WEIGHTED QUASI-VARIATIONAL INEQUALITIES AND CONSTRAINED NASH EQUILIBRIUM PROBLEMS

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Abstract. The weighted quasi-variational inequalities over product of sets (for short, WQVIP) and system of weighted quasi-variational inequalities (for short, SWQVI) are introduced. It is shown that these two problems are equivalent. A relationship between SWQVI and system of vector quasi-variational inequalities is given. The concept of normalized solutions of WQVIP and SWQVI is introduced. A relationship between solution (respectively, normalized solution) of SWQVI and solution of weighted constrained Nash equilibrium problem (respectively, normalized weight Nash equilibrium) is also given. The scalar quasi-equilibrium problem (for short, QEP), which includes WQVIP as a particular case, is also considered. By introducing the concept of densely pseudomonotonicity of bifunctions, the existence of a solution of QEP is established. As a consequence, existence results for solutions of WQVIP and constrained Nash equilibrium problems for vector valued functions are derived.

1. INTRODUCTION

The theory of vector variational inequalities which is started in 1980 by Giannessi [17], is one of the most important tools to study vector optimization problems; See, for example [10, 13, 18, 21, 27] and references therein. Goh and Yang [19]

applied weighted sum method (called scalarization method) to vector variational inequalities, presented new relationships between vector variational inequalities and vector optimization problems, and established sufficient and necessary conditions for reducing a vector variational inequality to a scalar variational inequality. The basic idea behind the weighted sum method is to scalarize a set of objectives into a single objective by multiplying each objective with a user-supplied weight.

Nash equilibrium problem [23] and constrained Nash equilibrium problem [24] also known as Debreu type equilibrium problem [14] are fundamental problems in the study of game theory and mathematical economics. In the last three decades, these problems have been extensively studied in the literature; See, for example [11, 29] and references therein. In the recent past, much attention has been paid on the game theory with vector payoffs. Specially, the study of existence of Pareto equilibria in game theory with vector payoffs has attracted much attention, for example, see [3, 12, 25, 26, 28, 30] and references therein. In the last decade, the weighted sum method is used to study the existence of solutions of Nash equilibrium problems and constrained Nash equilibrium problems; see, for example [3, 15, 26, 28] and references therein.

In the recent past, Nash equilibrium and constrained Nash equilibrium problems for vector valued functions have been studied by using systems of vector variational inequalities (for short, SVVI) and systems of vector quasi-variational inequalities (for short, SVQVI), respectively; See, for example [1, 4, 7, 8] and references therein. Ansari et al. [6] applied the weighted sum method to SVVI and introduced weighted variational inequalities over product of sets and system of weighted variational inequalities (for short, SWVI). It is noticed that the weighted variational inequality problem over product of sets and the problem of system of weighted variational inequalities are equivalent. They gave a relationship between SWVI and SVVI. They established several existence results for solutions of above mentioned problems under several kinds of weighted monotonicities. They also introduced the weighted generalized variational inequalities over product of sets, that is, weighted variational inequalities for multivalued maps and system of weighted generalized variational inequalities, and proved some existence results for their solutions. One of the main goals of this paper is to achieve the existence results for solutions of SVQVI and constrained Nash equilibrium problems for vector valued functions by appying weighted sum method.

The present paper is organized as follows. In the next section, we first recall the formulations of SVQVI, and then we define weighted quasi-variational inequality problem over product of sets (for short, WQVIP) and system of weighted quasi-variational inequalities (for short, SWQVI). We also introduce the concept of normalized solutions of WQVIP and SWQVI. We present a relationship between the solutions (respectively, normalized solutions) of above mentioned problems. Some

notations, definitions and results are also recalled. Section 3 deals with constrained Nash equilibrium problems for vector valued functions. We prove that normalized solution of SWQVI provides a sufficient condition for a weak Pareto equilibrium / Pareto equilibrium of constrained Nash equilibrium problem for vector valued functions. The last section is devoted to the existence theory of above mentioned problems. We first introduce the concept of densely pseudomonotonicity of a bifunction and then establish an existence result for a solution of (scalar) quasi-equilibrium problem which includes WQVIP as a special case. As a consequence, we derive existence results for solutions of WQVIP and constrained Nash equilibrium problems for vector valued functions.

2. Preliminaries and Formulations

For each given $m \in \mathbb{N}$, we denote by $\mathbb{R}_+^m$ the non-negative orthant of $\mathbb{R}^m$, that is,

$$\mathbb{R}_+^m = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m : u_j \geq 0, \text{ for } j = 1, \ldots, m\},$$

so that $\mathbb{R}_+^m$ has a nonempty interior with the topology induced in terms of convergence of vectors with respect to the Euclidean metric. That is,

$$\text{int } \mathbb{R}_+^m = \{u = (u_1, \ldots, u_m) \in \mathbb{R}^m : u_j > 0, \text{ for } j = 1, \ldots, m\}.$$

We denote by $\mathbb{T}_+^m$ and int $\mathbb{T}_+^m$ the simplex of $\mathbb{R}_+^m$ and its relative interior, respectively, that is,

$$\mathbb{T}_+^m = \{u = (u_1, \ldots, u_m) \in \mathbb{R}_+^m : \sum_{j=1}^m u_j = 1\}, \text{ and}$$

$$\text{int } \mathbb{T}_+^m = \{u = (u_1, \ldots, u_m) \in \text{int } \mathbb{R}_+^m : \sum_{j=1}^m u_j = 1\}.$$

Let I be a finite index set, that is, $I = \{1, \ldots, n\}$ and for each $i \in I$, let ℓ_i be a positive integer. For each $i \in I$, let X_i be a real topological vector space with its dual X_i^*, K_i a nonempty convex subset of X_i, $X = \prod_{i \in I} X_i$ and $K = \prod_{i \in I} K_i$. For each $x \in X$, $x_i \in X_i$ denotes the ith coordinate and we write $x = (x_i)_{i \in I}$. For each $i \in I$ and each $j = 1, \ldots, \ell_i$, let $f_{i_j} : K \to X_i^*$ be a map. For each $i \in I$, let $A_i : K \to 2^{K_i}$ be a multivalued map with nonempty values. For each $i \in I$ and for all $x, y \in K$, we denote

$$F_i(x) := \big(f_{i_1}(x), \ldots, f_{i_{\ell_i}}(x)\big)$$

and

$$\langle F_i(x), x_i - y_i \rangle := \big(\langle f_{i_1}(x), x_i - y_i \rangle, \ldots, \langle f_{i_{\ell_i}}(x), x_i - y_i \rangle\big),$$

where $\langle \cdot, \cdot \rangle$ denotes the continuous pairing between X_i^* and X_i.

