TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 2, pp. 339-360, February 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

STRONG CONVERGENCE THEOREMS BY THE HYBRID METHOD FOR FAMILIES OF NONEXPANSIVE MAPPINGS IN HILBERT SPACES

K. Nakajo, K. Shimoji and W. Takahashi

Abstract. Let C be a nonempty closed convex subset of a real Hilbert space and let $\{T_n\}$ be a family of mappings of C into itself such that the set of all common fixed points of $\{T_n\}$ is nonempty. We consider a sequence $\{x_n\}$ generated by the hybrid method in mathematical programming and give the conditions of $\{T_n\}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

1. Introduction

Throughout this paper, let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ and let **N** and **R** be the set of all positive integers and the set of all real numbers, respectively. Haugazeau [7] introduced a sequence $\{x_n\}$ generated by the hybrid method, that is, let $\{T_n\}$ be a family of mappings of H into itself with $\cap_{n=0}^{\infty} F(T_n) \neq \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n and let $\{x_n\}$ be a sequence generated by

$$(1)\qquad \begin{cases} x_0 = x \in H\,, \\ y_n = T_n x_n\,, \\ C_n = \{z \in H \mid (x_n - y_n, y_n - z) \geq 0\}\,, \\ Q_n = \{z \in H \mid (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for each $n \in \mathbf{N} \cup \{0\}$, where $P_{C_n \cap Q_n}$ is the metric projection onto $C_n \cap Q_n$. He proved a strong convergence theorem when $T_n = P_{n(mod\ m)+1}$ for every $n \in$

Received October 5, 2004; Accepted April 20, 2005.
2000 *Mathematics Subject Classification*: 47H05, 47H09, 47H20.
Key words and phrases: Strong convergence, Hybrid method, Proximal point algorithm, Nonexpansive, W-mapping, Nonexpansive semigroup, Splitting method, Variational inequality.

$\mathbf{N} \cup \{0\}$, where P_i is the metric projection onto a nonempty closed convex subset C_i of H for each $i = 1, 2, \cdots, m$ and $\cap_{i=1}^m C_i \neq \emptyset$. Later, Solodov and Svaiter [21] proved a strong convergence theorem for a maximal monotone operator and Bauschke and Combettes [4] proved the following theorem: Let $\{T_n\}$ be a family of mappings of H into itself with $\cap_{n=0}^\infty F(T_n) \neq \emptyset$ which satisfies the following conditions: (I) $(x - T_n x, T_n x - z) \geq 0$ for every $n \in \mathbf{N} \cup \{0\}$, $x \in H$ and $z \in F(T_n)$; (II) (coherent) for every bounded sequence $\{z_n\}$ in H, there holds that $\sum_{n=0}^\infty \|z_{n+1} - z_n\|^2 < \infty$ and $\sum_{n=0}^\infty \|z_n - T_n z_n\|^2 < \infty$ imply $\omega_w(z_n) \subset \cap_{n=0}^\infty F(T_n)$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$. Then, $\{x_n\}$ generated by (1) converges strongly to $z_0 = P_F(x_0)$, where $F = \cap_{n=0}^\infty F(T_n)$. On the other hand, Nakajo and Takahashi [13] proved the following theorem: Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$(2) \qquad \begin{cases} x_0 = x \in C\,, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n\,, \\ C_n = \{z \in C \,|\, \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \,|\, (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for each $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}(x_0)$. Later, Nakajo and Takahashi [14], Kikkawa and Takahashi [11], Atsushiba and Takahashi [3], Iiduka, Takahashi and Toyoda [9] and Iiduka and Takahashi [10] studied strong convergence of $\{x_n\}$ generated by type (2). And recently, Nakajo, Shimoji and Takahashi [15] studied strong convergence by type (1) and (2).

Motivated by Bauschke and Combettes [4] and Nakajo, Shimoji and Takahashi [15], in this paper, we consider unification of types of (1) and (2) and prove a strong convergence theorem.

2. Preliminaries and Lemmas

We write $x_n \rightharpoonup x$ to indicate that a sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize strong convergence. We know that H satisfies Opial's condition [16], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ holds for every $y \in H$ with $y \neq x$. It is known that $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$ for each $x, y \in H$ and $\lambda \in \mathbf{R}$. We also know that the norm is lower semicontinuous, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$ holds. Further, it is known that for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and

$\|x_n\| \to \|x\|$, $x_n \to x$ holds. Let C be a nonempty closed convex subset of H and let T be a mapping of C into itself. T is said to be firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for every $x, y \in C$, where I is the identity mapping. T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. If T is firmly nonexpansive, T is nonexpansive. We know that P_C is firmly nonexpansive. It is known that $F(T)$ is closed and convex if T is a nonexpansive mapping of C into itself.
An operator $A : H \longrightarrow 2^H$ is said to be monotone if $(x_1 - x_2, y_1 - y_2) \geq 0$ whenever $y_1 \in Ax_1$ and $y_2 \in Ax_2$. A monotone operator A is said to be maximal if the graph of A is not properly contained in the graph of any other monotone operator. It is known that a monotone operator A is maximal if and only if $R(I + \lambda A) = H$ for every $\lambda > 0$, where $R(I + \lambda A)$ is the range of $I + \lambda A$. It is also known that a monotone operator A is maximal if and only if for $(u, v) \in H \times H$, $(x - u, y - v) \geq 0$ for every $(x, y) \in A$ implies $v \in Au$. For a maximal monotone operator A, we know that $A^{-1}0 = \{x \in H \,|\, 0 \in Ax\}$ is closed and convex. If A is monotone, then we can define, for each $\lambda > 0$, a mapping $J_\lambda : R(I + \lambda A) \longrightarrow D(A)$ by $J_\lambda = (I + \lambda A)^{-1}$, where $D(A)$ is the domain of A. J_λ is called the resolvent of A. We also define the Yosida approximation A_λ by $A_\lambda = (I - J_\lambda)/\lambda$; see [24, 25] for more details. The following are the fundamental results for resolvents of monotone operators; see [17, 24, 25].

Lemma 2.1. *Let $A : H \longrightarrow 2^H$ be a monotone operator and $\lambda > 0$. Then, the following hold:*

(*i*) $F(J_\lambda) = A^{-1}0$;

(*ii*) $\|J_\lambda x - J_\lambda y\|^2 \leq \|x - y\|^2 - \|(I - J_\lambda)x - (I - J_\lambda)y\|^2$ *for every* $x, y \in R(I + \lambda A)$.

Let $\alpha > 0$ and let C be a nonempty closed convex subset of H. An operator $A : C \longrightarrow H$ is said to be α-inverse-strongly-monotone [5, 12, 14] if $(x - y, Ax - Ay) \geq \alpha\|Ax - Ay\|^2$ for all $x, y \in C$. We have the following lemma for inverse-strongly-monotone operators; see [14].

Lemma 2.2. *Let $\alpha > 0$. Let $A : H \longrightarrow H$ be an α-inverse-strongly-monotone operator with $D(A) = H$ and let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A + B)^{-1}0 \neq \emptyset$. Then the following hold:*

(*i*) *A is maximal monotone;*

(*ii*) *$A + B$ is maximal monotone and $(A + B)^{-1}0$ is closed and convex;*

(*iii*) *for every $\lambda \in [0, 2\alpha]$, $I - \lambda A : H \longrightarrow H$ is nonexpansive;*

(*iv*) *for every $\lambda \in (0, \infty)$, $T_\lambda \equiv J^B_\lambda(I - \lambda A)$ is well defined and $(A + B)^{-1}0 = F(T_\lambda)$, where $J^B_\lambda = (I + \lambda B)^{-1}$ and $F(T_\lambda)$ is the set of all fixed points of T_λ;*

(*v*) *for every $\lambda \in (0, 2\alpha]$, T_λ is nonexpansive.*

Let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Then, an element x in C is a solution of the variational inequality of A if $(y-x, Ax) \geq 0$ for all $y \in C$. It is known that for $\lambda > 0$, $x \in C$ is a solution of the variational inequality of A if and only if $x = P_C(I-\lambda A)x$. We denote by $VI(C, A)$ the set of all solutions of the variational inequality of A. We know that $VI(C, A)$ is a closed convex subset of C if A is monotone and continuous. We also have the following result for inverse-strongly -monotone operators.

