TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 2, pp. 417-429, February 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

ASYMPTOTIC REGULARITY OF LINEAR POWER BOUNDED OPERATORS

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Abstract. Let T be a linear power bounded operator on a Banach space X and let $S_\lambda = (1-\lambda)I+\lambda T$ be the averaged map of T, where $\lambda \in (0,1)$. It is shown that S_λ is asymptotically regular on X; that is, $\lim_{n\to\infty} \|S_\lambda^n x - S_\lambda^{n+1}x\| = 0$ for every $x \in X$. Hence the sequence $\{S_\lambda^n x\}$ converges strongly provided it has a weak cluster point.

1. Introduction

Let X be a Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping (i.e., $\|Tx-Ty\| \le \|x-y\|$ for all $x, y \in C$). A point $x \in C$ is a fixed point of T provided $Tx = x$. Construction of fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in signal processing and image recovery (see, e.g., [4, 11, 12, 14, 16, 17]). However, the sequence $\{T^n x\}$ of iterates of the mapping T at a point $x \in C$ may not behave well, in general. This means that it may not converge even in the weak topology. One way to overcome this difficulty is to use the averaged mappings which are given by $S_\lambda := (1-\lambda)I + \lambda T$, where I is the identity mapping on X and λ is a number in $(0, 1)$. By a result of Ishikawa [10], each averaged map S_λ is *asymptotically regular* [2]: $\lim_{n\to\infty} \|S_\lambda^{n+1}x - S_\lambda^n x\| = 0$ for all $x \in C$ provided $\{T^n x\}$ is bounded. However, this does not mean that the iterates $\{T_\lambda^n x\}$ converge (either strongly or weakly) to a fixed point of T, in general. Some additional conditions are needed to impose, for example, uniform convexity and Fréchet differentiability of the norm of X (see Reich [13]). It would be simpler for linear nonexpansive

Received November 11, 2004; Revised March 23, 2005; Accepted April 6, 2005.
2000 *Mathematics Subject Classification*: Primary 47B44; Secondary 47H10.
Key words and phrases: Mean ergodic theorem, Linear power bounded operator, Asymptotic regularity, Averaged mapping.
Supported in part by the South African National Research Foundation.

mappings. Indeed, Dotson [5] proved the following result.

Theorem D. *If X is a uniformly convex Banach space and if T is a linear nonexpansive mapping on X, then for each $x \in X$, the iterates $\{S_\lambda^n x\}$ of the averaged map S_λ converges strongly to a fixed point of T.*

Another important result for nonexpansive mappings is Baillon's mean ergodic theorem [1].

Theorem B. *If X is a Hilbert space and if $T : C \to C$ is a nonexpansive mapping with a fixed point, then for each $x \in C$, the means*

$$S_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} T^i x, \quad n \geq 1$$

converge weakly to a fixed point of T.

Theorem B has been extended to a uniformly convex Banach space with a Fréchet differentiable norm (see Bruck [3]). But it is still unclear if Theorem B is valid in a uniformly convex Banach space. On the other hand, Ky Fan asked (cf. [9]) whether or not there exists the limit as $n \to \infty$ of the sequence $\{\|S_n(x) - p\|\}$ in a Hilbert space, where p is a fixed point of T. This question seems unsolved.

It is the purpose of this paper to study some convergence results for linear power bounded operators on a Banach space. More precisely, we will prove that if T is a linear power bounded operator on a Banach space X, then each averaged mapping S_λ is asymptotically regular on X; hence the sequence $\{S_\lambda^n x\}$ converges strongly to a fixed point of T provided it contains a weak cluster point, which is the case if the space X is reflexive. We have therefore weakened the uniform convexity assumption on the space X in Theorem D to reflexivity of the space X. (The example to be given in Section 3 shows that reflexivity of X is unremovable; thus our extension of Theorem D is sharp.)

We have also weakened the nonexpansiveness of T in Theorem D to power boundedness of T. In proving the strong convergence of $\{S_\lambda^n x\}$ we shall employ the abstract mean ergodic theorem of Eberlein [6, 7] which seems not widely known (see [8]), but powerful.