We consider the following *systems of vector quasi-variational inequalities*:

$$\text{(SVQVI)} \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I, \ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ \langle F_i(\bar{x}), \bar{x}_i - y_i \rangle \notin \mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\}, \quad \text{for all } y_i \in A_i(\bar{x}), \end{cases}$$

where $\bar{x}_i$ is the ith component of $\bar{x}$ and $\mathbf{0}$ is the zero vector of $\mathbb{R}^{\ell_i}$.

$$\text{(SVQVI)}_w \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I, \ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ \langle F_i(\bar{x}), \bar{x}_i - y_i \rangle \notin \text{int } \mathbb{R}_+^{\ell_i}, \quad \text{for all } y_i \in A_i(\bar{x}). \end{cases}$$

It is clear that every solution of (SVQVI) is a solution of (SVQVI)$_w$, but the converse is not true in general. (SVQVI)$_w$ is introduced and studied by Ansari et al. [1] in a more general setting. They proved the existence of a solution of (SVQVI)$_w$ (for the infinite set I) by using maximal element theorems for a family of multivalued maps. They used (SVQVI)$_w$ as a tool to prove the existence of a solution of system of vector quasi-optimization problems which includes the constrained Nash equilibrium problem for vector valued functions.

Of course, if for each $i \in I$ and for all $x \in K$, $A_i(x) = K_i$, (SVQVI) and (SVQVI)$_w$ are called *systems of vector variational inequalities* studied by Ansari et al. [7]. They used these systems to prove the existence of a solution of Nash equilibrium problem for vector valued functions.

If for each $i \in I$, $\ell_i = 1$, then (SVQVI) and (SVQVI)$_w$ reduce to the problem of system of quasi-variational inequalities (for short, SQVI) considered and studied in [2, 29], see also references therein.

One of the main motivations of this paper is to study the existence of solutions of (SVQVI) and (SVQVI)$_w$ so that by using the technique of Ansari et al. [1] one can derive the existence results for a solution of constrained Nash equilibrium problem for vector valued functions. For this purpose, we introduce the following *weighted quasi-variational inequality problem* over product of sets (for short, WQVIP): find $\bar{x} \in K$ w.r.t. the weight vector $W = (W_1, \ldots, W_n) \in \prod_{i \in I} \left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\} \right)$ such that $\bar{x} \in A(\bar{x}) = \prod_{i \in I} A_i(\bar{x})$ and

$$\sum_{i \in I} W_i \cdot \langle F_i(\bar{x}), \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all } y_i \in A_i(\bar{x}), \ i \in I,$$

where $\cdot$ denotes the inner product on $\mathbb{R}^{\ell_i}$. The solution set of (WQVIP) is denoted by K^w.

We also introduce the following *problem of system of weighted quasi-variational*

inequalities:

$$(\text{SWQVI}) \quad \begin{cases} \text{Find } \bar{x} \in K \text{ w.r.t. the weight vector } W = (W_1, \ldots, W_n) \\ \text{such that for each } i \in I, \ W_i \in \mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\}, \ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ W_i \cdot \langle F_i(\bar{x}), \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all } y_i \in A_i(\bar{x}). \end{cases}$$

We denote by K_s^w the solution set of (SWQVI).

If for each $i \in I$, $W_i \in \mathbb{T}_+^{\ell_i}$, then the solution of (WQVIP) and (SWQVI) are called *normalized*, where $\mathbb{T}_+^{\ell_i}$ is the simplex of $\mathbb{T}^{\ell_i}$. The normalized solutions sets of (WQVIP) and (SWQVI) are denoted by K_n^w and K_{sn}^w, respectively.

If for each $i \in I$ and for all $x \in K$, $A_i(x) = K_i$ then (WQVIP) and (SWQVI) reduce to weighted variational inequality problem over product of sets (for short, WVIP) and system of weighted variational inequalities (for short, SWVI), respectively. These problems are introduced and studied by Ansari et al. [6]. They have pointed out that (WVIP) and (SWVI) are equivalent. The concept of a normalized solution of (SWVI) is introduced and its relationship with the solution of (SVVI) is given. They defined several kinds of weighted monotonicities and established several existence results for solutions of (WVIP), (SWVI), (SVVI)$_w$ and (SVVI).

The following lemma shows that the solution sets of (WQVIP) and (SWQVI) are equal.

Lemma 2.1. *For a given weight vector* $W = (W_1, \ldots, W_n) \in \prod_{i \in I} \left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\} \right)$ *(respectively,* $W = (W_1, \ldots, W_n) \in \prod_{i \in I} \mathbb{T}_+^{\ell_i}$*),* $K^w = K_s^w$ *(respectively,* $K_n^w = K_{sn}^w$*).*

Proof. Obviously, $K_s^w \subseteq K^w$.

The converse part can be easily proved by using the arguments as in the proof of Lemma 1 in [9]. ∎

Next we establish the following lemma which shows that (SVQVI)$_w$ or (SVQVI) can be solved by using (SWQVI).

Lemma 2.2. *Each normalized solution* $\bar{x} \in K$ *with weight vector* $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n \mathbb{T}_+^{\ell_i}$ *(respectively,* $W = (W_1, \ldots, W_n) \in \prod_{i=1}^n (int\ \mathbb{T}_+^{\ell_i})$*) of (SWQVI) is a solution of (SVQVI)$_w$ (respectively, (SVQVI)).*

Proof. It is similar to the proof of Lemma 2.2 in [6] and therefore we omit it. ∎

In view of Lemmas 2.1 and 2.2, we have the following result.

Lemma 2.3. *Each normalized solution $\bar{x} \in K$ with weight vector $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} \mathbb{T}_+^{\ell_i}$ (respectively, $W = (W_1, \ldots, W_n) \in \prod_{i=1}^{n} (int\ \mathbb{T}_+^{\ell_i}))$ of (WQVIP) is a solution of $(SVQVI)_w$ (respectively, (SVQVI)).*

We close this section by recalling known definition and a result.

Definition 2.1. Let U be a nonempty subset of a topological vector space E. A multivalued map $T : U \to 2^U$ is said to be a *KKM map* provided $co(M) \subseteq T(M) = \bigcup_{x \in M} T(x)$ for each finite subset M of U, where $co(M)$ denotes the convex hull of M.

The following Fan-KKM theorem [16] will be used in the sequel.

Theorem 2.1. [16] *Let U be a nonempty subset of a Hausdorff topological vector space E. Assume that $T : U \to 2^U \setminus \{\emptyset\}$ is a KKM map satisfying the following conditions:*

(i) *For each $x \in U$, $T(x)$ is closed;*

(ii) *For at least one $x \in U$, $T(x)$ is compact.*

Then $\bigcap_{x \in U} T(x) \neq \emptyset$.