Lemma 2.3. *Let $\alpha > 0$ and C be a nonempty closed convex subset of H. Let $A : C \longrightarrow H$ be an α-inverse-strongly-monotone operator with $VI(C, A) \neq \emptyset$. Then, for every $\lambda > 0$, $x \in C$ and $z \in VI(C, A)$, $\|P_C(I-\lambda A)x - z\|^2 \leq \|x-z\|^2 - \frac{2\alpha-\lambda}{2\alpha}\|x - P_C(I-\lambda A)x\|^2$.*

Proof. Let $\lambda > 0$, $x \in C$ and $z \in VI(C, A)$. We have

$$\begin{aligned}
&\|P_C(I-\lambda A)x - z\|^2 \\
&\quad \leq \|(I-\lambda A)x - (I-\lambda A)z\|^2 - \|(I-P_C)(I-\lambda A)x - (I-P_C)(I-\lambda A)z\|^2 \\
&\quad = \|(x-z) - \lambda(Ax - Az)\|^2 - \|(x - P_C(I-\lambda A)x) - \lambda(Ax-Az)\|^2 \\
&\quad \leq \|x-z\|^2 - 2\alpha\lambda\|Ax-Az\|^2 + 2\lambda\|Ax-Az\| \\
&\qquad \cdot\|x - P_C(I-\lambda A)x\| - \|x - P_C(I-\lambda A)x\|^2 \\
&\quad = \|x-z\|^2 - 2\alpha\lambda\Big\{\|Ax-Az\| - \frac{1}{2\alpha}\|x - P_C(I-\lambda A)x\|\Big\}^2 \\
&\qquad -\frac{2\alpha-\lambda}{2\alpha}\|x - P_C(I-\lambda A)x\|^2 \\
&\quad \leq \|x-z\|^2 - \frac{2\alpha-\lambda}{2\alpha}\|x - P_C(I-\lambda A)x\|^2.
\end{aligned}$$

∎

Let C be a nonempty closed convex subset of H. Let $\{S_n\}$ be a family of mappings of C into itself and let $\{\beta_{n,k} : n, k \in \mathbf{N}, 1 \leq k \leq n\}$ be a sequence of real numbers such that $0 \leq \beta_{i,j} \leq 1$ for every $i, j \in \mathbf{N}$ with $i \geq j$. Then, for any $n \in \mathbf{N}$, Takahashi [19, 23, 25] introduced a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n} &= \beta_{n,n}S_n + (1-\beta_{n,n})I, \\
U_{n,n-1} &= \beta_{n,n-1}S_{n-1}U_{n,n} + (1-\beta_{n,n-1})I, \\
&\vdots \\
U_{n,k} &= \beta_{n,k}S_kU_{n,k+1} + (1-\beta_{n,k})I, \\
&\vdots \\
U_{n,2} &= \beta_{n,2}S_2U_{n,3} + (1-\beta_{n,2})I, \\
W_n = U_{n,1} &= \beta_{n,1}S_1U_{n,2} + (1-\beta_{n,1})I.
\end{aligned}$$

Such a mapping W_n is called the W-mapping generated by $S_n, S_{n-1}, \cdots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \cdots, \beta_{n,1}$. The following lemma was proved by Takahashi and Shimoji [26] (see also [25]).

Lemma 2.4. *Let C be a nonempty closed convex subset of H. Let $S_1, S_2, \cdots, S_n$ be nonexpansive mappings of C into itself with $\cap_{i=1}^{n} F(S_i) \neq \emptyset$ and let $\beta_{n,1}, \beta_{n,2}, \cdots \beta_{n,n}$ be real numbers with $0 < \beta_{n,i} < 1$ for every $i = 2, \cdots, n$ and $0 < \beta_{n,1} \leq 1$. Let W_n be the W-mapping generated by $S_n, S_{n-1}, \cdots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \cdots, \beta_{n,1}$. Then, $F(W_n) = \cap_{i=1}^{n} F(S_i)$.*

We have that if $\beta_{n,k} = \beta_k$ $(\forall n = k, k+1, \cdots)$ for every $k \in \mathbf{N}$ such that $0 < \beta_k \leq b < 1$ $(\forall k \in \mathbf{N})$ for some $b \in (0,1)$ and $\{S_n\}$ is a family of nonexpansive mappings of C into itself with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$, $\lim_{n\to\infty} U_{n,k}x$ exists for every $x \in C$ and $k \in \mathbf{N}$; see [19]. By this, we define a mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a W is called the W-mapping generated by $S_1, S_2, \cdots$ and $\beta_1, \beta_2, \cdots$. We have that $F(W) = \cap_{i=1}^{\infty} F(S_i)$; see [19].

Let C be a nonempty closed convex subset of H. A family $\mathcal{S} = \{T(s) \mid 0 \leq s < \infty\}$ of mappings of C into itself is called a one-parameter nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s+t) = T(s)T(t)$ for every $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for each $s \geq 0$ and $x, y \in C$;

(iv) for all $x \in C$, $s \longmapsto T(s)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \cap_{0\leq s<\infty} F(T(s))$. It is known that $F(\mathcal{S})$ is closed and convex. The following lemma was proved by Shimizu and Takahashi [18]; see also [2, 6, 20].

Lemma 2.5. *Let C be a nonempty bounded closed convex subset of H and let $\mathcal{S} = \{T(s) \mid 0 \leq s < \infty\}$ be a one-parameter nonexpansive semigroup on C. Then, for any $h \geq 0$,*

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t} \int_0^t T(s)x\,ds - T(h) \left(\frac{1}{t} \int_0^t T(s)x\,ds \right) \right\| = 0\,.$$

Let S be a semigroup and let $B(S)$ be the Banach space of all bounded real valued functions on S with supremum norm. Then, for every $s \in S$ and $f \in B(S)$, we can define $r_s f \in B(S)$ and $l_s f \in B(S)$ by $(r_s f)(t) = f(ts)$ and $(l_s f)(t) = f(st)$ for each $t \in S$, respectively. We also denote by r_s^* and l_s^* the conjugate operators of r_s and l_s, respectively. Let D be a subspace of $B(S)$ containing constants and let μ be an element of D^*. A linear functional μ is called a mean on D if $\|\mu\| = \mu(1) = 1$. Let C be a nonempty closed convex subset of H. A family $\mathcal{S} = \{T(s) \mid s \in S\}$ of mappings of C into itself is called a nonexpansive semigroup on C if it satisfies the following conditions:

(i) $T(st) = T(s)T(t)$ for all $s, t \in S$;

(ii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for every $s \in S$ and $x, y \in C$.

It is known that $F(\mathcal{S})$ is closed and convex. Takahashi [22] proved the following; see also [8].

Lemma 2.6. *Let S be a semigroup. Let C be a nonempty closed convex subset of H and let $\mathcal{S} = \{T(s) \mid s \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let D be a subspace of $B(S)$ such that D contains constants and $(T(\cdot)x, y) \in D$ for every $x \in C$ and $y \in H$. Then, for any mean μ on D and $x \in C$, there exists a unique element $T_\mu x$ in C such that $(T_\mu x, z) = \mu_s(T(s)x, z)$ for all $z \in H$. And T_μ is a nonexpansive mapping of C into itself.*

Further, Atsushiba and Takahashi [1] proved the following.