We also answer Fan's question for linear nonexpansive mappings in a Banach space. The full question for nonlinear nonexpansive mappings remains open.

2. Preliminaries

Let X be a Banach space and T a bounded linear operator on X. Recall that T is *power bounded* if there is a constant $M > 0$ for which $\|T^n\| \leq M$ for all integers $n \geq 0$; in particular, T is nonexpansive if $\|T\| \leq 1$.

Let G be a semigroup of linear operators on X. Let G^* be the set of all operators of the form $\sum a_j T_j,\ a_j \geq 0,\ \sum a_j = 1,\ T_j \in G$ and let $O(x)$ be the orbit of G and x, i.e., $O(x) = \{T^* x : T^* \in G^*\}$. Recall that a family $\{T_\alpha\}$ of linear bounded operators in X is an *almost invariant integral system* for G ([6, 7]) if the following properties hold:

(a) $T_\alpha x \in \overline{O}(x), x \in X$;

(b) $\|T_\alpha\| \leq M$;

(c) $\lim_\alpha (T T_\alpha - T_\alpha)x = \lim_\alpha (T_\alpha T - T_\alpha)x = 0$, $T \in G$, $x \in X$.

The semigroup G is said to be *ergodic* if it possesses an almost invariant integral system. The following abstract ergodic theorem is due to W. F. Eberlein [6, 7].

Theorem E. *If G is ergodic, if x is a member of X and if $\{T_\alpha\}$ is an almost invariant integral system for G, then the following statements on an element $y \in X$ are equivalent:*

(i) $y \in \overline{O}(x)$, $Ty = y$, $T \in G$;

(ii) $y = \lim_\alpha T_\alpha x$;

(iii) $y = \lim_\alpha T_\alpha x$ *weakly;*

(iv) y *is a weak cluster point of* $\{T_\alpha x\}$.

For the proof of Theorem 2 in the next section, we need the following result.

Lemma 1. *Let $\{a_n\}_{n=1}^\infty$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+m} \leq a_n + b_n a_m, \quad n,\, m \geq 1, \tag{1}$$

where $\{b_n\}_{n=1}^\infty$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = 1$. Then the $\lim_{n\to\infty} a_n/n$ exists.

Proof. For an arbitrary $\varepsilon > 0$, there is an integer n_0 with the property

$$b_n < 1 + \varepsilon \quad \text{for all } n \geq n_0 \text{ and } \quad \frac{a_{n_0}}{n_0} < c + \varepsilon,$$

where

$$c = \liminf_{n\to\infty} \frac{a_n}{n}.$$

Repeatedly using (1) we get, for $k \geq 1$,

$$a_{kn_0} \leq \left(1 + \sum_{i=1}^{k-1} b_{in_0}\right) a_{n_0} < [1 + (k-1)(1+\varepsilon)] n_0 (c + \varepsilon). \tag{2}$$

Now let $n > n_0$ and write $n = kn_0 + r$ with $k \geq 1$ and $0 \leq r < n_0$. From (1) and (2) it follows that

$$\begin{aligned} \frac{a_n}{n} = \frac{a_{kn_0+r}}{kn_0+r} &\leq \frac{a_{kn_0} + b_{kn_0} a_r}{kn_0 + r} \\ &\leq \frac{[1+(k-1)(1+\varepsilon)]n_0(c+\varepsilon)}{kn_0} + \frac{b_{kn_0} a_r}{kn_0 + r} \\ &\leq \left(\frac{1}{k} + \left(1 - \frac{1}{k}\right)(1+\varepsilon)\right)(c+\varepsilon) + \frac{(1+\varepsilon)a_r}{n}. \end{aligned}$$

Letting $n \to \infty$ gives

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq (1+\varepsilon)(c+\varepsilon).$$

This implies that

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq c$$

and hence $\lim_{n\to\infty} a_n/n$ exists. ∎

2. Convergence Results

For a given bounded linear operator T on a Banach space X, let G be the discrete semigroup $\{T^n : n \geq 0\}$. Suppose that T is power bounded; thus $\|T^n\| \leq M$ for some constant $M > 0$ and for all integers $n \geq 0$. As before we put

$$S_n x = \frac{1}{n} \sum_{i=0}^{n-1} T^i x, \quad n \geq 1.$$

Since T is linear we have

$$\|TS_n x - S_n x\| = \|S_n T x - S_n x\| = \frac{1}{n}\|T^n x - x\| \to 0 \ (n \to \infty).$$