3. CONSTRAINED NASH EQUILIBRIUM PROBLEMS

For each $i \in I$, let $\Phi_i = (\phi_{i_1}, \phi_{i_2}, \ldots \phi_{i_{\ell_i}}) : K \to \mathbb{R}^{\ell_i}$ be a vector valued function and let $K^i = \prod_{j \in I,\ j \neq i} K_j$ and we write $K = K^i \times K_i$. For $x \in K$, x^i denotes the projection of x onto K^i and hence we write $x = (x^i, x_i)$. For each $i \in I$, let A_i be the same as defined in the previous section.

The *constrained Nash equilibrium problems* for vector valued functions are defined as follows:

$$(\text{CNEP}) \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I, \ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ \Phi_i(\bar{x}^i, \bar{x}_i) - \Phi_i(\bar{x}^i, y_i) \notin \mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\}, \quad \text{for all } y_i \in A_i(\bar{x}). \end{cases}$$

$$(\text{CNEP})_w \quad \begin{cases} \text{Find } \bar{x} \in K \text{ such that for each } i \in I, \ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\ \Phi_i(\bar{x}^i, \bar{x}_i) - \Phi_i(\bar{x}^i, y_i) \notin \text{int } \mathbb{R}_+^{\ell_i}, \quad \text{for all } y_i \in A_i(\bar{x}). \end{cases}$$

A solution $\bar{x} \in K$ of (CNEP) (respectively, $(\text{CNEP})_w$) is called a *Pareto equilibrium* (respectively, a *weak Pareto equilibrium*).

It is clear that each Pareto equilibrium is certainly a weak Pareto equilibrium, but converse may not be true.

Now we define the following *weighted constrained Nash equilibrium problem:*

(WCNEP)
$$
\begin{cases}
\text{Find } \bar{x} \in K \text{ w.r.t. the weight vector } W = (W_1, \ldots, W_n) \\[4pt]
\text{such that for each } i \in I, \ W_i \in \mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\}, \ \bar{x}_i \in A_i(\bar{x}) \text{ and} \\[4pt]
W_i \cdot \Phi_i(\bar{x}^i, \bar{x}_i) \leq W_i \cdot \Phi_i(\bar{x}^i, y_i), \quad \text{for all } y_i \in A^i(\bar{x}),
\end{cases}
$$

where $\cdot$ denotes the inner product on $\mathbb{R}^{\ell_i}$. In particular, when $W_i \in \mathbb{T}_+^{\ell_i}$ for each $i \in I$, the solution $\bar{x} \in K$ of (WCNEP) is called a *normalized weight Nash equilibrium* w.r.t. the weight vector W.

The following lemma of Ding [15] tells us the relationships between solutions of (WCNEP) and (CNEP), and solutions of (WCNEP) and $(\text{CNEP})_w$ under certain circumstances.

Lemma 3.1. [16] *Each normalized weight Nash equilibrium $\bar{x} \in K$ w.r.t. the weight vector $W = (W_1, \ldots, W_n) \in \mathbb{T}_+^{\ell_1} \times \ldots \times \mathbb{T}_+^{\ell_n}$ (respectively, $W = (W_1, \ldots, W_n) \in \text{int } \mathbb{T}_+^{\ell_1} \times \ldots \times \text{int } \mathbb{T}_+^{\ell_n}$) of (WCNEP) is a weak Pareto equilibrium, that is, a solution of $(\text{CNEP})_w$ (respectively, a Pareto equilibrium, that is, a solution of (CNEP)).*

To show that every solution of (SWQVI) is a solution of (WCNEP), we recall the following definitions.

Definition 3.1. [31] For each $i \in I$, let X_i and Z be normed spaces. The function $\varphi : X = \prod_{i \in I} X_i \to Z$ is said to be *partial Gâteaux differentiable* at $x = (x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n) \in X$ w.r.t. the jth variable x_j if

$$
\langle D_{x_j} \varphi(x), h_j \rangle = \lim_{t \to 0} \frac{\varphi(x_1, \ldots, x_{j-1}, x_j + th_j, x_{j+1}, \ldots, x_n) - \varphi(x)}{t} \text{ exists,}
$$

for all $h_j \in X_j$. $D_{x_j} \varphi(x) \in L(X_j, Z)$ is called *partial Gâteaux derivative of φ at $x \in X$ w.r.t. the jth variable x_j*, where $L(X_j, Z)$ denotes the space of all continuous linear maps from X_i to Z.

φ is called *partial Gâteaux differentiable on X* if it is partial Gâteaux differentiable at each point of X w.r.t. each variable.

If the partial Gâteaux derivatives $D_{x_j} \varphi(x)$ exist for each $j = 1, \ldots, n$ and for all $x \in K \subseteq X$, then the mapping

$$
D_{x_j} \varphi : K \subseteq X \to L(X_j, Z) \quad \text{defined by} \quad x \mapsto D_{x_j} \varphi(x)
$$

for each $j = 1, \ldots, n$, is called the *partial Gâteaux derivative of φ on K* (see, pp. 135 in [31]).

Definition 3.2. Let M a nonempty convex subset of a normed space E. A Gâteaux differentiable function $\varphi : M \to \mathbb{R}$ is said to be *convex* if and only if for all $x, y \in M$,

$$\varphi(y) - \varphi(x) - \langle D_x \varphi(x), y - x \rangle \geq 0,$$

where $D_x \varphi(x)$ denotes the Gâteaux derivative of φ at x.

Proposition 3.1. *For each $i \in I$, let X_i be a normed space, K_i a nonempty, open and convex subset of X_i and $A_i : K \to 2^{K_i}$ a multivalued map with nonempty convex values. For each $i \in I$ and each $j = 1, \ldots, \ell_i$, let $\phi_{i_j} : K \to \mathbb{R}$ be partial Gâteaux differentiable on K and convex in each argument. Then $\bar{x} \in K$ is a solution of (SWQVI) w.r.t. the weight vector $W = (W_1, \ldots, W_n) \in \prod_{i \in I} \left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\} \right)$ (respectively, normalized solution of (SWQVI) w.r.t. the weight vector $W \in \prod_{i \in I} \mathbb{T}_+^{\ell_i}$) with $f_{i_j}(x) = D_{x_i} \phi_{i_j}(x)$ for each $i \in I$, $j = 1, \ldots, \ell_i$ and for all $x \in K$ if and only if it is a solution of (WCNEP) w.r.t. the same weight vector W (respectively, normalized weight Nash equilibrium w.r.t. the same weight vector $W \in \prod_{i \in I} \mathbb{T}_+^{\ell_i}$).*

