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NULL BOUNDARY CONTROLLABILITY FOR A FOURTH ORDER SEMILINEAR EQUATION

Hsiao-Fang Chou and Yung-Jen Lin Guo

Abstract. We consider the null boundary controllability for a one-dimensional fourth order semilinear equation. We show that if the initial data is continuous and sufficiently small, then the fourth order semilinear equation is controllable.

1. INTRODUCTION

We consider the following initial boundary value problem for a one-dimensional fourth order semilinear equation

$$w_t + w_{xxxx} = f(w,x) \quad \text{on} \quad (0,1)\times(0,\infty), \tag{1.1}$$

$$w(0,t) = 0, \quad w_x(0,t) = 0 \quad \text{for} \quad t \geq 0, \tag{1.2}$$

$$w(x,0) = w_0(x) \quad \text{for} \quad x \in [0,1], \tag{1.3}$$

$$w(1,t) = g(t), \quad w_x(1,t) = h(t) \quad \text{for} \quad t \geq 0, \tag{1.4}$$

where $f(s,x)$ is a function defined in $\mathcal{N}\times[0,1]$ where $\mathcal{N}$ is a neighborhood of the origin, satisfying

$$| \; f(s_1,x_1) - f(s_2,x_2) \; | \leq K[\; | \; s_1 - s_2 \; | + | \; x_1 - x_2 \; |^{\alpha}]$$

for $s_1, s_2 \in \mathcal{N}$, $\quad x_1, x_2 \in [0,1]$ for some constants $K > 0, \quad 0 < \alpha < 1$, is analytic in both arguments in a neighborhood of the origin, belongs to Gevrey class 2 in s, varying contionuously with respect to x, and satisfies $f(0,x) = 0, \quad D_1 f(0,x) = 0$ for all $x \in [0,1]$. This work is devoted to syudying the null boundary controllability

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problem for (1.1)-(1.4); that is, given $T > 0$, is it possible, for every initial data w_0 which is small enough and in an appropriate space, to find corresponding controllers $g(t)$ and $h(t)$ so that the solution of the resulting problem (1.1)-(1.4) vanishes at time T for all $x \in [0, 1]$?

Here we use the method based on the work of Y. -J. L. Guo and W. Littman [5, 6], in which the control problem is transformed to two well-posed problems. For our case, the method proceeds roughly as follows:

(1) Extend the domain of f to be $\mathcal{N} \times [0, 2]$ and the initial data w_0 to be [0,2] so that all properties of the extended f and w_0 are preserved and $w_0(x) \equiv 0$ in a neighborhood of 2.

(2) With the modified function f and the initial data w_0, solve the initial boundary-value problem:

$$v_t + v_{xxxx} = f(v, x) \quad \text{on} \quad (0, 2) \times (0, \infty), \tag{1.5}$$

$$v(0, t) = 0, \quad v_x(0, t) = 0 \quad \text{for} \quad t \geq 0, \tag{1.6}$$

$$v(2, t) = 0, \quad v_x(2, t) = 0 \quad \text{for} \quad t \geq 0, \tag{1.7}$$

$$v(x, 0) = w_0(x) \quad \text{for} \quad x \in (0, 2), \tag{1.8}$$

(3) Choose a cut-off function ψ satisfying $\psi(t) = 1$ for $t \leq T/2$ and $\psi(t) = 0$ for $t \geq T$. Let

$$\xi(t) = v_{xx}(0, t) \cdot \psi(t), \quad \zeta(t) = v_{xxx}(0, t) \cdot \psi(t),$$

where v is the solution of problem (1.5)-(1.8) in step (2).

(4) Solve the Cauchy problem

$$u_{xxxx} = -u_t + f(u, x) \quad \text{for} \quad t \geq T_0, \quad x > 0, \tag{1.9}$$

$$\begin{aligned} &u(0, t) = 0, u_x(0, t) = 0, u_{xx}(0, t) = \xi(t), u_{xxx}(0, t) \\ &\quad = \zeta(t) \quad \text{for} \quad t \geq T_0, \end{aligned} \tag{1.10}$$

in the x-direction to obtain a solution which vanishes for $t \geq T$ and equals the solution v for $t \leq T/2$, where T_0 is a positive constant.