This implies that

$$\lim_{n\to\infty} \|T^m S_n x - S_n x\| = \lim_{n\to\infty} \|S_n T^m x - S_n x\| = 0$$

for all integers $m \geq 0$. So $\{S_n\}$ is an almost invariant integral system for G and the following theorem (see [15, Chapter VII, section 3] for a more general case in locally convex linear space) is a consequence of Eberlein's abstract mean ergodic theorem (Theorem E).

Theorem 1. *Let T be a power bounded linear operator on a Banach space X. If $\{S_n x\}$ has a weak cluster point, then it converges strongly to a fixed point of T.*

We next turn to investigate the existence of the $\lim_{n\to\infty} \|S_n x - p\|$, where p is a fixed point of T. We do not know the full answer to Fan's question mentioned in section one. The result below is only partial answer in the case of those linear mappings which are *almost* nonexpansive, by which we mean linear operators T such that $\lim_{n\to\infty} \|T^n\| = 1$.

Theorem 1. *Let T be a bounded linear operator on a Banach space X which is almost nonexpansive. Then for each fixed point p of T, there exists the* $\lim_{n\to\infty} \|S_n x - p\|$.

Proof. Since T is linear, it suffices to prove the theorem for the case where $p = 0$. Put

$$a_n = a_n(x) = \|nS_n x\| = \left\|\sum_{i=0}^{n-1} T^i x\right\|.$$

Then for all integers $n, m \geq 0$,

$$\begin{aligned} a_{n+m} &= \left\|\textstyle\sum_{i=0}^{n+m-1} T^i x\right\| \\ &= \|nS_n x + mS_m T^n x\| = \|nS_n x + T^n(mS_m x)\| \\ &\leq a_n + b_n a_m, \end{aligned}$$

where $b_n = \|T^n\| \to 1$ as $n \to \infty$. By Lemma 1 we see that $\lim_{n\to\infty} \|S_n x\|$ exists. ∎

Remark 1. If $\{S_n x\}$ has a weak cluster point, then it converges strongly by Theorem 1. In this case, Theorem 2 is a consequence of Theorem 1. But Theorem 2 indicates that even if $\{S_n x\}$ does not have a weak cluster point, the $\lim_{n\to\infty} \|S_n x - p\|$ always exists. We do not know if Theorem 2 holds for a linear power bounded operator T without assuming that $\{S_n x\}$ has a weak cluster point.

Next we present the main result of this paper; that is, the asymptotic regularity of the averaged mappings. Recall that by an averaged map we mean a map S which can be written as $S = (1-\lambda)I + \lambda T$, where λ is a number in $(0, 1)$ and I and T are the identity operator and a linear power bounded operator on the Banach space X, respectively. We sometimes write S_λ for S to emphasize the dependence of S upon λ. We also call S_λ the averaged map associated with T. Recall also that S_λ is asymptotically regular on X if $\lim_{n\to\infty} \|S_\lambda^n x - S_\lambda^{n+1} x\| = 0$ for every $x \in X$.

For convenience we include Stirling's formula as a lemma.

Lemma 2. (*Stirling's Formula*)

$$\frac{\sqrt{x}(2x)!}{(2^x x!)^2} \approx \frac{1}{\sqrt{\pi}} \quad (\text{as } x \to \infty).$$

Theorem 3. Let S_λ be an averaged mapping associated with a linear power bounded operator T. Then S_λ is asymptotically regular on X.