Proof. For the sake of simplicity, for each $i \in I$ and for all $x, y \in K$, we denote

$$D_{x_i} \Phi_i(x) := \left(D_{x_i} \phi_{i_1}(x), \ldots, D_{x_i} \phi_{i_{\ell_i}}(x) \right)$$

and

$$\langle D_{x_i} \Phi_i(x), x_i - y_i \rangle := \left(\langle D_{x_i} \phi_{i_1}(x), x_i - y_i \rangle, \ldots, \langle D_{x_i} \phi_{i_{\ell_i}}(x), x_i - y_i \rangle \right).$$

Suppose that $\bar{x} \in K$ is a solution of (SWQVI) w.r.t. the weight vector $W = (W_1, \ldots, W_n) \in \prod_{i \in I} \left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\} \right)$ with $F_i(\bar{x}) = D_{\bar{x}_i} \Phi_i(\bar{x})$. Then for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$W_i \cdot \langle D_{\bar{x}_i} \Phi_i(\bar{x}), \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all } y_i \in A_i(\bar{x}).$$

Since for each $i \in I$ and each $j = 1, \ldots \ell_i$, ϕ_{i_j} is convex in each argument, we have

$$\Phi_i(\bar{x}^i, y_i) - \Phi_i(\bar{x}^i, \bar{x}_i) + \langle D_{\bar{x}_i} \Phi_i(\bar{x}), \bar{x}_i - y_i \rangle \in \mathbb{R}_+^{\ell_i}.$$

This implies that

$$W_i \cdot \Phi_i(\bar{x}^i, y_i) - W_i \cdot \Phi_i(\bar{x}^i, \bar{x}_i) + W_i \cdot \langle D_{\bar{x}_i} \Phi_i(\bar{x}), \bar{x}_i - y_i \rangle \geq 0,$$

that is, for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$0 \geq W_i \cdot \langle D_{\bar{x}_i} \Phi_i(\bar{x}), \bar{x}_i - y_i \rangle \geq W_i \cdot \Phi_i(\bar{x}^i, \bar{x}_i) - W_i \cdot \Phi_i(\bar{x}^i, y_i).$$

Hence $\bar{x} \in K$ is a solution of (WCNEP).

Conversely, let $\bar{x} \in K$ be a solution of (WCNEP) w.r.t. the weight vector $W = (W_1, \ldots, W_n) \in \prod_{i \in I} \left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\} \right)$. Then for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$ and

$$W_i \cdot \Phi_i(\bar{x}^i, y_i) - W_i \cdot \Phi_i(\bar{x}^i, \bar{x}_i) \geq 0, \quad \text{for all } y_i \in A_i(\bar{x}).$$

Since $\bar{x}_i, y_i \in A_i(\bar{x})$ and $A_i(\bar{x})$ is convex, we have $\bar{x}_i + t(y_i - \bar{x}_i) \in A_i(\bar{x})$ for all $t \in [0, 1]$. Therefore, we have

$$W_i \cdot \Phi_i(\bar{x}^i, \bar{x}_i + t(y_i - \bar{x}_i)) - W_i \cdot \Phi_i(\bar{x}^i, \bar{x}_i) \geq 0$$

or

$$W_i \cdot \left[\Phi_i(\bar{x}^i, \bar{x}_i + t(y_i - \bar{x}_i)) - \Phi_i(\bar{x}^i, \bar{x}_i) \right] \geq 0.$$

This implies that

$$W_i \cdot \left[\lim_{t \to 0} \frac{\Phi_i(\bar{x}^i, \bar{x}_i + t(y_i - \bar{x}_i)) - \Phi_i(\bar{x}^i, \bar{x}_i)}{t} \right] \geq 0$$

and thus

$$W_i \cdot \langle D_{\bar{x}_i} \Phi_i(\bar{x}), y_i - \bar{x}_i \rangle \geq 0 \text{ or } W_i \cdot \langle D_{\bar{x}_i} \Phi_i(\bar{x}), \bar{x}_i - y_i \rangle \leq 0, \quad \text{for all } y_i \in A_i(\bar{x}),$$

which means that $\bar{x} \in K$ is a solution of (SWQVI). ■

Remark 3.1.

(a) The convexity assumption on $A_i(x)$ is not needed for the first part of above proposition.

(b) Under the assumption of Proposition 3.1, it is easy to show that $\bar{x} \in K$ is a normalized solution of (SWQVI) w.r.t. the weight vector $W \in \prod_{i \in I} \left(\text{int } \mathbb{T}_+^{\ell_i} \right)$ with $F_i(x) = D_{x_i} \Phi_i(x)$ if and only if it is a normalized weight Nash equilibrium w.r.t. weight vector $W \in \prod_{i \in I} \left(\text{int } \mathbb{T}_+^{\ell_i} \right)$.

From Lemma 3.1, Proposition 3.1 and Remark 3.1 (b) we have the following result.

Proposition 3.2. *For each $i \in I$, let X_i be a normed space, K_i a nonempty, open and convex subset of X_i and $A_i : K \to 2^{K_i}$ a multivalued map with nonempty convex values. For each $i \in I$ and each $j = 1, \ldots, \ell_i$, let $\phi_{i_j} : K \to \mathbb{R}$ be partial Gâteaux differentiable on K and convex in each argument. Then every normalized solution $\bar{x} \in K$ of (SWQVI) w.r.t. the weight vector $W \in \prod_{i \in I} \mathbb{T}_+^{\ell_i}$ (respectively, $W \in \prod_{i \in I}(\text{int } \mathbb{T}_+^{\ell_i})$) with $F_i(x) = D_{x_i} \Phi_i(x)$ is a weak Pareto equilibrium, that is, a solution of $(CNEP)_w$ w.r.t. the same weight vector W (respectively, a Pareto equilibrium, that is, a solution of $(CNEP)$ w.r.t. the same weight vector W).*

4. Existence Results

In view of Lemma 2.3 and Proposition 3.2, the existence of a solution of (WQVIP) provides the existence of solution of (SWQVI) as well as the solution of constrained Nash equilibrium problems. So, this section deals with the existence of solution of (WQVIP) under densely pseudomonotonicity assumption.