Lemma 2.7. *Let C be a nonempty bounded closed convex subset of H and let S be a semigroup. Let $\mathcal{S} = \{T(s) \mid s \in S\}$ be a nonexpansive semigroup on C and let D be a subspace of $B(S)$ containing constants and invariant under l_s for all $s \in S$. Suppose that for every $x \in C$ and $z \in H$, the function $t \mapsto (T(t)x, z)$ is in D. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - l_s^*\mu_n\| = 0$ for each $s \in S$. Then, $\lim_{n\to\infty} \sup_{x\in C} \|T_{\mu_n}x - T(t)T_{\mu_n}x\| = 0$ for all $t \in S$.*

3. Strong Convergence Theorems

Let C be a nonempty closed convex subset of H and let $\{T_n\}$ be a family of mappings of C into itself with $\cap_{n=0}^\infty F(T_n) \neq \emptyset$ which satisfies the following condition: There exists $\{a_n\} \subset (-1, \infty)$ such that

$$\|T_n x - z\|^2 \leq \|x - z\|^2 - a_n\|(I - T_n)x\|^2 \tag{3}$$

for every $n \in \mathbf{N} \cup \{0\}$, $x \in C$ and $z \in F(T_n)$. Then, we know that $\cap_{n=0}^\infty F(T_n)$ is closed (see [15]). We also have that $\cap_{n=0}^\infty F(T_n)$ is convex. In fact, let $n \in \mathbf{N} \cup \{0\}$ and let $z_1, z_2 \in F(T_n)$, $0 \leq \alpha \leq 1$ and $x = \alpha z_1 + (1-\alpha)z_2$. Suppose that $x \neq T_n x$. For some $\beta \in (0, 1)$ with $a_n > -\beta$, we get

$$\begin{aligned}\|\beta x+(1-\beta)T_nx-z_1\|^2 &= \beta\|x-z_1\|^2+(1-\beta)\|T_nx-z_1\|^2-\beta(1-\beta)\|x-T_nx\|^2\\ &\leq \beta\|x-z_1\|^2+(1-\beta)\{\|x-z_1\|^2-a_n\|x-T_nx\|^2\}\\ &\quad -\beta(1-\beta)\|x-T_nx\|^2\\ &= \|x-z_1\|^2-(1-\beta)(a_n+\beta)\|x-T_nx\|^2 < \|x-z_1\|^2.\end{aligned}$$

Similarly, $\|\beta x+(1-\beta)T_nx-z_2\| < \|x-z_2\|$ holds. So, we obtain

$$\begin{aligned}\|z_1-z_2\| &\leq \Big\| z_1-\{\beta x+(1-\beta)T_nx\} \Big\|+\Big\| \{\beta x+(1-\beta)T_nx\}-z_2 \Big\|\\ &< \|x-z_1\|+\|x-z_2\| = (1-\alpha)\|z_1-z_2\|+\alpha\|z_1-z_2\| = \|z_1-z_2\|.\end{aligned}$$

This is a contradiction. Therefore, $F(T_n)$ is convex. Let us define a sequence $\{x_n\}$ as follows:

$$(4)\begin{cases} x_0 = x \in C\,,\\ y_n = T_nP_C(x_n+\varepsilon_n)\,,\\ C_n = \{z \in C \mid \|y_n-z\|^2 \leq \|x_n+\varepsilon_n-z\|^2-a_n\|P_C(x_n+\varepsilon_n)-y_n\|^2\}\,,\\ Q_n = \{z\in C \mid (x_n-z, x_0-x_n) \geq 0\}\,,\\ x_{n+1} = P_{C_n\cap Q_n}(x_0)\end{cases}$$

for each $n \in \mathbf{N}\cup\{0\}$, where $\{\varepsilon_n\} \subset H$ and $\liminf_{n\to\infty} a_n > -1$. Now, we get the following.

Theorem 3.1. *The followings hold:*

(i) *A sequence $\{x_n\}$ generated by (4) is well defined and $\{x_n\} \subset C$;*

(ii) *assume that $\sum_{n=0}^{\infty}\|\varepsilon_n\|^2 < \infty$ and for every bounded sequence $\{z_n\}$ in C, there holds that $\sum_{n=0}^{\infty}\|z_{n+1}-z_n\|^2 < \infty$ and $\sum_{n=0}^{\infty}\|z_n-T_nz_n\|^2 < \infty$ imply $\omega_w(z_n) \subset \cap_{n=0}^{\infty}F(T_n)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_F(x_0)$, where $F = \cap_{n=0}^{\infty}F(T_n)$;*

(iii) *assume that $\lim_{n\to\infty}\|\varepsilon_n\| = 0$ and for every bounded sequence $\{z_n\}$ in C, there holds that $\lim_{n\to\infty}\|z_n-T_nz_n\| = 0$ implies $\omega_w(z_n) \subset \cap_{n=0}^{\infty}F(T_n)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_F(x_0)$.*

Proof. As in the proof of [15, Theorem 4.2], we get that $\{x_n\}$ is well defined, $\{x_n\} \subset C$ and $F \subset C_n\cap Q_n$ for each $n \in \mathbf{N}\cup\{0\}$. So, the proof of (i) is complete. We have, for $z_0 = P_F(x_0)$,

$$\|x_{n+1}-x_0\| \leq \|z_0-x_0\| \tag{5}$$

and

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &= \|(x_{n+1}-x_0)+(x_0-x_n)\|^2\\ &= \|x_{n+1}-x_0\|^2+2(x_{n+1}-x_0,x_0-x_n)+\|x_0-x_n\|^2\\ &= \|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2+2(x_{n+1}-x_n,x_0-x_n)\\ &\le \|x_{n+1}-x_0\|^2-\|x_n-x_0\|^2\end{aligned}$$

for every $n \in \mathbf{N}\cup\{0\}$ by $(x_0-x_n,x_{n+1}-x_n)\le 0$. So, we obtain that $\{x_n\}$ is bounded and the limit

$$\lim_{n\to\infty}\|x_n-x_0\| \tag{6}$$

exists. We get

$$\sum_{n=0}^{\infty}\|x_{n+1}-x_n\|^2<\infty. \tag{7}$$

From $\liminf_{n\to\infty} a_n > -1$, there exists $a \in (0,1)$ such that $a_n \ge -a$ for all $n\in\mathbf{N}\cup\{0\}$. Let $\beta\in(0,\frac{1-a}{a})$ and $\alpha=\frac{1-a(1+\beta)}{a}(>0)$. We have

$$\begin{aligned}&\|P_C(x_n+\varepsilon_n)-y_n\|^2\\ &\quad\le \|x_n+\varepsilon_n-y_n\|^2\le(\|x_n+\varepsilon_n-x_{n+1}\|+\|x_{n+1}-y_n\|)^2\\ &\quad\le \Big(1+\frac{1}{\alpha}\Big)\|x_n+\varepsilon_n-x_{n+1}\|^2+(1+\alpha)\|x_{n+1}-y_n\|^2\\ &\quad\le \Big(1+\frac{1}{\alpha}\Big)\|x_n+\varepsilon_n-x_{n+1}\|^2+(1+\alpha)\|x_n+\varepsilon_n-x_{n+1}\|^2\\ &\qquad -(1+\alpha)a_n\|P_C(x_n+\varepsilon_n)-y_n\|^2\end{aligned}$$

which implies

$$\begin{aligned}a\beta\|P_C(x_n+\varepsilon_n)-y_n\|^2 &= \{1-(1+\alpha)a\}\|P_C(x_n+\varepsilon_n)-y_n\|^2\\ &\le \{1+(1+\alpha)a_n\}\|P_C(x_n+\varepsilon_n)-y_n\|^2\\ &\le \Big(2+\alpha+\frac{1}{\alpha}\Big)\|x_n+\varepsilon_n-x_{n+1}\|^2\end{aligned}$$

for each $n\in\mathbf{N}\cup\{0\}$. So, we get

$$\|P_C(x_n+\varepsilon_n)-y_n\|^2\le\frac{2\big(2+\alpha+\frac{1}{\alpha}\big)}{a\beta}(\|x_n-x_{n+1}\|^2+\|\varepsilon_n\|^2) \tag{8}$$

for every $n\in\mathbf{N}\cup\{0\}$.