Proof. We first prove the case where $\lambda = 1/2$. In this case we write $S = S_{1/2}$; that is, $S = \frac{1}{2}(I+T)$. We have

$$\begin{aligned} S^n x &= \frac{1}{2}(I+T)^n x \\ &\frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}T^j x \\ TS^n x &= S^n T x \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}T^{j+1}x \\ &\frac{1}{2^n}\sum_{j=1}^{n+1}\binom{n}{j-1}T^j x. \end{aligned}$$

We may assume that $n = 2k$ is even, the case where n is odd being similar. Rearranging the terms and using the fact that $\binom{n}{j} = \binom{n}{n-j}$, we obtain

$$\begin{aligned} S^n x - S^n T x &= \frac{1}{2^n}\left\{\left[\binom{n}{0}x - \binom{n}{n}T^{n+1}x\right] + \sum_{j=1}^{n}\left[\binom{n}{j} - \binom{n}{j-1}\right]T^j x\right\} \\ &= \frac{1}{2^n}\left\{(x - T^{n+1}x) + \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right](T^j x - T^{n-j+1}x)\right\}. \end{aligned}$$

Since $\binom{n}{j} - \binom{n}{j-1} > 0$ for $1 \le j \le k$, we derive that from the last equation that (with $d = \sup\{\|T^i x - T^j x\| : i, j \ge 0\}$ being the diameter of the sequence $\{T^n x\}$),

$$\begin{aligned} \|S^n x - S^n T x\| &\le \frac{d}{2^n}\left\{1 + \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right]\right\} \\ &= \frac{d}{2^n}\binom{n}{k} = \frac{d}{2^n}\frac{n!}{(k!)^2} \\ &\approx \frac{d}{\sqrt{\pi k}} = \frac{d}{\sqrt{\pi n/2}} \quad (\text{by Stirling's formula}) \\ &\to 0 \ (n \to \infty). \end{aligned}$$

Now consider the case: $0 < \lambda < 1/2$. This can be reduced to the case of $\lambda = 1/2$. Indeed, for a given $0 < \lambda < 1/2$, let $\mu = 2\lambda < 1$ and let $\tilde{T} = (1-\mu)I + \mu T$.

Then $\tilde{T}$ is linear and still power bounded. In fact,

$$\begin{aligned}\|\tilde{T}^n\| &= \left\|\sum_{i=0}^{n}\binom{n}{i}(1-\mu)^{n-i}\mu^i T^i\right\| \\ &\leq \sum_{i=0}^{n}\binom{n}{i}(1-\mu)^{n-i}\mu^i\|T^i\| \\ &\leq \sum_{i=0}^{n}\binom{n}{i}(1-\mu)^{n-i}\mu^i M \\ &= M,\end{aligned}$$

where $M > 0$ is a constant such that $\|T^n\| \leq M$ for all integers $n \geq 0$. Now rewrite S_λ as $S_\lambda = \frac{1}{2}(I + \tilde{T})$ and apply the result just proved above to $\tilde{T}$ to obtain the asymptotic regularity of S_λ for $0 < \lambda < 1/2$.

Next consider the case: $1/2 < \lambda < 1$. We first show a general result. That is, if V is any linear power bounded operator on X which commutes T and which is asymptotically regular, and if $S = (V + T)/2$, then S is also asymptotically regular. This is, given $x \in X$, we need to prove that

$$\lim_{n\to\infty}\|S^n x - S^{n+1}x\| = 0,$$

or sufficiently,

$$\lim_{n\to\infty}\|S^n x - TS^n x\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|S^n x - VS^n x\| = 0.$$

We first show that $\lim_{n\to\infty}\|S^n x - TS^n x\| = 0$. We have

$$S^n x = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}V^{n-j}T^j x.$$

$$TS^n x = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}V^{n-j}T^{j+1}x = \frac{1}{2^n}\sum_{j=1}^{n+1}\binom{n}{j-1}V^{n-j+1}T^j x.$$

Hence

$$\begin{aligned}S^n x - TS^n x = &\ \frac{1}{2^n}(V^n x - T^{n+1}x) \\ &+\frac{1}{2^n}\sum_{j=1}^{n}\left[\binom{n}{j}V^{n-j} - \binom{n}{j-1}V^{n-j+1}\right]T^j x\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^n}(V^n x - T^{n+1}x) \\
&\quad + \frac{1}{2^n}\sum_{j=1}^{n}\left[\binom{n}{j} - \binom{n}{j-1}\right] V^{n-j}T^j x \\
&\quad + \frac{1}{2^n}\sum_{j=1}^{n}\binom{n}{j-1} T^j\left(V^{n-j}x - V^{n-j+1}x\right) \\
&=: A_n + B_n + C_n.
\end{aligned}
$$