Let U be a nonempty convex subset of a real Hausdorff topological vector space E. Let $B : U \to 2^U$ be a multivalued map with nonempty values and $\psi : U \times U \to \mathbb{R}$ a bifunction. We consider the following more general problem, known as *quasi-equilibrium problem* which contains (WQVIP) as a special case.

$$(\text{QEP}) \quad \begin{cases} \text{Find } \bar{x} \in U \text{ such that } \bar{x} \in B(\bar{x}) \text{ and} \\ \psi(\bar{x}, y) \geq 0, \quad \text{for all } y \in B(\bar{x}). \end{cases}$$

Definition 4.1. [22] A subset U^0 of U is said to be *segment-dense in U* if for all $x \in U$, there can be found $x^0 \in U^0$ such that x is a cluster point of the set $[x, x^0] \cap U^0$, where $[x, x^0]$ denotes the line segment joining x and x^0 including end points.

Now we extend the notion of densely pseudomonotonicity introduced by Luc [22] to bifunctions.

Definition 4.2. A bifunction $\psi : U \times U \to \mathbb{R}$ is said to be

(i) *pseudomonotone on U* if for all $x, y \in U$, we have

$$\psi(x, y) \geq 0 \quad \text{implies} \quad \psi(y, x) \leq 0;$$

(ii) *densely pseudomonotone on U* if there exists a segment-dense set U^0 in U such that ψ is pseudomonotone on U^0.

Definition 4.3. A function $g : U \to \mathbb{R}$ is said to be *hemicontinuous* if for all $x, y \in U$, the map $t \mapsto g(y + t(x - y))$ is continuous, that is, g is continuous along the line segment. Upper and lower hemicontinuity are defined analogously.

Lemma 4.1. *Let U be a convex subset of E and U^0 a segment-dense set in U. Let $B : U \to 2^U$ be a multivalued map with nonempty values. If $\psi : U \times U \to \mathbb{R}$ is lower hemicontinuous in the first argument, then the following problems are equivalent:*

$$(\text{DQEP}) \quad \begin{cases} \text{Find } \bar{x} \in U \text{ such that } \bar{x} \in B(\bar{x}) \text{ and} \\ \psi(y, \bar{x}) \leq 0, \quad \text{for all } y \in B(\bar{x}) \end{cases}$$

and

$$(DQEP)^0 \quad \begin{cases} \textit{Find } \bar{x} \in U \textit{ such that } \bar{x} \in B(\bar{x}) \textit{ and} \\ \psi(y, \bar{x}) \leq 0, \quad \textit{for all } y \in B^0(\bar{x}), \end{cases}$$

where $B^0 : U \to 2^{U^0}$ is a multivalued map with nonempty values such that $B^0(x) = B(x) \cap U^0$ for all $x \in U$.

The solution sets of (DQEP) and (DQEP)^0 are denoted by U_d and U_d^0, respectively.

Proof. Obviously, $U_d \subseteq U_d^0$.

To prove the reverse implication, let $\bar{x} \in U$ be a solution of (DQEP)^0. Then

(4.1) $\qquad \bar{x} \in B(\bar{x})$ and $\psi(y, \bar{x}) \leq 0, \quad$ for all $y \in B^0(\bar{x})$.

Since for each $x \in U$, $B^0(x) = B(x) \cap U^0 \subseteq B(x) \cap U$ and U^0 is segment-dense set in U, we have $B(x) \cap U^0$ is segment-dense in $B(x) \cap U$, that is, $B^0(x)$ is segment-dense in $B(x)$ for all $x \in U$. Then for each $z \in B(\bar{x})$, we can find $z^0 \in B^0(\bar{x})$ and a net $\{z_\alpha\}_{\alpha \in \Lambda}$ in $[z, z^0] \cap B^0(\bar{x})$ converging to z. From (4.1), we get

$$\psi(z_\alpha, \bar{x}) \leq 0, \quad \text{for all } \alpha \in \Lambda.$$

Since $\psi(.,.)$ is lower hemicontinuous in the first argument and z_α converges to z, we have

$$\psi(z, \bar{x}) \leq 0, \quad \text{for all } z \in B(\bar{x}).$$

Hence $U_d^0 \subseteq U_d$. ∎

Now we will show that (DQEP) is equivalent to (QEP) under certain assumptions.

Lemma 4.2. *Let U be a convex subset of E and U^0 a segment-dense set in U. Let $B : U \to 2^U$ be a multivalued map with nonempty convex values. If $\psi : U \times U \to \mathbb{R}$ is densely pseudomonotone, hemicontinuous in the first argument, and semistrictly quasiconvex in the second argument such that $\psi(x, x) = 0$ for all $x \in U$. Then (DQEP) is equivalent to (QEP).*

Proof. Let $\bar{x} \in U$ be a solution of (QEP). Then by densely pseudomonotonicity of ψ, $\bar{x} \in U$ is a solution of (DQEP)^0. Lemma 4.1 implies that $\bar{x} \in U$ is a solution of (DQEP).

For the converse, let $\bar{x} \in U$ be a solution of (DQEP). Then

$$\bar{x} \in B(\bar{x}) \quad \text{and} \quad \psi(y, \bar{x}) \leq 0, \quad \text{for all } y \in B(\bar{x}).$$

Since for all $x \in U$, $B(x)$ is convex, we have $z_t = \bar{x} + t(y - \bar{x}) \in B(\bar{x})$ for all $t \in [0, 1]$. Then, in particular, we have $\psi(z_t, \bar{x}) \leq 0$. Since ψ is semistrictly quasiconvex in the second argument, we obtain

$$\psi(z_t, z_t) \leq \max\{\psi(z_t, y), \ \psi(z_t, \bar{x})\}.$$

If $\psi(z_t, y) < \psi(z_t, \bar{x})$ then we obtain $\psi(z_t, \bar{x}) = 0$ and therefore $\psi(z_t, z_t) < 0$ in view of semistrictly quasiconvexity. However, this contradicts to our assumption that $\psi(z_t, z_t) = 0$. Hence $\psi(z_t, y) \geq 0$ and by hemicontinuity of ψ in the first argument, we obtain $\psi(\bar{x}, y) \geq 0$ for all $y \in B(\bar{x})$. Hence $\bar{x} \in U$ is a solution of (QEP). ∎

Now we are ready to prove the existence of a solution of (QEP).