(ii) Assume that $\sum_{n=0}^{\infty} \|\varepsilon_n\|^2 < \infty$. If $z_n = P_C(x_n + \varepsilon_n)$, we have that $\{z_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \|z_n - T_n z_n\|^2 = \sum_{n=0}^{\infty} \|P_C(x_n + \varepsilon_n) - y_n\|^2 < \infty$$

from (7) and (8). Further we obtain

$$\|z_n - z_{n+1}\|^2 \leq \|(x_n + \varepsilon_n) - (x_{n+1} + \varepsilon_{n+1})\|^2 \leq 3\|x_n - x_{n+1}\|^2 + 3\|\varepsilon_n\|^2 + 3\|\varepsilon_{n+1}\|^2$$

for all $n \in \mathbf{N} \cup \{0\}$ which implies $\sum_{n=0}^{\infty} \|z_n - z_{n+1}\|^2 < \infty$. Therefore, we have $\omega_w(z_n) \subset F$. As $\lim_{n\to\infty} \|x_n - z_n\| = 0$ from $\|x_n - z_n\| \leq \|\varepsilon_n\|$ for every $n \in \mathbf{N} \cup \{0\}$, we get $\omega_w(x_n) \subset F$. So, assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to $w_1 \in F$. We have

$$\|x_0 - z_0\| \leq \|x_0 - w_1\| \leq \lim_{i\to\infty} \|x_0 - x_{n_i}\| \leq \|x_0 - z_0\|$$

by the lower semicontinuity of the norm, (5) and (6). Thus, we obtain $\lim_{i\to\infty} \|x_{n_i} - x_0\| = \|x_0 - w_1\| = \|x_0 - z_0\|$. This implies

$$x_{n_i} \to w_1 = z_0.$$

Therefore, we have $x_n \to z_0$. So, the proof of (ii) is complete.
(iii) Assume $\lim_{n\to\infty} \|\varepsilon_n\| = 0$. If $z_n = P_C(x_n + \varepsilon_n)$, we get that $\{z_n\}$ is bounded, $\lim_{n\to\infty} \|x_n - z_n\| = 0$ and $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ by (7) and (8). Therefore, we obtain $\omega_w(x_n) \subset F$. As in the proof of (ii), we have $x_n \to z_0$. So, the proof of (iii) is complete. ∎

The following is the result proved by Bauschke and Combettes [4].

Theorem 3.2. *Let $\{T_n\}$ be a family of mappings of H into itself with $\cap_{n=0}^{\infty} F(T_n) \neq \emptyset$ which satisfies the following conditions: (I) $(x - T_n x, T_n x - z) \geq 0$ for every $n \in \mathbf{N} \cup \{0\}$, $x \in H$ and $z \in F(T_n)$; (II) (coherent) for every bounded sequence $\{z_n\}$ in H, there holds that $\sum_{n=0}^{\infty} \|z_{n+1} - z_n\|^2 < \infty$ and $\sum_{n=0}^{\infty} \|z_n - T_n z_n\|^2 < \infty$ imply $\omega_w(z_n) \subset \cap_{n=0}^{\infty} F(T_n)$. Then, $\{x_n\}$ generated by (1) converges strongly to $z_0 = P_F(x_0)$, where $F = \cap_{n=0}^{\infty} F(T_n)$.*

Proof. If $C = H$, $a_n = 1$ and $\varepsilon_n = 0$ for every $n \in \mathbf{N} \cup \{0\}$ in Theorem 3.1, we have

$$\|T_n x - z\|^2 \leq \|x - z\|^2 - \|x - T_n x\|^2 \iff (x - T_n x, T_n x - z) \geq 0$$

for each $n \in \mathbf{N} \cup \{0\}$, $x \in H$ and $z \in F(T_n)$ and $C_n = \{z \in H \mid (x_n - y_n, y_n - z) \geq 0\}$ for all $n \in \mathbf{N} \cup \{0\}$. Further, $\sum_{n=0}^{\infty} \|\varepsilon_n\|^2 < \infty$. Therefore, $\{x_n\}$ converges strongly to $z_0 = P_F(x_0)$ from (ii) in Theorem 3.1. ∎

The following is a generalization of the result proved by Solodov and Svaiter [21].

Theorem 3.3. *Let $A : H \longrightarrow 2^H$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in H\,, \\ y_n = J_{\lambda_n}(x_n + \varepsilon_n)\,, \\ C_n = \{z \in H \mid (x_n - y_n + \varepsilon_n, y_n - z) \geq 0\}\,, \\ Q_n = \{z \in H \mid (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{\lambda_n\} \subset (0, \infty)$ and $J_{\lambda_n} = (I + \lambda_n A)^{-1}$ for each $n \in \mathbf{N} \cup \{0\}$. If (i) $\liminf_{n\to\infty} \lambda_n > 0$ and $\lim_{n\to\infty} \|\varepsilon_n\| = 0$ or (ii) $\sum_{n=0}^{\infty} \lambda_n^2 = \infty$ and $\sum_{n=0}^{\infty} \|\varepsilon_n\|^2 < \infty$, then, $\{x_n\}$ converges strongly to $z_0 = P_{A^{-1}0}(x_0)$.

Proof. If $C = H$ and $T_n = J_{\lambda_n}$ for all $n \in \mathbf{N} \cup \{0\}$ in Theorem 3.1, we have $F(T_n) = A^{-1}0$ and $a_n = 1$ for every $n \in \mathbf{N} \cup \{0\}$ by (i) and (ii) in Lemma 2.1. So we get $C_n = \{z \in H \mid (x_n - y_n + \varepsilon_n, y_n - z) \geq 0\}$ for each $n \in \mathbf{N} \cup \{0\}$.
(i) Assume that $\liminf_{n\to\infty} \lambda_n > 0$ and $\lim_{n\to\infty} \|\varepsilon_n\| = 0$. There exists $\lambda > 0$ with $\lambda_n \geq \lambda$ for each $n \in \mathbf{N} \cup \{0\}$. Let $\{z_n\}$ be a bounded sequence in H which satisfies $\lim_{n\to\infty} \|z_n - J_{\lambda_n} z_n\| = 0$. And suppose that $z_n \rightharpoonup w$. For all $(u, v) \in A$, we obtain

$$\left(J_{\lambda_n} z_n - u, \frac{z_n - J_{\lambda_n} z_n}{\lambda_n} - v \right) \geq 0$$

which implies

$$\begin{aligned} (J_{\lambda_n} z_n - u, -v) &\geq \frac{1}{\lambda_n}(J_{\lambda_n} z_n - u, J_{\lambda_n} z_n - z_n) \\ &\geq -\frac{1}{\lambda_n}\|J_{\lambda_n} z_n - u\| \cdot \|J_{\lambda_n} z_n - z_n\| \end{aligned} \tag{9}$$

for every $n \in \mathbf{N} \cup \{0\}$. As a sequence $\{\frac{1}{\lambda_n}\|J_{\lambda_n} z_n - u\|\}$ is bounded, we have $(w - u, -v) \geq 0$ for each $(u, v) \in A$. Therefore, $w \in A^{-1}0$ from maximality of A. By (iii) in Theorem 3.1, $\{x_n\}$ converges strongly to z_0.
(ii) Assume that $\sum_{n=0}^{\infty} \lambda_n^2 = \infty$ and $\sum_{n=0}^{\infty} \|\varepsilon_n\|^2 < \infty$. Let $\{z_n\}$ be a bounded sequence in H which satisfies $\sum_{n=0}^{\infty} \|z_{n+1} - z_n\|^2 < \infty$ and $\sum_{n=0}^{\infty} \|z_n - J_{\lambda_n} z_n\|^2 < \infty$. And suppose that $z_n \rightharpoonup w$. We get $\lim_{n\to\infty} \|z_n - J_{\lambda_n} z_n\| = \liminf_{n\to\infty} \frac{1}{\lambda_n}\|z_n - J_{\lambda_n} z_n\| = 0$ by $\sum_{n=0}^{\infty} \lambda_n^2 (\frac{1}{\lambda_n}\|z_n - J_{\lambda_n} z_n\|)^2 < \infty$. From (9), we obtain

$$(w - u, -v) \geq -\liminf_{n\to\infty} \frac{1}{\lambda_n}\|J_{\lambda_n} z_n - z_n\| \cdot \|J_{\lambda_n} z_n - u\| = 0$$

for all $(u, v) \in A$. So, we have $w \in A^{-1}0$. By (ii) in Theorem 3.1, $\{x_n\}$ converges strongly to z_0. ∎

The following is a genaralization of the result proved by Nakajo and Takahashi [13].