(5) Setting $g(t) = u(1, t)$ and $h(t) = u_x(1, t)$, we acquire the desired boundary control functions.

We can conduct our study with the standard measures except step 4, in which we use the nonlinear Cauchy-Kowalevski Theorem to solve the problem (1.9)-(1.10). To apply this theorem, we need ξ and ζ to be of Gevrey class 2 in t for $t > 0$; that is, there exist four positive constants C_i, H_i, $i = 1, 2$ such that

$$\left|\frac{\partial^n \xi}{\partial t^n}(t)\right| \le C_1 H_1^n (2n)! \quad \text{for} \quad t \ge 0, \quad n \ge 0,$$

$$\left|\frac{\partial^n \zeta}{\partial t^n}(t)\right| \le C_2 H_2^n (2n)! \quad \text{for} \quad t \ge 0, \quad n \ge 0.$$

In addition, we have to estimate the length of the x-interval of existence for the solution $u(x,t)$ of (1.9)-(1.10) by rechecking the constants in the proof of the nonlinear Cauchy-Kowalevski Theorem. If the solution $u(x,t)$ exists beyond $x = 1$, then the controllers are obtained simply by reading the values of the derivatives of $v(x,t)$ and $u(x,t)$ at $x = 1$, where $v(x,t)$ and $u(x,t)$ are solutions of (1.5)-(1.8) and (1.9)-(1.10) respectively. In [6], the authors consider the null boundary controllability for second order semilinear heat equations and acquire the results when the initial data is bounded continuous and sufficiently small. To guarantee the existence in the whole unit x-interval for the problem similar to problem (1.9)-(1.10), the smallness condition on the initial data cannot be eliminated in general. In [4], the author consider the exact boundary controllability for a second order linear heat equation with coefficients depending on the space variable and the time variable. With the aid of the Gevrey Class 2 properties for the coefficients, the linearity of the differential equation and the continuity of the initial data, one can show that the x-interval of existence for the problem similar to problem (1.9)-(1.10) is greater than 1 and the equation is controllable without the "sufficiently small" assumption on the initial data. In [5], the author consider the null boundary controllability for a linear fourth order parabolic equation and assume that the initial data is continuous. The linearity and the simplicity of the coefficients of the equation will ensure that the existence in the entire unit x-interval for the problem similar to problem (1.9)-(1.10). In this work, we consider a fourth order semilinear equation. The assumption that the initial data is continuously differentiable and small enough will help us to show that the x-interval of existence for problem (1.9)-(1.10) is greater than 1 and therefore the equation is controllable.

We remark that the controllers $g(t)$ and $h(t)$ one can seek are not necessarily unique. Null boundary controllability may also be secured by other continuous controllers.

The controllability theory of the linear heat equation has been considerably studied for several decades. Fattorini and Russell have, for example, initiated lots of decisive developments and presented them in numerous articles (see, e.g. [1, 2]). Most pieces of the research are dedicated to acquiring controllable results for parabolic equations. For the null boundary controllability of second order semilinear parabolic equations and a fourth order parabolic equation, see [5] and [6]. The case we consider here is a fourth order semilinear equation.

The paper is organized as follows. Section 2 contains the detailed restatement of the nonlinear Cauchy-Kowalevski Theorem, which is used to solve the Cauchy problem (1.9)-(1.10) and estimate the interval of existence. At last we obtain the null boundary controllability result for (1.1)-(1.4) in Section 3.

2. Solutins of the Cauchy Problem in the x-Direction

In this section, we shall solve the following Cauchy problem by using the nonlinear Cauchy-Kowalevski Theorem:

$$u_{xxxx} = -u_t + f(u,x) \quad \text{for} \quad x > 0, \quad t \geq T_0, \tag{2.1}$$

$$u(0,t)=0, u_x(0,t)=0, u_{xx}(0,t)=\xi(t), u_{xxx}(0,t)=\zeta(t) \;\text{ for }\; t \geq T_0, \tag{2.2}$$

where $\xi(t)$ and $\zeta(t)$ are Gevrey class 2 functions in t for $t > 0$ and T_0 is a positive constant. We will prove that the solution exists and the x-interval of existence is greater than 1, provided that $\xi(t)$ and $\zeta(t)$ are sufficiently small.

