$

Note that

$$\lim_{i \to \infty} \bar{\Lambda}(x_{n_i}, \rho_{n_i}) = \bar{l} \text{ and } \lim_{i \to \infty} \hat{\Lambda}(x_{n_i}, \rho_{n_i}) = \hat{l}.$$

Since $\lim_{n \to \infty} \rho_n$ exists, $\hat{x}$ and $\bar{x}$ both are solutions to problem (1), and h' is sequentially continuous from the weak topology to the strong topology, we deduce that

$$\hat{l} - \bar{l} = \lim_{i \to \infty} (\hat{\Lambda}(x_{n_i}, \rho_{n_i}) - \bar{\Lambda}(x_{n_i}, \rho_{n_i})) = \lim_{i \to \infty} R(x_{n_i})$$
$$= \lim_{i \to \infty} \{h(\hat{x}) - h(\bar{x}) - \langle h'(x_{n_i}), \eta(\hat{x}, \bar{x}) \rangle + \rho_{n_i}[\langle T(\hat{x}), \eta(\bar{x}, \hat{x}) \rangle + f(\bar{x}) - f(\hat{x})]\}$$
$$\geq h(\hat{x}) - h(\bar{x}) - \langle h'(\bar{x}), \eta(\hat{x}, \bar{x}) \rangle.$$

Hence it follows from the η-strong convexity of h that

$$\hat{l} \geq \bar{l} + (\mu/2) \|\hat{x} - \bar{x}\|^2.$$

By interchanging the role of $\bar{x}$ and $\hat{x}$ and of the subsequences $\{x_{n_i}\}$ and $\{x_{m_j}\}$, the same calculations yield

$$\bar{l} \geq \hat{l} + (\mu/2) \|\hat{x} - \bar{x}\|^2.$$

Then, $0 \leq (\mu/2) \|\hat{x} - \bar{x}\|^2 \leq \hat{l} - \bar{l}$ and $0 \leq (\mu/2) \|\hat{x} - \bar{x}\|^2 \leq \bar{l} - \hat{l}$. This implies that $\bar{x} = \hat{x}$. ∎

Remark 3.1. Compared with Theorem 3.1 in Ansari and Yao (Ref. 1), our Theorem 3.1 improves and generalizes their Theorem 3.1 in the following aspects: (i) Since the η-cocoercive condition on T is the η-Dunn property, the η-cocoercive condition on T is replaced by our weaker assumption that T has the joint pseudo-Dunn property with respect to η and f. (ii) Our Theorem 3.1 removes the restriction: $h(y) - h(x)\langle h'(x), \eta(y, x) \rangle \leq \gamma \|y - x\|^2 \ \forall x, y \in K$ for some $\gamma > 0$. (iii) Our Theorem 3.1 removes the boundedness assumption on K. In addition, our Theorem

3.1 extends Theorem 4.1 in Farouq (Ref. 11) to the case of the mixed variational-like inequality problem (1).

Corollary 3.1. *Suppose that problem (1) has a solution x^*. Let $f : K \to R$ be a lower semicontinuous and convex functional and let $T : K \to H$ be jointly strongly pseudomontone with constant e (x^* is then unique) and Lipschitz continuous with constant L. Assume that there hold the following conditions:*

(i) *$\eta : K \times K \to H$ is Lipschitz continuous with constant λ such that*

    (a) *$\eta(x, y) + \eta(y, x) = 0, \ \forall x, y \in K$,*

    (b) *$\eta(x, y) = \eta(x, z) + \eta(z, y), \ \forall x, y, z \in K$,*

    (s) *$\eta(\cdot, \cdot)$ is affine in the first variable,*

    (d) *for each fixed $y \in K$, $\eta(y, \cdot) : K \to H$ is sequentially continuous form the weak topology to the weak topology;*

(ii) *$h : K \to R$ is η-strongly convex with constant μ and its derivative h' is sequentially continuous from the weak topology to the strong topology;*

(iii) *for each fixed $n \geq 0$ and $z \in K$, $\{x \in K : \langle \rho_n T(z) + h'(x) - h'(z), \eta(y, x) \rangle + \rho_n[f(y) - f(x)] \geq 0\}$ is bounded for at least one $y \in K$.*

Then there exists a unique solution $x_{n+1} \in K$ to (6) for each given iterate x_n. If

(16) $\alpha < \rho_n < 2e\mu/(L^2\lambda^2 + \beta), \ \forall n \geq 0, \quad$ *for some $\alpha, \beta > 0$,*

then $\{x_n\}$ converges strongly to x^*. Moreover, if h' is Lipschitz continuous with constant B, then we have the posteriori error estimation:

(17) $\|x_{n+1} - x^*\| \leq (B\lambda/e\rho_n + L\lambda/e)\|x_{n+1} - x_n\|.$

Proof. We consider the variation of the function Φ in (11) for one stage of Algorithm 3.1,
$$\Delta_n^{n+1} = \Phi(x_{n+1}) - \Phi(x_n).$$
By using the same notation and similar calculations to those in the proof of Theorem 3.1, we get
$$\Delta_n^{n+1} = s_1 + s_2,$$
with $s_1 \leq -(\mu/2)\|x_{n+1} - x_n\|^2$ and

$$s_2 \leq \rho_n\langle T(x_n), \eta(x^*, x_{n+1})\rangle + \rho_n(f(x^*) - f(x_{n+1}))$$
$$= \rho_n\langle T(x_n) - T(x_{n+1}), \eta(x^*, x_{n+1})\rangle + \rho_n\langle T(x_{n+1}), \eta(x^*, x_{n+1})\rangle$$
$$+ \rho_n(f(x^*) - f(x_{n+1})).$$