Theorem 3.4. *Let $A : H \longrightarrow 2^H$ be a maximal monotone operator such that $A^{-1}0 \neq \emptyset$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in H\,, \\ y_n = J_{\lambda_n}(x_n + \varepsilon_n)\,, \\ C_n = \{z \in H \mid \|y_n - z\| \leq \|x_n + \varepsilon_n - z\|\}\,, \\ Q_n = \{z \in H \mid (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{\lambda_n\} \subset (0, \infty)$ and $J_{\lambda_n} = (I + \lambda_n A)^{-1}$ for each $n \in \mathbf{N} \cup \{0\}$. If (i) $\liminf_{n\to\infty} \lambda_n > 0$ and $\lim_{n\to\infty} \|\varepsilon_n\| = 0$ or (ii) $\sum_{n=0}^{\infty} \lambda_n^2 = \infty$ and $\sum_{n=0}^{\infty} \|\varepsilon_n\|^2 < \infty$, then, $\{x_n\}$ converges strongly to $z_0 = P_{A^{-1}0}(x_0)$.

Proof. If $C = H$ and $T_n = J_{\lambda_n}$ for all $n \in \mathbf{N} \cup \{0\}$ in Theorem 3.1, we can select $a_n = 0$ for every $n \in \mathbf{N} \cup \{0\}$. As in the proof of Theorem 3.3, $\{x_n\}$ converges strongly to z_0. ∎

The following is the result proved by Nakajo and Takahashi [13].

Theorem 3.5. *Let C be a nonempty closed convex subset of H and let T be a nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C\,, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n\,, \\ C_n = \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \mid (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for each $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}(x_0)$.

Proof. If $T_n = \alpha_n I + (1 - \alpha_n)T$ and $\varepsilon_n = 0$ for all $n \in \mathbf{N} \cup \{0\}$ in Theorem 3.1, we have that $F(T_n) = F(T)$ and $a_n = 0$ for every $n \in \mathbf{N} \cup \{0\}$. Let $\{z_n\}$ be a bounded sequence in C which satisfies $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$. We obtain

$\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. So, by Opial's condition, we get $\omega_w(z_n) \subset F(T)$. Therefore, $\{x_n\}$ converges strongly to z_0 from (ii) or (iii) in Theorem 3.1. ∎

The following is a generalization of the result proved by Nakajo and Takahashi [14].

Theorem 3.6. *Let $\alpha > 0$. Let $A : H \longrightarrow H$ be an α-inverse-strongly-monotone operator with $D(A) = H$ and let $B : H \longrightarrow 2^H$ be a maximal monotone operator such that $(A+B)^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in H\,, \\ y_n = J^B_{\lambda_n}(I - \lambda_n A)x_n\,, \\ C_n = \{z \in H \mid \|y_n - z\| \le \|x_n - z\|\}\,, \\ Q_n = \{z \in H \mid (x_n - z, x_0 - x_n) \ge 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{\lambda_n\} \subset (0, 2\alpha]$ with $\sum_{n=0}^{\infty} \lambda_n^2 = \infty$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{(A+B)^{-1}0}(x_0)$.

Proof. If $C = H$, $T_n = J^B_{\lambda_n}(I - \lambda_n A)$ and $\varepsilon_n = 0$ for all $n \in \mathbf{N} \cup \{0\}$ in Theorem 3.1, we have $F(T_n) = (A+B)^{-1}0$ and $a_n = 0$ for every $n \in \mathbf{N} \cup \{0\}$ by (iv) and (v) in Lemma 2.2. Let $\{z_n\}$ be a bounded sequence in H which satisfies $\sum_{n=0}^{\infty} \|z_{n+1} - z_n\|^2 < \infty$ and $\sum_{n=0}^{\infty} \|z_n - J^B_{\lambda_n}(I - \lambda_n A)z_n\|^2 < \infty$. We obtain $\lim_{n\to\infty} \|z_n - J^B_{\lambda_n}(I - \lambda_n A)z_n\| = 0$ and $\liminf_{n\to\infty} \frac{1}{\lambda_n}\|z_n - J^B_{\lambda_n}(I - \lambda_n A)z_n\| = 0$ from $\sum_{n=0}^{\infty} \lambda_n^2 \{\frac{1}{\lambda_n}\|z_n - J^B_{\lambda_n}(I - \lambda_n A)z_n\|\}^2 < \infty$. Assume that $z_n \rightharpoonup w$. As in the proof of [14, Theorem 3.1], we get

$$\left(u_n - u, \frac{z_n - u_n}{\lambda_n} - Az_n - (v - Au)\right) \ge 0$$

for every $(u, v) \in A + B$ and $n \in \mathbf{N} \cup \{0\}$, where $u_n = J^B_{\lambda_n}(I - \lambda_n A)z_n$ for all $n \in \mathbf{N} \cup \{0\}$. So, we have

$$\begin{aligned} (u_n - u, -v) &\ge \left(u_n - u, \frac{u_n - z_n}{\lambda_n} + (Az_n - Au)\right) \\ &= \frac{1}{\lambda_n}(u_n - u, (I - \lambda_n A)u_n - (I - \lambda_n A)z_n) + (u_n - u, Au_n - Au) \\ &\ge -\frac{1}{\lambda_n}\|u_n - u\| \cdot \|(I - \lambda_n A)u_n - (I - \lambda_n A)z_n\| \\ &\ge -\frac{1}{\lambda_n}\|u_n - u\| \cdot \|u_n - z_n\| \end{aligned}$$

by (iii) in Lemma 2.2 which implies

$$(w-u,-v) \geq -\liminf_{n\to\infty} \frac{1}{\lambda_n}\|u_n-u\|\cdot\|u_n-z_n\| = 0$$

for every $(u,v) \in A+B$ since a sequence $\{u_n - u\}$ is bounded. Therefore, $w \in (A+B)^{-1}0$ as $A+B$ is maximal monotone from (ii) in Lemma 2.2. Therefore, $\{x_n\}$ converges strongly to z_0 by (ii) in Theorem 3.1. ■

The following is a generalization of the result proved by Iiduka, Takahashi and Toyoda [9].

Theorem 3.7. *Let $\alpha > 0$ and let C be a nonempty closed convex subset of H. Let $A : C \longrightarrow H$ be an α-inverse-strongly-monotone operator such that $VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C\,, \\ y_n = P_C(I-\lambda_n A)x_n\,, \\ C_n = \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \mid (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{\lambda_n\} \subset (0, 2\alpha]$ with $\sum_{n=0}^{\infty} \lambda_n^2 = \infty$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{VI(C,A)}(x_0)$.

Proof. If $T_n = P_C(I-\lambda_n A)$ and $\varepsilon_n = 0$ for all $n \in \mathbf{N}\cup\{0\}$ in Theorem 3.1, we have $F(T_n) = VI(C,A)$ for every $n \in \mathbf{N}\cup\{0\}$. Further, we get $a_n = 0$ for each $n \in \mathbf{N}\cup\{0\}$ from Lemma 2.3. Let $\{z_n\}$ be a bounded sequence in C which satisfies $\sum_{n=0}^{\infty}\|z_{n+1}-z_n\|^2 < \infty$ and $\sum_{n=0}^{\infty}\|z_n - v_n\|^2 < \infty$, where $v_n = P_C(I-\lambda_n A)z_n$ for all $n \in \mathbf{N}\cup\{0\}$. We get $\lim_{n\to\infty}\|z_n - v_n\| = 0$ and $\liminf_{n\to\infty}\frac{1}{\lambda_n}\|z_n - v_n\| = 0$ from $\sum_{n=0}^{\infty}\lambda_n^2\{\frac{1}{\lambda_n}\|z_n - v_n\|\}^2 < \infty$. Assume that $z_n \rightharpoonup w$. For every $u \in C$, we have

$$(z_n - \lambda_n A z_n - v_n, v_n - u) \geq 0$$

which implies

$$\begin{aligned}(Au, u-v_n) &\geq (Av_n - Au, v_n - u) + \frac{1}{\lambda_n}((I-\lambda_n A)v_n - (I-\lambda_n A)z_n, v_n - u) \\ &\geq -\frac{1}{\lambda_n}\|v_n - u\|\cdot\|(I-\lambda_n A)v_n - (I-\lambda_n A)z_n\|\end{aligned}$$

for all $n \in \mathbf{N}\cup\{0\}$ since A is monotone. And we obtain

$$\|(I-\lambda_n A)v_n - (I-\lambda_n A)z_n\|^2 \leq \|v_n - z_n\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Av_n - Az_n\|^2 \leq \|v_n - z_n\|^2$$

and hence

$$(Au, u - v_n) \geq -\frac{1}{\lambda_n}\|v_n - u\| \cdot \|v_n - z_n\|$$

for every $u \in C$ and $n \in \mathbf{N} \cup \{0\}$. So, we get $(Au, u - w) \geq 0$ for each $u \in C$ as a sequence $\{v_n - u\}$ is bounded. Since A is continuous, we obtain $(u - w, Aw) \geq 0$ for all $u \in C$, that is, $w \in VI(C, A)$. Therefore, $\{x_n\}$ converges strongly to z_0 by (ii) in Theorem 3.1. ∎

The following are the results by Iiduka and Takahashi [10].