A generalization of the well-known Cauchy-Kowalevski Theorem, the nonlinear Cauchy-Kowalevski Theorem attributed to Ovcyannikov has been exploited variously to obtain existence results in the study of the nonlinear abstract Cauchy problem

$$\begin{aligned} \frac{du}{dx} &= F(u,x), \quad |x| < \eta, \quad \eta > 0, \\ u(0) &= u_0. \end{aligned}$$

Here the solutions are sought, as functions of the variable x, in a scale of Banach space $\{X_s\}$. The nonlinear Cauchy-Kowalevski Theorem is reduced to the Cauchy-Kowalevski Theorem when all data are real analytic. To solve problem (2.1)-(2.2), the method used in [6] is again employed here. We begin by considering a 1-parameter family of Banach spaces $\{X_s\}$ where the parameter s is allowed to vary in [0,1].

Definition 2.1. $\{X_s\}_{0\leq s\leq 1}$ is a scale of Banach spaces if for any $s \in [0,1]$, X_s is a linear subspace of X_0 and if $s' \leq s$ then $X_s \subset X_{s'}$ and the natural injection of X_s into $X_{s'}$ has norm less than or equal to 1.

We denote by $\|\cdot\|_{X_s}$ the norm of X_s.

Since it is necessary to estimate the parameters in the nonlinear Cauchy-Kowalevski Theorem to obtain the interval of existence, we shall restate the Theorem here.

For each i, $i = 1, \cdots, m$, let $\{X_s^i\}_{0\leq s\leq 1}$ be a scale of Banach spaces with norm $\|\cdot\|_{X_s^i}$. Consider the system of differential equations

$$\frac{du_i}{dx} = F_i(u_1, u_2, \cdots, u_m, x), \quad |x| < \eta, \quad \eta > 0, \quad i = 1, \cdots, m, \tag{2.3}$$

$$u_i(0) = u_{i,0}, \quad i = 1, \cdots, m, \tag{2.4}$$

where the u_i, as functions of the variable x, are in X_s^i, $i = 1, \cdots, m$.

We need the following hypotheses.

(**H1**) $u_{i,0} \in X_s^i$ for every $s \in [0,1]$ and satisfies

$$\|u_{i,0}\|_{X_s^i} \le R_{i,0},$$

for some $R_{i,0} < \infty$ and for $i = 1, \cdots, m$.

(**H2**) There are $R_i > R_{i,0} \ge 0, i = 1, \cdots, m$, $\eta > 0$, such that for every pair of numbers s, s' with $0 \le s' < s \le 1$, the mapping $F_i(u_1, \cdots, u_m, x), i = 1, \cdots, m$, is continuous from the set

$$\left\{u_1 \in X_s^1 \mid \|u_1\|_{X_s^1} < R_1\right\} \times \cdots \times \left\{u_m \in X_s^m \mid \|u_m\|_{X_s^m} < R_m\right\} \times \{x \mid |x| < \eta\}$$

into $X_{s'}^i$.

(**H3**) There are constants C_i, $i = 1, \cdots, m$, such that for every pair of numbers s, s' with $0 \le s' < s \le 1$, for all $\|u_j\|_{X_s^j} < R_j$, $\|v_j\|_{X_s^j} < R_j$, $j = 1, \cdots, m$, and for all x, $|x| < \eta$, we have

$$\begin{aligned} &\|F_i(u_1, u_2, \cdots, u_m, x) - F_i(v_1, v_2, \cdots, v_m, x)\|_{X_{s'}^i} \\ \le\ & \frac{C_i}{(s-s')^{\alpha_i}}\Big[\vartheta_i^1 \|u_1 - v_1\|_{X_s^1} + \cdots + \vartheta_i^m \|u_m - v_m\|_{X_s^m}\Big], \ \ i = 1, \cdots, m, \end{aligned}$$

where the number ϑ_i^j is set to be zero if F_i is independent of u_j and to be one otherwise, for some parameters $\alpha_i \ge 0$, $i = 1, \cdots, m$, such that for any collection of m^2 numbers c_i^j, the degree of $P(\lambda, \mu)$ with respect to λ, μ is at most m, where the expression $P(\lambda, \mu)$ of two variables λ, μ is defined by

$$P(\lambda, \mu) = det\left(\lambda I - \left[\mu_i^{\alpha}\vartheta_i^j c_i^j\right]\right),$$

with I the $m \times m$ identity matrix and the degree is defined to be the highest degree among all monomials in $P(\lambda, \mu)$.