By using (1) with $y = x_{n+1}$ and the joint strong pseudomonotonicity of T, we get

$$(18) \qquad \langle T(x_{n+1}), \eta(x_{n+1}, x^*) \rangle + f(x_{n+1}) - f(x^*) \geq e\|x_{n+1} - x^*\|^2.$$

Thus, we have

$$s_2 \leq -e\rho_n\|x_{n+1} - x^*\|^2 + \rho_n\langle T(x_n) - T(x_{n+1}), \eta(x^*, x_{n+1})\rangle.$$

Therefore,

$$\begin{aligned}
\Delta_n^{n+1} &\leq -(\mu/2)\|x_{n+1} - x_n\|^2 - e\rho_n\|x_{n+1} - x^*\|^2 \\
&\quad + \lambda\rho_n\|T(x_n) - T(x_{n+1})\|\|x^* - x_{n+1}\| \\
&\leq -(\mu/2)\|x_{n+1} - x_n\|^2 - e\rho_n\|x_{n+1} - x^*\|^2 + \rho_n L\lambda\|x_{n+1} - x_n\|\|x_{n+1} - x^*\|.
\end{aligned}$$

By using the inequality

$$\rho_n L\lambda\|x_{n+1} - x_n\|\|x_{n+1} - x^*\| \leq (\omega/2)\|x_{n+1} - x_n\|^2 + (\rho_n^2 L^2\lambda^2/2\omega)\|x_{n+1} - x^*\|^2,$$

with $\omega = \mu$, it follows from condition (16) that

$$(19) \quad \Delta_n^{n+1} \leq \rho_n^2(-e/\rho_n + L^2\lambda^2/2\mu)\|x_{n+1} - x^*\|^2 \leq (-\alpha^2\beta/2\mu)\|x_{n+1} - x^*\|^2.$$

Under this condition, Δ_n^{n+1} is negative unless $x_{n+1} = x^*$. The sequence $\Phi(x_n)$ is strictly decreasing. But since it is positive, it must converge and the difference between two consecutive terms tends to zero, that is, $\Delta_n^{n+1} \to 0$ as $n \to \infty$. Therefore, it follows from (19) that $\{x_n\}$ converges strongly to x^*.

Now, by using (10) with $y = x^*$ and (18), we infer that

$$\begin{aligned}
\langle h'(x_{n+1}) &- h'(x_n), \eta(x^*, x_{n+1})\rangle + \rho_n\langle T(x_n) - T(x_{n+1}), \eta(x^*, x_{n+1})\rangle \\
&\geq \rho_n[\langle T(x_{n+1}), \eta(x_{n+1}, x^*)\rangle + f(x_{n+1}) - f(x^*)] \\
&\geq e\rho_n\|x_{n+1} - x^*\|^2.
\end{aligned}$$

Then by using the Schwarz inequality and the Lipschitz assumptions on h' and T, we get

$$B\lambda\|x_{n+1} - x_n\|\|x_{n+1} - x^*\| + \rho_n L\lambda\|x_{n+1} - x_n\|\|x_{n+1} - x^*\| \geq e\rho_n\|x_{n+1} - x^*\|^2.$$

We obtain inequality (17) after division by $\|x_{n+1} - x^*\|$ which is assumed to be nonzero; otherwise, the result is trivial. ∎

Remark 3.2. Corollary 3.1 extends Farouq's Corollary 5.1 in Ref. 11 to the case of the mixed variational-like inequality problem (1).

REFERENCES

1. Q. H. Ansari and J. C. Yao, Iterative Schemes for Solving Mixed Variational-Like Inequalities, *Journal of Optimization Theory and Applications*, **108** (2001), 527-541.

2. X. P. Ding, Random Mixed Variational-Like Inequalities in Topological Vector Spaces, *Journal of Sichuan Normal University*, **20** (1997), 1-13.

3. N. H. Dien, Some Remarks on Variational-Like and Quasi-Variational-Like Inequalities, *Bulletin of the Australian Mathematical Society*, **46** (1992), 335-342.

4. Q. H. Ansari and J. C. Yao, Nonlinear Variational Inequalities for Pseudomonotone Operators with Applications, *Advances in Nonlinear Variational Inequalities*, **3** (2000), 61-69.

5. J. Parida, M. Sahoo and A. Kumar, A Variational-Like Inequality Problem, *Bulletin of the Australian Mathematical Society*, **39** (1989), 225-231.

6. A. H. Siddiqi, A. Khaliq and Q. H. Ansari, On Variational-Like Inequalities, *Annales des Sciences Mathematiques du Quebec*, **18** (1994), 39-48.

7. X. Q. Yang and G. Y. Chen, A Class of Nonconvex Functions and Prevariational Inequalities, *Journal of Mathematical Analysis Applications*, **169** (1992), 359-373.

8. L. C. Zeng and J. C. Yao, A Class of Variational-Like Inequality Problems and Its Equivalence with the Least Element Problems, *Journal of Nonlinear and Convex Analysis*, **4** (2004), to appear.

9. J. C. Yao, Variational Inequalities with Generalized Monotone Operators, *Mathematics of Operations Research*, **19** (1994), 691-705.

10. J. C. Yao, Multivalued Variational Inequalities with K-Pseudomonotone Operators, *Journal of Optimization Theory and Applications*, **83** (1994), 391-403.

11. N. El Farouq, Pseudomonotone Variational Inequalities: Convergence of the Auxiliary Problem Method, *Journal of Optimization Theory and Applications*, **111** (2001), 305-326.

12. G. Cohen, Auxiliary Problem Principle Extended to Variational Inequalities, *Journal of Optimization Theory and Applications*, **49** (1988), 325-333.

13. K. Fan, A Generalization of Tychonoff's Fixed-Point Theorem, *Mathematische Annalen*, **142** (1961), 305-310.

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