Theorem 4.1. *Let U be a convex subset of E and U^0 a segment-dense set in U. Let $B : U \to 2^U$ be a multivalued map with nonempty convex values such that $B^{-1}(y^0)$ is open in U for all $y^0 \in U^0$. Let the set $F = \{x \in U : x \in B(x)\}$ be nonempty and closed . Let $\psi : U \times U \to \mathbb{R}$ be densely pseudomonotone, hemicontinuous in the first argument, and explicitly quasiconvex (i.e., semistrictly quasiconvex and quasiconvex) and lower semicontinuous in the second argument such that $\psi(x, x) = 0$ for all $x \in U$. Assume there exist a nonempty compact subset $V \subseteq U$ and $\tilde{y} \in U$ such that for all $x \in U \setminus V$, $\tilde{y} \in B(x)$ and $\psi(x, \tilde{y}) < 0$. Then (QEP) has a solution.*

Proof. For all $x \in U$, we define two multivalued maps $P_1, P_2 : U \to 2^U$ by

$$P_1(x) = \{y \in U : \psi(x, y) < 0\} \quad \text{and} \quad P_2(x) = \{y \in U : \psi(y, x) > 0\}.$$

For all $x, y \in U$ and for each $i = 1, 2$, we also define $Q_i : U \to 2^U$ and $T_i : U \to 2^U$ by

$$Q_i(x) = \begin{cases} B(x) \cap P_i(x), & \text{if} \quad x \in F \\ B(x), & \text{if} \quad x \in U \setminus F \end{cases}$$

and

$$T_i(y) = U \setminus Q_i^{-1}(y).$$

For each $i = 1, 2$ and for all $y \in U$, we have (see, for Example [20])

$$Q_i^{-1}(y) = \left[(U \setminus F) \cup P_i^{-1}(y) \right] \cap B^{-1}(y)$$

and therefore

$$T_i(y) = \left[F \cap (U \setminus P_i^{-1}(y)) \right] \cup \left[U \setminus B^{-1}(y) \right].$$

Rest of the proof is divided into the following five steps.

(a) We show that T_1 is a KKM map on U.

Assume to the contrary that T_1 is not a KKM map on U. Then there exist a finite set $\{y_1, \ldots, y_n\}$ in U and $t_1, \ldots, t_n \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that $\hat{y} = \sum_{i=1}^n t_i y_i \notin T_1(y_i)$ for all $i = 1, \ldots, n$, that is,

$$\hat{y} \in Q_1^{-1}(y_i) \quad \Leftrightarrow \quad y_i \in Q_1(\hat{y}) \quad \text{for all } i = 1, \ldots, n.$$

If $\hat{y} \in F$, then $Q_1(\hat{y}) = B(\hat{y}) \cap P_1(\hat{y})$ and therefore

$$y_i \in P_1(\hat{y}) \quad \text{and} \quad y_i \in B(\hat{y}) \quad \text{for all } i = 1, \ldots, n.$$

Hence

$$\psi(\hat{y}, y_i) < 0 \quad \text{and} \quad y_i \in B(\hat{y}) \quad \text{for all } i = 1, \ldots, n.$$

Since ψ is semistrictly quasiconvex in the second argument, it is also quasiconvex in the second argument. Thus from the previous inequality we deduce that

$$0 = \psi(\hat{y}, \hat{y}) = \psi\left(\hat{y}, \sum_{i=1}^n t_i y_i\right) < 0,$$

a contradiction.

If $\hat{y} \in U \setminus F$, then $\hat{y} \notin B(\hat{y})$. By the definition of T_1, we have $Q_1(\hat{y}) = B(\hat{y})$ and therefore $y_i \in Q_1(\hat{y}) = B(\hat{y})$ for all $i = 1, \ldots, n$. Since $B(\hat{y})$ is convex, we obtain $\hat{y} \in B(\hat{y})$, again a contradiction. Hence T_1 is a KKM map.

(b) We show that $T_1(\tilde{y}) = U \setminus Q_1^{-1}(\tilde{y}) \subseteq V$, where $\tilde{y}$ and V are the same as in the hypothesis.

Indeed if $x \in T_1(\tilde{y}) \setminus V$, then $x \in \left[F \cap (U \setminus P_1^{-1}(\tilde{y}))\right] \cup \left[U \setminus B^{-1}(\tilde{y})\right]$, that is, either $x \in F \cap (U \setminus P_1^{-1}(\tilde{y}))$ or $x \in U \setminus B^{-1}(\tilde{y})$.

If $x \in F \cap (U \setminus P_1^{-1}(\tilde{y}))$ then $x \in F$ and $x \in U \setminus P_1^{-1}(\tilde{y})$, that is, $x \in B(x)$ and $\psi(x, \tilde{y}) \geq 0$, a contradiction of our assumption that $\psi(x, \tilde{y}) < 0$.

If $x \in U \setminus B^{-1}(\tilde{y})$, then $x \notin B^{-1}(\tilde{y})$ if and only if $\tilde{y} \notin B(x)$, again a contradiction of our assumption that $\tilde{y} \in B(x)$. Hence $T_1(\tilde{y}) \subseteq V$.

(c) We claim that $\bigcap_{y \in U} \overline{T_1(y)} \neq \emptyset$.

Since V is compact, $\overline{T_1(\tilde{y})}$ is also compact. Moreover, $\mathrm{co}\{y_1, \ldots, y_n\} \subseteq \bigcup_{i=1}^n T_1(y_i) \subseteq \bigcup_{i=1}^n \overline{T_1(y_i)}$ for each finite subset $\{y_1, \ldots, y_n\}$ of U. Then by Theorem 2.1, we get $\bigcap_{y \in U} \overline{T_1(y)} \neq \emptyset$.

(d) Next we show that $\bigcap_{y \in U^0} \overline{T_1(y)} \subseteq \bigcap_{y \in U^0} T_2(y)$. Let $z \in \bigcap_{y \in U^0} \overline{T_1(y)}$, then $z \in \overline{T_1(y)}$ for all $y \in U^0$. For an arbitrary element $y^0 \in U^0$, we have to show that $z \in T_2(y^0)$.

Since $z \in \overline{T_1(y^0)}$, there exists a net $\{z_\alpha\}_{\alpha \in \Lambda} \subseteq T_1(y^0)$ such that $\{z_\alpha\}$ converges to z. Since $\{z_\alpha\}_{\alpha \in \Lambda} \subseteq T_1(y^0)$, we have

$$\{z_\alpha\}_{\alpha \in \Lambda} \subseteq U \setminus Q_1^{-1}(y^0) = \left[F \cap (U \setminus P_1^{-1}(y^0))\right] \cup \left[U \setminus B^{-1}(y^0)\right].$$

Then either $\{z_\alpha\} \subseteq F \cap (U \setminus P_1^{-1}(y^0))$ or $\{z_\alpha\} \subseteq U \setminus B^{-1}(y^0)$.