(**H4**) $F_i(0, \cdots, 0, x)$ is a continuous function of x, $|x| < \eta$, with values in X_s^i for every $s < 1$ and satisfies

$$\|F_i(0, \cdots, 0, x)\|_{X_s^i} \le \frac{K_i}{(1-s)^{\alpha_i}}, \quad 0 \le s < 1,$$

for some constants K_i, $i = 1, \cdots, m$, with α_i defined in (H3).

Then we have the following existence and uniqueness theorem for solutions of (2.3)-(2.4).

Theorem 1.1 [6]. *Under the preceding hypotheses* (H1)-(H4) *there is a positive constant* ρ *such that the Cauchy problem* (2.3)-(2.4) *has a unique solution* $\{u_i(x), i = 1, \cdots, m\}$, *which are continuously differentiable functions of* x, $|x| < \rho(1-s)$, *with values in* X_s^i *such that* $\|u_i(x)\|_{X_s^i} < R_i$ *for every* $s < 1/2$.

Remark 2.1. The proof of Theorem 2.1 [6] offers us an estimate of the interval of existence. For $m = 5$, which is the case we will consider, the constant ρ in Theorem 2.1 is any positive constant satisfying

$$\begin{aligned}
&\rho < \frac{1}{8\sqrt[4]{2(4C_5 + N)}}, \\
&8(2\rho)^2(R_{3,0} + 8\rho R_{4,0}) < \frac{R_1}{2}, \\
&4\rho(R_{3,0} + 8\rho R_{4,0}) < \frac{R_2}{2}, \\
&4\rho[R_{4,0} + 64(4C_5 + N)(2\rho)^3 R_{3,0}] < \frac{R_3 - R_{3,0}}{2}, \\
&32(4C_5 + N)(2\rho)^3(R_{3,0} + 8\rho R_{4,0}) < \frac{R_4 - R_{4,0}}{2}, \\
&16C_5(2\rho)^3(R_{3,0} + 8\rho R_{4,0}) < \frac{R_5}{2},
\end{aligned}$$

where C_i, R_i, $R_{i,0}$ are constants in the hypotheses (H1)-(H4), for $i = 1, 2, \cdots, 5$ and N is a constant depending on R_1. We note that the interval [0, ρ) is not necessary the largest length of the interval of existence because we have not always chosen the best possible constants here.

To apply Theorem 2.1 to solve the Cauchy problem (2.1)-(2.2), we choose the following scale of Banach spaces.

Definition 2.2. Let K be a compact interval and let θ_0 and θ_1 be two positive constants such that $\theta_0 < \theta_1 < \infty$. Given $s \in [0, 1]$, we define the space $B_s(K)$ to be the set of all $C^\infty(K)$ functions ϕ satisfying

$$\|\phi\|_{B_s} \equiv \sup_{n\geq 0} \max_{t\in K} \frac{\widetilde{n}^4 \theta(s)^n}{\lambda(2n)!} \left|\phi^{(n)}(t)\right| < \infty, \tag{2.5}$$

where $1/\theta(s) = (1-s)/\theta_0 + s/\theta_1$, $\widetilde{n} = \max(n, 1)$, and λ is any positive constant satisfying

$$\lambda \leq 1/[2 + 2^4 \sum_{k=1}^{\infty} (1/k)^4].$$

It is easy to see that $\| \cdot \|_{B_s}$ in (2.5) is a norm on $B_s(K)$ which makes $\{B_s(K)\}_{0 \leq s \leq 1}$ a scale of Banach spaces.

The Gevrey class 2 functions which play an important role in this work are defined as follows.