Theorem 3.8. *Let $\alpha > 0$ and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself and let $A : C \longrightarrow H$ be an α-inverse-strongly-monotone operator such that $F(T) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C\,, \\ y_n = P_C(I - \lambda_n A)Tx_n\,, \\ C_n = \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \mid (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 2\alpha)$ with $a \leq b$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(T) \cap VI(C,A)}(x_0)$.

Proof. If $T_n = P_C(I - \lambda_n A)T$ and $\varepsilon_n = 0$ for all $n \in \mathbf{N} \cup \{0\}$ in Theorem 3.1, we have $F(T_n) = F(T) \cap VI(C, A)$ for every $n \in \mathbf{N} \cup \{0\}$. In fact, $F(T) \cap VI(C, A) \subset F(T_n)$ is trivial. Let $z \in F(T_n)$ and $u \in F(T) \cap VI(C, A)$. We get

$$\begin{aligned} \|z - u\|^2 &= \|P_C(I - \lambda_n A)Tz - u\|^2 \leq \|Tz - u\|^2 - \frac{2\alpha - \lambda_n}{2\alpha}\|Tz - P_C(I - \lambda_n A)Tz\|^2 \\ &\leq \|z - u\|^2 - \frac{2\alpha - b}{2\alpha}\|Tz - P_C(I - \lambda_n A)Tz\|^2 \end{aligned}$$

from Lemma 2.3. So we obtain $Tz = P_C(I - \lambda_n A)Tz$ which implies $Tz = z$. And we have $P_C(I - \lambda_n A)z = P_C(I - \lambda_n A)Tz = z$. Therefore, $F(T_n) \subset F(T) \cap VI(C, A)$. And we get $a_n = 0$ for each $n \in \mathbf{N} \cup \{0\}$ by nonexpansivity of T and Lemma 2.3. Let $\{z_n\}$ be a bounded sequence in C which satisfies

$\lim_{n\to\infty}\|z_n - T_n z_n\| = 0$ and let $u \in F(T)\cap VI(C,A)$. We obtain

$$\begin{aligned}\|z_n-u\|^2 &\le (\|z_n-T_nz_n\|+\|T_nz_n-u\|)^2\\ &= \|z_n-T_nz_n\|^2+2\|z_n-T_nz_n\|\\ &\quad\cdot\|P_C(I-\lambda_nA)Tz_n-u\|+\|P_C(I-\lambda_nA)Tz_n-u\|^2\\ &\le \|z_n-T_nz_n\|^2+2\|z_n-T_nz_n\|\cdot\|Tz_n-u\|\\ &\quad+\left\{\|Tz_n-u\|^2-\frac{2\alpha-\lambda_n}{2\alpha}\|Tz_n-P_C(I-\lambda_nA)Tz_n\|^2\right\}\\ &\le \|z_n-T_nz_n\|^2+2\|z_n-T_nz_n\|\cdot\|z_n-u\|\\ &\quad+\|z_n-u\|^2-\frac{2\alpha-b}{2\alpha}\|Tz_n-T_nz_n\|^2\end{aligned}$$

for all $n\in\mathbf{N}$ from Lemma 2.3. So, we have

$$\frac{2\alpha-b}{2\alpha}\|Tz_n-T_nz_n\|^2 \le \|z_n-T_nz_n\|^2+2\|z_n-T_nz_n\|\cdot\|z_n-u\|$$

for every $n\in\mathbf{N}$ and hence $\lim_{n\to\infty}\|Tz_n-T_nz_n\|=0$. Therefore, $\lim_{n\to\infty}\|z_n - Tz_n\|=0$ by $\lim_{n\to\infty}\|z_n-T_nz_n\|=0$. From Opial's condition, we get $\omega_w(z_n)\subset F(T)$. Further, we obtain

$$\begin{aligned}&\|z_n-P_C(I-\lambda_nA)z_n\|\\ &\quad\le \|z_n-Tz_n\|+\|Tz_n-P_C(I-\lambda_nA)Tz_n\|\\ &\qquad+\|P_C(I-\lambda_nA)Tz_n-P_C(I-\lambda_nA)z_n\|\\ &\quad\le 2\|z_n-Tz_n\|+\|Tz_n-P_C(I-\lambda_nA)Tz_n\|\end{aligned}$$

for every $n\in\mathbf{N}$ by nonexpansivity of $P_C(I-\lambda_nA)$ and hence $\lim_{n\to\infty}\|z_n-P_C(I-\lambda_nA)z_n\|=0$. As in the proof of Theorem 3.7, we have $\omega_w(z_n)\subset VI(C,A)$. So, $\omega_w(z_n)\subset F(T)\cap VI(C,A)$. Therefore, $\{x_n\}$ converges strongly to z_0 by (ii) or (iii) in Theorem 3.1. ■

Theorem 3.9. *Let $\alpha>0$ and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself and let $A: C\longrightarrow H$ be an α-inverse-strongly-monotone operator such that $F(T)\cap VI(C,A)\neq\emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x\in C\,,\\ y_n = TP_C(I-\lambda_nA)x_n\,,\\ C_n=\{z\in C\,|\,\|y_n-z\|\le\|x_n-z\|\}\,,\\ Q_n=\{z\in C\,|\,(x_n-z,x_0-x_n)\ge 0\}\,,\\ x_{n+1}=P_{C_n\cap Q_n}(x_0)\end{cases}$$

for every $n \in \boldsymbol{N} \cup \{0\}$*, where* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 2\alpha)$ *with* $a \leq b$*. Then,* $\{x_n\}$ *converges strongly to* $z_0 = P_{F(T)\cap VI(C,A)}(x_0)$*.*

Proof. If $T_n = TP_C(I - \lambda_n A)$ and $\varepsilon_n = 0$ for all $n \in \mathbf{N} \cup \{0\}$ in Theorem 3.1, we have $F(T_n) = F(T) \cap VI(C, A)$ for every $n \in \mathbf{N} \cup \{0\}$. In fact, similarly in Theorem 3.8, we get

$$\|z - u\|^2 \leq \|P_C(I - \lambda_n A)z - u\|^2 \leq \|z - u\|^2 - \frac{2\alpha - b}{2\alpha}\|z - P_C(I - \lambda_n A)z\|^2$$

for $z \in F(T_n)$ and $u \in F(T) \cap VI(C, A)$. Hence we obtain $z = P_C(I - \lambda_n A)z$ and further $z = Tz$, too. And we have $a_n = 0$ for each $n \in \mathbf{N}\cup\{0\}$ by Lemma 2.3. Let $\{z_n\}$ be a bounded sequence in C which satisfies $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ and let $u \in F(T) \cap VI(C, A)$. Similarly in Theorem 3.8, we get

$$\begin{aligned}\|z_n - u\|^2 \leq\ & \|z_n - T_n z_n\|^2 + 2\|z_n - T_n z_n\| \\ & \cdot\|z_n - u\| + \|z_n - u\|^2 - \frac{2\alpha - b}{2\alpha}\|z_n - P_C(I - \lambda_n A)z_n\|^2\end{aligned}$$

for all $n \in \mathbf{N}$. So, we obtain $\lim_{n\to\infty} \|z_n - P_C(I - \lambda_n A)z_n\| = 0$ which implies $\omega_w(z_n) \subset VI(C, A)$. Further, we have

$$\|z_n - Tz_n\| \leq \|z_n - T_n z_n\| + \|T_n z_n - Tz_n\| \leq \|z_n - T_n z_n\| + \|P_C(I - \lambda_n A)z_n - z_n\|$$

for every $n \in \mathbf{N}$. Therefore, $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$. By Opial's condition, we get $\omega_w(z_n) \subset F(T)$. So, $\{x_n\}$ converges strongly to z_0 from (ii) or (iii) in Theorem 3.1. ■

The following theorem contains the result proved by Kikkawa and Takahashi [11].