If $\{z_\alpha\} \subseteq F \cap (U \setminus P_1^{-1}(y^0))$, then $\{z_\alpha\} \subseteq F$ and $\{z_\alpha\} \subseteq U \setminus P_1^{-1}(y^0)$. It follows that $\{z_\alpha\} \subseteq F$ and $\psi(z_\alpha, y^0) \geq 0$. Since F is closed and $z_\alpha \to z$, we have $z \in F$, that is, $z \in B(z)$. By densely pseudomonotonicity of ψ, we obtain

$$z \in B(z) \quad \text{and} \quad \psi(y^0, z_\alpha) \leq 0.$$

By lower semicontinuity of ψ in the second argument, we get $\psi(y^0, z) \leq 0$. This implies that $z \in B(z)$ and $y^0 \notin P_2(z)$, that is, $z \notin (U \setminus F) \cup P_2^{-1}(y^0)$ and hence $z \notin Q_2^{-1}(y^0)$. Therefore, $z \in B(z)$ and $z \in U \setminus Q_2^{-1}(y^0) = T_2(y^0)$.

Let $\{z_\alpha\} \subseteq U \setminus B^{-1}(y^0)$. Since $B^{-1}(y^0)$ is open in U for all $y^0 \in U^0$, $U \setminus B^{-1}(y^0)$ is closed in U. Since $z_\alpha \to z$, we have $z \in U \setminus B^{-1}(y^0)$. Hence $z \notin B^{-1}(y^0) \Leftrightarrow y^0 \notin B(z)$ which implies that

$$y^0 \notin Q_2(z) \Leftrightarrow z \notin Q_2^{-1}(y^0) \Leftrightarrow z \in U \setminus Q_2^{-1}(y^0) \Leftrightarrow z \in T_2(y^0).$$

(e) From the above step and Lemma 4.1, we have

$$\bigcap_{y \in U} \overline{T_1(y)} \subseteq \bigcap_{y \in U^0} \overline{T_1(y)} \subseteq \bigcap_{y \in U^0} T_2(y) = \bigcap_{y \in U} T_2(y).$$

From (c), we get $\bigcap_{y \in U} T_2(y) \neq \emptyset$. Hence there exists $\bar{x} \in U$ such that

$$\bar{x} \in \bigcap_{y \in U} \left[U \setminus Q_2^{-1}(y)\right] = U \setminus \bigcup_{y \in U} Q_2^{-1}(y).$$

This implies that $Q_2(\bar{x}) = \emptyset$. Now two possibilities are there.

If $\bar{x} \in U \setminus F$, then $Q_2(\bar{x}) = B(\bar{x}) \neq \emptyset$, a contradiction. Otherwise, $\bar{x} \in F$ then $\emptyset = Q_2(\bar{x}) = B(\bar{x}) \cap P_2(\bar{x})$. Therefore, $\bar{x} \in B(\bar{x})$ such that $\psi(y, \bar{x}) \leq 0$ for all $y \in B(\bar{x})$. From Lemma 4.2, $\bar{x} \in U$ is a solution of (QEP). ∎

Remark 4.1.

(a) If B is a closed map then the set $F = \{x \in U : x \in B(x)\}$ is closed.

(b) In most of the papers appeared in the literature on quasi-equilibrium problems and their generalizations (see, for example [1-3, 9, 20, 29] and references therein), $B^{-1}(y)$ is open in U for all $y \in U$ is assumed. But in Theorem 4.1, we assumed that $B^{-1}(y)$ is open in U only for all $y \in U^0$.

Now we derive the existence results for a solution of (WQVIP).

Definition 4.4. Let $W = (W_1, \ldots, W_n) \in \prod_{i \in I} \left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\}\right)$ be a weight vector. A family $\{f_{i_j}\}_{i \in I, \, j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ is said to be

(i) *weighted monotone w.r.t. the weight vector W* if for all $x, y \in K$, we have

$$\sum_{i \in I} W_i \cdot \langle F_i(x) - F_i(y), x_i - y_i \rangle \geq 0;$$

(ii) *weighted pseudomonotone w.r.t. the weight vector W* if for all $x, y \in K$, we have

$$\sum_{i \in I} W_i \cdot \langle F_i(x), y_i - x_i \rangle \geq 0 \implies \sum_{i \in I} W_i \cdot \langle F_i(y), y_i - x_i \rangle \geq 0;$$

(iii) *weighted densely pseudomonotone on K* if there exists a segment-dense subset $K^0 \subseteq K$ such that the family $\{f_{i_j}\}_{i \in I, \, j=1}^{\ell_i}$ is weighted pseudomonotone on K^0.

(iv) *weighted hemicontinuous w.r.t. the weight vector W* if for all $x, y \in K$ and $t \in [0, 1]$, the mapping $t \mapsto \sum_{i \in I} W_i \cdot \langle F_i(y + t(x-y)), y_i - x_i \rangle$ is continuous.

Theorem 4.2. *Let $W = (W_1, \ldots, W_n) \in \prod_{i \in I} \left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\}\right)$ be a given weight vector. For each $i \in I$, let K_i be a nonempty convex subset of a Housdorff topological vector space X_i, K_i^0 a segment-dense set in K and $A_i : K \to 2^{K_i}$ a multivalued map with nonempty convex values such that $A_i^{-1}(y_i^0)$ is open for all $y_i^0 \in K_i^0$. Let $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$ such that the set $F = \{x \in K : x \in A(x)\}$ is nonempty and closed. Let the family $\{f_{i_j}\}_{i \in I, \, j=1}^{\ell_i}$ of functions $f_{i_j} : K \to X_i^*$ be weighted hemicontinuous and weighted densely pseudomonotone w.r.t. the same weight vector W on K. Assume there exist a nonempty compact subset $D \subseteq K$ and $\tilde{y} \in K$ such that for all $x \in K \setminus D$, $\tilde{y} \in A(x)$ and $\sum_{i \in I} W_i \cdot \langle F_i(x), x_i - \tilde{y}_i \rangle > 0$. Then (WQVIP) has a solution.*

Proof. For each $x, y \in K$, define $\psi : K \times K \to \mathbb{R}$ by

$$\psi(x, y) = \sum_{i \in I} W_i \cdot \langle F_i(x), y_i - x_i \rangle$$

and $B : K \to 2^K$ by $B(x) = A(x)$. Then the hemicontinuity, densely pseudomonotonicity of ψ and the coercivity condition of Theorem 4.1 follow directly from the assumptions. Since for each $i \in I$ and for all $x \in K$, $A_i(x)$ is nonempty convex, we have $B(x) = A(x) = \prod_{i \in I} A_i(x)$ is nonempty convex. Also, since for all $y_i^0 \in K_i^0$, $A^{-1}(y^0) = \bigcap_{i \in I} A_i^{-1}(y_i^0)$ and $A_i^{-1}(y_i^0)$ is open in K for each $i \in I$ and for all $y_i^0 \in K_i^0$, we obtain $A^{-1}(y^0)$ is open in K. This implies that $B^{-1}(y^0)$ is open in K for all $y^0 \in K^0$. Hence all the conditions of Theorem 4.1 are satisfied and thus the conclusion follows from Theorem 4.1. ∎

Remark 4.2.