Definition 2.3. Let Ω be a subset of $\mathbf{R}^n$ and $\delta > 0$. A C^∞ function f in Ω is said to be of Gevrey class δ in Ω (in short, $f \in \gamma^\delta(\Omega)$) if there exist positive constants C and H such that

$$|D_x^\alpha f(x)| \leq CH^{|\alpha|}(\delta|\alpha|)!$$

for all multi-indices α and for all $x \in \Omega$, where $\alpha! = \Gamma(\alpha + 1)$ and Γ is the usual gamma function.

It is obvious that a function of Gevrey class δ in Ω is bounded.

Now we can easily deduce (a) and (b) of the following proposition which describe the relationship between the space $B_s(K)$ and the Gevrey class 2 functions. We also have (c) by [6, Proposition 4.5].

Proposition 2.1. *Let K be a compact intervel. Then*

(a) *The space $B_s(K)$ is contained in γ^2 for all $s \in [0, 1]$.*

(b) *Suppose $\phi : \mathbf{R} \to \mathbf{R}$ is an infinitely differentiable function defined in K and there are positive constants C and H such that*

$$|\phi^{(j)}(t)| \leq CH^j(2j)!,$$

for all t and for all $j = 1, 2, \cdots$. If the constant θ_1 in definig $B_s(K)$ satisfies $\theta_1 < 1/H$, then $\phi \in B_s(K)$ for all $s \in [0, 1]$.

(c) *Suppose that $f(z, x)$ is a real-valued function on $\mathbf{R}^2$ which is of Gevrey class 2 in its first argument for all $(z, x) \in I \times \{x \in \mathbf{R} | |x| \leq \omega\}$, where I is any compact z-intervel and ω is some positive number. Define a map F on $B_s(K) \times \{x \in \mathbf{R} | |x| \leq \omega\}$ by*

$$F(u, x)(t) = f(u(t), x).$$

Then F is a map from $B_s(K) \times \{x \in \mathbf{R} | |x| \leq \omega\}$ into $B_{s'}(K)$, $0 \leq s' \leq s \leq 1$.

According to [6,position 4.2], the partial differentiation $\partial/\partial t$ defines a bounded linear operator from $B_s(K)$ into $B_{s'}(K)$ for $0 \leq s' < s \leq 1$ with norm less than or equal to $C/(s - s')^2$, where C is a positive constant which can be taken as $(4/e)^2\theta_0/(\theta_1 - \theta_0)^2$. Note that we can make the constant C as small as we desire by taking the constant θ_0 sufficiently small while keeping the constant θ_1 fixed in the definiton of $B_s(K)$.

Following is the main result of this section.

Theorem 2.2. *Let the function $f(u,x)$ belong to Gevrey class 2 locally in its first argument, varying continuously with respect to x, and satisfy $f(0,x)=0$ and $D_1 f(0,x)=0$ for all $x\in[0,2]$. Suppose that $\xi(t)$, $\zeta(t)\in\gamma^2([T_0,\infty))$ with support $[T_0,T]$ for some $T>T_0$. Then a classical solution $u(x,t)$ of* (2.1),(2.2) *exists and the x-interval of existence will be greater than 1 when $\xi(t)$ and $\zeta(t)$ are small enough.*

Proof. To apply Theorem 2.1, let

$$v_1=u,\ \ v_2=u_x,\ \ v_3=u_{xx},\ \ v_4=u_{xxx}\ \ and\ \ v_5=u_t$$

so that the problem (2.1)-(2.2) is converted to a first-order system of differential equations

$$\frac{dv_1}{dx}(x,\cdot)=v_2(x,\cdot), \tag{2.6}$$

$$\frac{dv_2}{dx}(x,\cdot)=v_3(x,\cdot), \tag{2.7}$$

$$\frac{dv_3}{dx}(x,\cdot)=v_4(x,\cdot), \tag{2.8}$$

$$\frac{dv_4}{dx}(x,\cdot)=-v_5(x,\cdot)+f(v_1(x,\cdot),x), \tag{2.9}$$

$$\frac{dv_5}{dx}(x,\cdot)=\frac{\partial}{\partial t}v_2(x,\cdot), \tag{2.10}$$

with the Cauchy data

$$v_1(0,\cdot)=0,\ v_2(0,\cdot)=0,\ v_3(0,\cdot)=\xi(\cdot),\ v_4(0,\cdot)=\zeta(\cdot),\ v_5(0,\cdot)=0. \tag{2.11}$$