Theorem 3.10. *Let* C *be a nonempty closed convex subset of* H*. Let* $\{S_n\}$ *be a family of nonexpansive mappings of* C *into itself with* $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$ *and let* $\{\beta_{n,k} : n, k \in \boldsymbol{N}, 1 \leq k \leq n\} \subset (0, 1)$ *be a sequence of real numbers such that (i)* $\beta_{n,k} = \beta_k\ (\forall n = k, k+1, \cdots)$ *for every* $k \in \boldsymbol{N}$ *such that* $0 < \beta_k \leq b < 1\ (\forall k \in \boldsymbol{N})$ *for some* $b \in (0, 1)$ *or (ii)* $a \leq \beta_{i,j} \leq b$ *for every* $i, j \in \boldsymbol{N}\ (i \geq j)$ *for some* $a, b \in (0, 1)$ *with* $a \leq b$*. Let* $W_n\ (n = 1, 2, \cdots)$ *be the W-mapping generated by* $S_n, S_{n-1}, \cdots, S_1$ *and* $\beta_{n,n}, \beta_{n,n-1}, \cdots, \beta_{n,1}$*. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in C\,, \\ y_n = W_n x_n\,, \\ C_n = \{z \in C \mid \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \mid (x_n - z, x_1 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_1) \end{cases}$$

for each $n \in \mathbf{N}$*. Then,* $\{x_n\}$ *converges strongly to* $z_0 = P_{\cap_{i=1}^{\infty} F(S_i)}(x_1)$.

Proof. If $T_n = W_n$ and $\varepsilon_n = 0$ for all $n \in \mathbf{N}$ in Theorem 3.1, we have $\cap_{n=1}^{\infty} F(T_n) = \cap_{n=1}^{\infty} F(W_n) = \cap_{i=1}^{\infty} F(S_i)$ and $a_n = 0$ for every $n \in \mathbf{N}$ by Lemma 2.4 and nonexpansivity of W_n. Let $\{z_n\}$ be a bounded sequence in C which satisfies $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$.
(i) Let W be the W-mapping generated by $S_1, S_2, \cdots$ and $\beta_1, \beta_2, \cdots$. Assume that $z_n \rightharpoonup w$. As in the proof of [11, Theorem 3.1], if we suppose that $w \neq Ww$,

$$\begin{aligned}
\liminf_{n\to\infty} \|z_n - w\| &< \liminf_{n\to\infty} \|z_n - Ww\| \\
&\leq \liminf_{n\to\infty} (\|z_n - W_n z_n\| + \|W_n z_n - W_n w\| + \|W_n w - Ww\|) \\
&\leq \liminf_{n\to\infty} (\|z_n - W_n z_n\| + \|z_n - w\| + \|W_n w - Ww\|) \\
&= \liminf_{n\to\infty} \|z_n - w\|
\end{aligned}$$

by Opial's condition. This is a contradiction. So, we get $\omega_w(z_n) \subset F(W) = \cap_{n=1}^{\infty} F(S_n)$.
(ii) We get $\lim_{n\to\infty} \|z_n - S_1 U_{n,2} z_n\| = 0$ from $0 < a \leq \beta_{n,1}$. Let $z \in \cap_{n=1}^{\infty} F(S_n)$. We obtain

$$\begin{aligned}
\|z_n - z\|^2 &\leq (\|z_n - S_1 U_{n,2} z_n\| + \|S_1 U_{n,2} z_n - z\|)^2 \\
&= \|z_n - S_1 U_{n,2} z_n\|(\|z_n - S_1 U_{n,2} z_n\| \\
&\quad + 2\|S_1 U_{n,2} z_n - z\|) + \|S_1 U_{n,2} z_n - z\|^2 \\
&\leq M\|z_n - S_1 U_{n,2} z_n\| + \|U_{n,2} z_n - z\|^2 \\
&= M\|z_n - S_1 U_{n,2} z_n\| + \beta_{n,2}\|S_2 U_{n,3} z_n - z\|^2 \\
&\quad + (1-\beta_{n,2})\|z_n - z\|^2 - \beta_{n,2}(1-\beta_{n,2})\|S_2 U_{n,3} z_n - z_n\|^2 \\
&\leq M\|z_n - S_1 U_{n,2} z_n\| + \|z_n - z\|^2 - \beta_{n,2}(1-\beta_{n,2})\|S_2 U_{n,3} z_n - z_n\|^2
\end{aligned}$$

for each $n \in \mathbf{N}$, where $M = \sup_{n\in\mathbf{N}}\{\|z_n - S_1 U_{n,2} z_n\| + 2\|S_1 U_{n,2} z_n - z\|\}$. So, we obtain $\lim_{n\to\infty} \|S_2 U_{n,3} z_n - z_n\| = 0$. By induction, we have

$$\lim_{n\to\infty} \|S_m U_{n,m+1} z_n - z_n\| = 0$$

for all $m \in \mathbf{N}$. Since

$$\begin{aligned}
\|z_n - S_m z_n\| &\leq \|z_n - S_m U_{n,m+1} z_n\| + \|S_m U_{n,m+1} z_n - S_m z_n\| \\
&\leq \|z_n - S_m U_{n,m+1} z_n\| + \|U_{n,m+1} z_n - z_n\| \\
&= \|z_n - S_m U_{n,m+1} z_n\| + \beta_{n,m+1}\|S_{m+1} U_{n,m+2} z_n - z_n\| \\
&\leq \|z_n - S_m U_{n,m+1} z_n\| + b\|S_{m+1} U_{n,m+2} z_n - z_n\|
\end{aligned}$$

for every $n \in \mathbf{N}$, we get $\lim_{n\to\infty} \|z_n - S_m z_n\| = 0$ for all $m \in \mathbf{N}$. By Opial's condition, $\omega_w(z_n) \subset F(S_m)$ for each $m \in \mathbf{N}$ which implies $\omega_w(z_n) \subset \cap_{n=1}^{\infty} F(T_n)$. Therefore, $\{x_n\}$ converges strongly to z_0 from (ii) or (iii) in Theorem 3.1. ■

The following is the result proved by Nakajo and Takahashi [13].

Theorem 3.11. *Let C be a nonempty closed convex subset of H and let $\mathcal{S} = \{T(s) \mid 0 \le s < \infty\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C\,, \\ y_n = \alpha_n x_n + (1-\alpha_n)\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(s)x_n\,ds\,, \\ C_n = \{z \in C \mid \|y_n - z\| \le \|x_n - z\|\}\,, \\ Q_n = \{z \in C \mid (x_n - z, x_0 - x_n) \ge 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \mathbf{N}\cup\{0\}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$ and $\{t_n\}$ is a positive real divergent sequence. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{S})}(x_0)$.