(a) Theorem 4.2 is an extension and generalization of Theorem 3.1 in [5] for (WQVIP) under densely pseudomonotonicity assumption.

(b) Theorem 4.2 along with Lemma 2.3 provides the existence of solutions of (SVQVI) and (SVQVI)$_w$ under densely pseudomonotonicity assumption.

(c) Under the hypotheses of Theorem 4.2 with the weight vector $W \in \prod_{i \in I} \mathbb{T}_+^{\ell_i}$, there exists a normalized solution of (WQVIP) and hence normalized solution of (SWQVI).

If we consider the scalar case and I is a singleton set, then we derive the following result for the existence of a solution of quasi-variational inequality problem.

Corollary 4.1. *Let K be a nonempty convex subset of a Housdorff topological vector space X, K^0 a segment-dense set in K and $A : K \to 2^K$ a multivalued map with nonempty convex values such that $A^{-1}(y^0)$ is open in K for all $y^0 \in K^0$. Let the set $F = \{x \in K : x \in A(x)\}$ be nonempty and closed. Let $f : K \to X^*$ be hemicontinuous and densely pseudomonotone on K. Assume there exist a nonempty compact subset $D \subseteq K$ and $\tilde{y} \in K$ such that for all $x \in K \setminus D$, $\tilde{y} \in A(x)$ and $\langle f(x), x - \tilde{y} \rangle > 0$. Then there exists a solution of the quasi-variational inequality problem, that is, $\bar{x} \in K$ such that $\bar{x} \in A(\bar{x})$ and*

$$\langle f(\bar{x}), \bar{x} - y \rangle \leq 0, \quad \text{for all } y \in A(\bar{x}).$$

Remark 4.3.

(a) Best our knowledge, there is no result on the existence of a solution of quasi-variational inequalities under densely pseudomonotonicity assumption. So Corollary 4.1 is a new result in the literature.

(b) Corollary 4.1 extends and generalizes Theorem 4.3 in [22] for quasi-variational inequalities without compactness assumption on K.

Lemma 4.3. *For each $i \in I$, let X_i be a normed space, K_i a nonempty, open and convex subset of X_i. For each $i \in I$ and each $j = 1, \ldots, \ell_i$, let $\phi_{i_j} : K \to \mathbb{R}$ be partial Gâteaux differentiable on K and convex in each argument. Then the family $\{D_{x_i}\phi_{i_j}\}_{i \in I, \; j=1}^{\ell_i}$ of partial Gâteaux derivative functions $D_{x_i}\phi_{i_j} : K \to X_i^*$ is weighted monotone w.r.t. the weight vector $W \in \prod_{i \in I}\left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\}\right)$.*

Proof. Since for each $i \in I$ and each $j = 1, \ldots, \ell_i$, ϕ_{i_j} is convex in each argument, we have

$$(4.2) \qquad \Phi_i(y) - \Phi_i(x) - \langle D_{x_i}\Phi_i(x), x_i - y_i\rangle \in \mathbb{R}_+^{\ell_i}, \quad \text{for all } x, y \in K.$$

By interchanging x and y, we obtain

$$(4.3) \qquad \Phi_i(x) - \Phi_i(y) - \langle D_{y_i}\Phi_i(y), y_i - x_i\rangle \in \mathbb{R}_+^{\ell_i}.$$

Adding (4.2) and (4.3), we get for each $i \in I$ and each $j = 1, \ldots, \ell_i$,

$$\langle D_{x_i}\Phi_i(x) - D_{y_i}\Phi_i(y), x_i - y_i\rangle \in \mathbb{R}_+^{\ell_i} + \mathbb{R}_+^{\ell_i} = \mathbb{R}_+^{\ell_i}.$$

This implies that

$$\sum_{i \in I} W_i \cdot \langle D_{x_i}\Phi_i(x) - D_{y_i}\Phi_i(y), x_i - y_i\rangle \geq 0, \quad \text{for all } x, y \in K.$$

Hence the family $\{D_{x_i}\phi_{i_j}\}_{i \in I, \; j=1}^{\ell_i}$ of partial Gâteaux derivative functions $D_{x_i}\phi_{i_j} : K \to X_i^*$ is weighted monotone w.r.t. the weight vector $W \in \prod_{i \in I}\left(\mathbb{R}_+^{\ell_i} \setminus \{\mathbf{0}\}\right)$. ∎

Finally, we prove the existence of solutions of (CNEP)$_w$ and (CNEP).

Theorem 4.3. *For each $i \in I$, let X_i be a normed space, K_i a nonempty, open and convex subset of X_i and K_i^0 a segment-dense set in K. For each $i \in I$, let $A_i : K \to 2^{K_i}$ be a multivalued map with nonempty convex values such that $A_i^{-1}(y_i^0)$ is open in K for all $y_i^0 \in K_i^0$. Let $A(x) = \prod_{i \in I} A_i(x)$ for all $x \in K$ such that the set $F = \{x \in K : x \in A(x)\}$ is nonempty and closed. For each $i \in I$ and each $j = 1, \ldots, \ell_i$, let $\phi_{i_j} : K \to \mathbb{R}$ be partial Gâteaux differentiable on K and convex in each argument such that the partial Gâteaux derivative map $D_{x_i}\phi_{i_j} : K \to X_i^*$ is weighted hemicontinuous w.r.t. the same weight vector $W \in \prod_{i \in I} \mathbb{T}_+^{\ell_i}$ (respectively, $W \in \prod_{i \in I}(\mathrm{int}\ \mathbb{T}_+^{\ell_i})$) on K. Assume there exist a nonempty compact subset $D \subseteq K$ and $\tilde{y} \in K$ such that for all $x \in K \setminus D$, $\tilde{y} \in A(x)$ and $\sum_{i \in I} W_i \cdot \langle D_{x_i}\Phi_i(x), x_i - \tilde{y}_i\rangle > 0$, where $D_{x_i}\Phi_i(x)$ is the same as defined in Section 3. Then there exists a weak Pareto equilibrium $\bar{x} \in K$, that is, a solution of (CNEP) $_w$ w.r.t. the same weight vector W (respectively, a Pareto*

equilibrium $\bar{x} \in K$, that is, a solution of (CNEP) w.r.t. the same weight vector W).

Proof. It follows from Theorem 4.2, Lemma 4.3 and Proposition 3.2.   ■

Remark 4.4.   In Theorem 4.3, we have not used any kind of continuity assumption on ϕ_{i_j} for each $\in I$

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