Put $a = \sup\{\|V^iT^jx - V^kT^lx\| : i,j,k,l \geq 0\}$. Note that a is finite since V and T are power bounded. It is a simple fact that

$$\|A_n\| \leq \frac{a}{2^n} \to 0 \quad \text{as } n \to \infty.$$

Without loss of generality we may assume that $n = 2k$ is an even integer, the case of an odd integer n being similar. Using again the fact that $\binom{n}{j} = \binom{n}{n-j}$, we have

$$
\begin{aligned}
&\sum_{j=1}^{n}\left[\binom{n}{j} - \binom{n}{j-1}\right] V^{n-j}T^j x \\
&= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right] V^{n-j}T^j x + \sum_{j=k+1}^{n}\left[\binom{n}{j} - \binom{n}{j-1}\right] V^{n-j}T^j x \\
&= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right] V^{n-j}T^j x + \sum_{j=1}^{k}\left[\binom{n}{k+j} - \binom{n}{k+j-1}\right] V^{n-k-j}T^{k+j} x \\
&= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right] V^{n-j}T^j x + \sum_{j=1}^{k}\left[\binom{n}{k-j} - \binom{n}{k-j+1}\right] V^{n-k-j}T^{k+j} x \\
&= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right] V^{n-j}T^j x + \sum_{j=1}^{k}\left[\binom{n}{j-1} - \binom{n}{j}\right] V^{j-1}T^{n-j+1} x \\
&= \sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right] \left(V^{n-j}T^j x - V^{j-1}T^{n-j+1}x\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|B_n\| &\leq \frac{a}{2^n}\sum_{j=1}^{k}\left[\binom{n}{j} - \binom{n}{j-1}\right] \\
&\leq \frac{a}{2^n}\binom{n}{k} \\
&\approx \frac{a}{\sqrt{\pi k}} = \frac{a}{\sqrt{\pi n/2}} \ \text{(by Stirling's formula)} \\
&\to 0 \ \text{ as } n \to \infty.
\end{aligned}
$$

To prove that $\|C_n\| \to 0$ as $n \to \infty$, we have (M is the constant such that $M \geq \|T^n\|$ for all $n \geq 0$)

$$\begin{aligned}\|C_n\| &\leq \frac{M}{2^n}\sum_{j=1}^{n}\binom{n}{j-1}\left\|V^{n-j}x - V^{n-j+1}x\right\| \\ &= \frac{M}{2^n}\sum_{j=0}^{n-1}\binom{n}{n-j-1}\left\|V^{j}x - V^{j+1}x\right\| \\ &= \frac{M}{2^n}\sum_{j=0}^{n-1}\binom{n}{j+1}\left\|V^{j}x - V^{j+1}x\right\|.\end{aligned}$$

Now since $\lim_{n\to\infty}\|V^n x - V^{n+1}x\| = 0$, for any given $\varepsilon > 0$, there is an integer $k_0 > 0$ such that $\|V^j x - V^{j+1}x\| < \varepsilon$ for all $j \geq k_0$. It follows that, for $n > k_0$,

$$\begin{aligned}\|C_n\| &\leq \frac{M}{2^n}\left\{\varepsilon\sum_{j=k_0}^{n-1}\binom{n}{j+1} + a\sum_{j=0}^{k_0-1}\binom{n}{j+1}\right\} \\ &< M\varepsilon + aM\sum_{j=0}^{k_0-1}\frac{1}{2^n}\binom{n}{j+1}.\end{aligned}$$

Observing that

$$\lim_{n\to\infty}\frac{1}{2^n}\binom{n}{j} = 0$$

for every fixed integer $j \geq 0$, we conclude that $\lim_{n\to\infty}\|C_n\| = 0$.