Proof. If $T_n x = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x\,ds$ $(\forall x \in C)$ and $\varepsilon_n = 0$ for all $n \in \mathbf{N}\cup\{0\}$ in Theorem 3.1, we have that $T_n : C \longrightarrow C$ for every $n \in \mathbf{N}\cup\{0\}$, $\cap_{n=0}^{\infty} F(T_n) = F(\mathcal{S})$ and $a_n = 0$ for each $n \in \mathbf{N}\cup\{0\}$. Let $\{z_n\}$ be a bounded sequence in C which satisfies $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$. At first, we have

$$(1-a)\left\| z_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds \right\| \le (1-\alpha_n)\left\| z_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds \right\| = \|z_n - T_n z_n\|$$

for all $n \in \mathbf{N}$ which implies

$$\lim_{n\to\infty}\left\| z_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds \right\| = 0.$$

And as in the proof of [13, Theorem 4.1], we have, for every $h \ge 0$,

$$\begin{aligned} \|z_n - T(h)z_n\| \le\ & \left\| z_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds \right\| \\ & + \left\| \frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds - T(h)\Big(\frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds\Big) \right\| \\ & + \left\| T(h)z_n - T(h)\Big(\frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds\Big) \right\| \\ \le\ & 2\left\| z_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds \right\| \end{aligned}$$

$$+\left\| \frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds - T(h)\Big(\frac{1}{t_n}\int_0^{t_n} T(s)z_n\,ds\Big)\right\|$$

for each $n \in \mathbf{N}$. So, we get

$$\lim_{n\to\infty} \|z_n - T(h)z_n\| = 0$$

for all $h \geq 0$ by Lemma 2.5. By Opial's condition, we obtain $\omega_w(z_n) \subset F(\mathcal{S})$. Therefore, $\{x_n\}$ converges strongly to z_0 from (ii) or (iii) in Theorem 3.1. ■

The following is the result proved by Atsushiba and Takahashi [3].

Theorem 3.12. *Let C be a nonempty closed convex subset of H and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) \,|\, t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let D be a subspace of $B(S)$ such that D contains constants, D is invariant under r_s for every $s \in S$ and $t \mapsto (T(t)x, y)$ is in D for each $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a sequence of means on D such that $\lim_{n\to\infty} \|\mu_n - r_s^*\mu_n\| = 0$ for all $s \in S$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C\,, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n\,, \\ C_n = \{z \in C \,|\, \|y_n - z\| \leq \|x_n - z\|\}\,, \\ Q_n = \{z \in C \,|\, (x_n - z, x_0 - x_n) \geq 0\}\,, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

for every $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in (0, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(\mathcal{S})}(x_0)$.

Proof. If $T_n = \alpha_n I + (1-\alpha_n)T_{\mu_n}$ and $\varepsilon_n = 0$ for all $n \in \mathbf{N} \cup \{0\}$ in Theorem 3.1, we have $T_n : C \longrightarrow C$, $a_n = 0$ for every $n \in \mathbf{N} \cup \{0\}$ and $\cap_{n=0}^{\infty} F(T_n) = F(\mathcal{S})$ from Lemmas 2.6 and 2.7. Let $\{z_n\}$ be a bounded sequence in C which satisfies $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$. At first, we get

$$(1-a)\|z_n - T_{\mu_n}z_n\| \leq (1-\alpha_n)\|z_n - T_{\mu_n}z_n\| = \|z_n - T_n z_n\|$$

for each $n \in \mathbf{N}$ which implies

$$\lim_{n\to\infty} \|z_n - T_{\mu_n}z_n\| = 0.$$

And for all $t \in S$,

$$\begin{aligned} \|z_n - T(t)z_n\| &\leq \|z_n - T_{\mu_n}z_n\| + \|T_{\mu_n}z_n - T(t)T_{\mu_n}z_n\| \\ &\quad + \|T(t)T_{\mu_n}z_n - T(t)z_n\| \\ &\leq 2\|z_n - T_{\mu_n}z_n\| + \|T_{\mu_n}z_n - T(t)T_{\mu_n}z_n\|. \end{aligned}$$

So, we obtain $\lim_{n\to\infty} \|z_n - T(t)z_n\| = 0$ for every $t \in S$ by Lemma 2.7. By Opial's condition, we get $\omega_w(z_n) \subset F(\mathcal{S})$. Therefore, $\{x_n\}$ converges strongly to z_0 by (ii) or (iii) in Theorem 3.1. ■

References

1. S. Atsushiba and W. Takahashi, Approximating common fixed points of nonexpansive semigroups by the Mann iteration process, *Ann. Univ. Mariae Curie-Sklodowska*, **51** (1997), 1-16.
2. S. Atsushiba and W. Takahashi, *A weak convergence theorem for nonexpansive semigroups by the Mann iteration process in Banach spaces*, in Nonlinear Analysis and Convex Analysis (W. Takahashi and T. Tanaka, Eds.), World Scientific, Singapore, 102-109, 1999.
3. S. Atsushiba and W. Takahashi, Strong convergence theorems for nonexpansive semigroups by a hybrid method, *J. Nonlinear Convex Anal.*, **3** (2002), 231-242.
4. H. H. Bauschke and P. L. Combettes, A weak-to-strong convergence principle for fejér-monotone methods in Hilbert spaces, *Math. Oper. Res.*, **26** (2001), 248-264.
5. F. E. Browder and W. V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.*, **20** (1967), 197-228.
6. R. E. Bruck, On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, *Israel J. Math.*, **38** (1981), 304-314.
7. Y. Haugazeau, *Sur les inéquations variationnelles et la minimisation de fonctionnelles convexes*, Thèse, Université de Paris, Paris, France, 1968.
8. N. Hirano, K. Kido and W. Takahashi, Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces, *Nonlinear Analysis*, **12** (1988), 1269-1281.
9. H. Iiduka, W. Takahashi and M. Toyoda, Approximation of solutions of variational inequalities for monotone mappings, *Panamerican Math. J.*, **14** (2004), 49-61.
10. H. Iiduka and W. Takahashi, *Strong convergence theorems by a hybrid method for nonexpansive mappings and inverse-strongly-monotone mappings*, in Fixed Point Theory and Applications (J. G. Falset, E. L. Fuster and B. Sims, Eds.), Yokohama Publishers, Yokohama, 81-94, 2004.
11. M. Kikkawa and W. Takahashi, Approximating fixed points of infinite nonexpansive mappings by the hybrid method, *J. Optim. Theory Appl.*, **117** (2003), 93-101.
12. F. Liu and M. Z. Nashed, Regularization of nonlinear ill-posed variational inequalities and convergence rates, *Set-Valued Anal.*, **6** (1998), 313-344.
13. K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, **279** (2003), 372-379.
14. K. Nakajo and W. Takahashi, Strong and weak convergence theorems by an improved splitting method, *Comm. Appl. Nonlinear Anal.*, **9** (2002), 99-107.

15. K. Nakajo, K. Shimoji and W. Takahashi, Weak and strong convergence theorems by Mann's type iteration and the hybrid method in Hilbert spaces, *J. Nonlinear Convex Anal.*, **4** (2003), 463-478.

16. Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591-597.

17. R. T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM J. Control Optim.*, **14** (1976), 877-898.

18. T. Shimizu and W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, *J. Math. Anal. Appl.*, **211** (1997), 71-83.

19. K. Shimoji and W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, *Taiwanese J. Math.*, **5** (2001), 387-404.

20. N. Shioji and W. Takahashi, Strong convergence theorems for continuous semigroups in Banach spaces, *Math. Japonica*, **50** (1999), 57-66.

21. M. V. Solodov and B. F. Svaiter, Forcing strong convergence of proximal point iterations in a Hilbert space, *Math. Programming Ser. A*, **87** (2000), 189-202.

22. W. Takahashi, A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space, *Proc. Amer. Math. Soc.*, **81** (1981), 253-256.

23. W. Takahashi, Weak and strong convergence theorems for families of nonexpansive mappings and their applications, *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, **51** (1997), 277-292.

24. W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

25. W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000(Japanese).

26. W. Takahashi and K. Shimoji, Convergence theorems for nonexpansive mappings and feasibility problems, *Math. Comput. Modelling*, *32* (2000), 1463-1471.

K. Nakajo
Faculty of Engineering,
Tamagawa University,
Tamagawa-gakuen, Machida-shi,
Tokyo, 194-8610, Japan
E-mail: nakajo@eng.Tamagawa.ac.jp

K. Shimoji
Department of Mathematical Sciences,
Faculty of Science,
University of the Ryukyus,
Nishihara-cho, Okinawa 903-0213, Japan
E-mail: shimoji@math.u-ryukyu.ac.jp

W. Takahashi
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology,
Oh-okayama, Meguro-ku,
Tokyo, 152-8552, Japan
E-mail: wataru@is.titech.ac.jp