Next we show that $\lim_{n\to\infty}\|S^n x - VS^n x\| = 0$. Noticing that

$$S^n x - VS^n x = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}T^j\left(V^{n-j}x - V^{n-j+1}x\right),$$

by repeating the argument above for the proof of $C_n \to 0$, we can obtain that $\lim_{n\to\infty}\|S^n x - VS^n x\| = 0$.

The proof can now be completed by the following observation: if V is a linear power bounded operator which commutes T and which is asymptotically regular, then similar to the previous case of $V = I$, we have for $\lambda \in (0, 1/2)$, the averaged operator $V_\lambda = (1-\lambda)V + \lambda T$ is also asymptotically regular.

As a matter of fact, this case can be reduced to the case of $\lambda = 1/2$. Putting $\mu = 2\lambda < 1$ and $\tilde{T} = (1-\mu)V + \mu T$, we can rewrite V_λ as $V_\lambda = (V + \tilde{T})/2$ which is seen to be asymptotically regular by applying the above result to $T := \tilde{T}$.

Finally since the the set of all the points of the form $k/2^n$, where $k = 1, 2, \cdots, 2^n - 1$, $n \geq 1$, is dense in $(0, 1)$, we see that for every $\lambda \in (0, 1)$, S_λ can be expressed in the form $S_\lambda = (1-\sigma)S_\mu + \sigma T$, where $0 < \sigma < 1/2$ and $\mu = k/2^n < \sigma$ (but close σ enough) for some $1 \le k \le 2^n - 1$ and $n \ge 1$. Hence S_λ is asymptotically regular. ∎

Remark 2. Theorem 3 is not valid if T is not power bounded. For example, if $T = \alpha I$ for some $\alpha > 1$, then $S_\lambda = (1-\lambda)I + \lambda T = \beta I$, where $\beta = 1-\lambda+\lambda\alpha > 1$. Hence for any $x \ne 0$, $\|S_\lambda^n x - S_\lambda^{n+1}x\| = \beta^n\|x\| \to \infty$ as $n \to \infty$.

Remark 3. We actually proved that $\|S_{1/2}^n x - S_{1/2}^{n+1}x\| = O(1/\sqrt{n})$. We conjecture that this is true for an arbitrary $\lambda \in (0,1)$; that is, $\|S_\lambda^n x - S_\lambda^{n+1}x\| = O(1/\sqrt{n})$.

Remark 4. The argument of Theorem 3 indeed shows that Theorem 3 actually holds true in a locally convex linear space. That is, let X be a locally convex linear topological space and let T be a linear continuous operator from X into X. Assume that the family of operators $\{T^n : n \ge 0\}$ is equi-continuous in the sense that, for any continuous semi-norm q on X, there exists a continuous semi-norm q' on X such that $\sup\{q(T^n x) : n \ge 1\} \le q'(x)$ for all $x \in X$. Then the averaged map $S_\lambda = (1-\lambda)I + \lambda T$ is asymptotically regular on X.

Theorem 4. *Let T be a linear power bounded operator on a Banach space X. Then for each $0 < \lambda < 1$, the sequence $\{S_\lambda^n x\}$ of iterates of the averaged mapping S_λ at x converges strongly to a fixed point of T provided $\{S_\lambda^n x\}$ has a weak cluster point.*

Proof. By Theorem 3, we have

$$\|TS_\lambda^n x - S_\lambda^n x\| = \|S_\lambda^n Tx - S_\lambda^n x\| \to 0 \ (n \to \infty).$$

Since T is linear and bounded, we further have that for each fixed integer $m \ge 0$,

$$\|T^m S_\lambda^n x - S_\lambda^n x\| = \|S_\lambda^n T^m x - S_\lambda^n x\| \to 0 \ (n \to \infty).$$

Hence $\{S_\lambda^n x\}$ forms an almost invariant integral system for the semigroup $G := \{T^n : n \ge 0\}$. By Theorem E, we conclude that $\{S_\lambda^n x\}$ converges strongly to a fixed point of T. ∎

Corollary. *Let T be a linear power bounded operator on a reflexive Banach space X. Then for each $0 < \lambda < 1$, the sequence $\{S_\lambda^n x\}$ of iterates of the averaged mapping S_λ at x converges strongly to a fixed point of T.*

The following example shows that without the assumption that $\{S_\lambda^n x\}$ have a weak cluster point in Theorem 4, or that the space X be reflexive in the Corollary, the conclusion in either Theorem 4 or the Corollary above may not be true.

Example. Let $X = \ell^1$ be the space of all absolutely summable sequences of real numbers, equipped with the norm

$$\|x\| = \sum_{n=1}^{\infty} |a_n| \quad \text{if} \;\; x = \sum_{n=1}^{\infty} a_n e_n,$$

where $\{e_n\}_{n=1}^{\infty}$ is the standard basis of ℓ^1; that is, for each $n \geq 1$, e_n is the vector whose $n-$th component is one and all else components are zero. Now define an operator $T : \ell^1 \to \ell^1$ by

$$Tx = \sum_{n=1}^{\infty} a_n e_{n+1}, \quad x = \sum_{n=1}^{\infty} a_n e_n \in \ell^1.$$

Namely, T is a shift operator on ℓ^1. It is not hard to see that T is an isometry. Indeed, for every $x = \sum_{n=1}^{\infty} a_n e_n \in \ell^1$, we have $\|Tx\| = \|\sum_{n=1}^{\infty} a_n e_{n+1}\| = \sum_{n=1}^{\infty} |a_n| = \|x\|$.

Now take $x = e_1 = (1, 0, \cdots, 0, \cdots)$. Let $S = (I+T)/2$ be the averaged map of T with $\lambda = 1/2$. Since it is easy to find that $T^j x = e_{j+1}$ for $j \geq 1$, it follows that

$$\begin{aligned} S^n x &= \frac{1}{2^n}(I+T)^n x \\ &= \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j} T^j x \\ &= \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j} e_{j+1}. \end{aligned}$$

Consequently,

$$\|S^n x\| = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j} = 1. \tag{2}$$

We use $(S^n x)_k$ to denote the $k-$th component of $S^n x$. Hence $(S^n x)_k = \binom{n}{k-1}$ if $k < n$ and $(S^n x)_k = 0$ if $k \geq n$. Suppose now that $\{S^n x\}$ converges strongly to some $y = \sum_{n=1}^{\infty} b_n e_n \in \ell^1$. Then for each fixed integer $k \geq 1$, we have

$$b_k = \lim_{n\to\infty} (S^n x)_k = \lim_{n\to\infty} \frac{1}{2^n}\binom{n}{k-1} = 0.$$

This implies that $y = 0$, which contradicts (3) as y is the strong limit of $\{S^n x\}$. Therefore, $\{S^n x\}$ does not strongly converge.

Since in ℓ^1, strong and weak convergences for sequences are equivalent, we see that $\{S^n x\}$ does not have a weak cluster point. This also shows that in Theorem 4, the assumption that $\{S_\lambda^n x\}$ have a weak cluster point is not removable.

Remark 5. As pointed out in Remark 4, Theorem 3 holds true in a locally convex linear space. It is however unclear if Theorem 4 holds true in a locally convex linear space as we do not know if Eberlein's abstract mean ergodic theorem holds true in the setting of locally convex linear spaces.

Remark 6. Let $z = z(\lambda)$ denote the limit of $\{S_\lambda^n x\}$ as obtained in Theorem 3. Then z is a fixed point of T. In another word, z solves the linear equation: $(I - T)z = 0$. Since T is power bounded, the spectral radius of T, $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = 1$. Thus the maximum possible eigenvalue for T is one. If 1 is not an eigenvalue of T, then the equation $(I-T)z = 0$ has only the trivial solution. In this case we have that $\{S_\lambda^n x\}$ converges strongly to 0, for every $\lambda \in (0,1)$. If, however, there is a $\lambda_0 \in (0,1)$ for which, $z_0 = z(\lambda_0) \neq 0$, then 1 is the maximum eigenvalue of T and z_0 is an eigenvector of T corresponding to the maximum eigenvalue 1 of T. We do not know else information for the set $\{z(\lambda) : 0 < \lambda < 1\}$. It is interesting to know if this set can generate the eigenspace of T corresponding to the (maximum) eigenvalue 1 of T.

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