For any finite positive number ε, let

$$K=[T_0,T+\varepsilon]\ \text{ and }\ D=[0,2]\times K.$$

Since $\xi(t)$, $\zeta(t)\in\gamma^2(D)$ and $f(u,x)$ belong to Gevrey class 2 locally in its first argument, there exist positive constants M_i, H_i, $i=1,2,3$ such that

$$\left|\partial_t^j\xi(t)\right|\le M_1H_1^j(2j)!,$$

$$\left|\partial_t^j\zeta(t)\right|\le M_2H_2^j(2j)!,$$

$$\left|\partial_u^j f(u,x)\right|\le M_3H_3^j(2j)!,$$

for all $t \in K$ and any nonnegative integers j. Choose two constants θ_0, θ_1 satisfying $0 < \theta_0 < \theta_1 < min(1/H_1, 1/H_2, 1/H_3)$. We use the same notation as in Theorem 2.1. Let $X_s^i = B_s(K)$, $i = 1, 2, \cdots, 5$, where $\{B_s(K)\}_{0 \le s \le 1}$ is the scale of Banach spaces as defined in Definition 2.2 with constants θ_0 and θ_1 and $R_{1,0} = R_{2,0} = R_{5,0} = 0$, $R_{3,0} = \|\xi\|_{B_1}$, $R_{4,0} = \|\zeta\|_{B_1}$. Then by Proposition 2.1 it is easy to check that $\xi(t)$, $\zeta(t) \in B_s(K)$ for all $s \in [0,1]$ and assumptions (H1)-(H4) of Theorem 2.1 are all satisfied with $C_i = 1$, for $i = 1, 2, 3$, some positive constant C_4 and $C_5 = (4/e)^2\theta_0/(\theta_1 - \theta_0)^2$ which can be chosen as small as we wish by taking the constant θ_0 sufficiently small while keeping the constant θ_1 fixed in the definition of $B_s(K)$. By Theorem 2.1, a constant $\rho > 0$ exists and the Cauchy problem (2.6)-(2.11) has a unique solution $\{v_i(x, \cdot), i = 1, 2, \cdots, 5\}$. So (2.1)-(2.2) has a C^4 solution $u(x, \cdot) \in B_0(K)$ for $|x| < \rho$.

According to the proof of the nonlinear Cauchy-Kowalevski Theorem in [6], since for any constants $R_1 > 0$, $R_5 > 0$ and for v_i, $\widetilde{v}_i \in B_s(K)$, $i = 1, 5$, $s \in [0,1]$ with $\|v_i\|_{B_s} < R_i$, $\|\widetilde{v}_i\|_{B_s} < R_i$, and $|x| < \eta$, where η can be any large number for our problem, we have, for $0 \le s' < s \le 1$,

$$\begin{aligned}\|[-v_5 + f(v_1, x)] - [-\widetilde{v}_5 + f(\widetilde{v}_1, x)]\|_{B_{s'}} &\le \|v_5 - \widetilde{v}_5\|_{B_s} + \|f(v_1, x) - f(\widetilde{v}_1, x)\|_{B_s} \\ &\le \|v_5 - \widetilde{v}_5\|_{B_s} + N\|v_1 - \widetilde{v}_1\|_{B_s},\end{aligned}$$

where N depending on R_1 is a constant which can become small enough when $R_1 \to 0$ by the assumption $D_1 f(0, x) = 0$, the length of the x-interval of existence ρ is any positive constant satisfying

$$\begin{aligned}&\rho < \frac{1}{8\sqrt[4]{2(4C_5 + N)}}, \\ &8(2\rho)^2(R_{3,0} + 8\rho R_{4,0}) < \frac{R_1}{2}, \\ &4\rho(R_{3,0} + 8\rho R_{4,0}) < \frac{R_2}{2}, \\ &4\rho[R_{4,0} + 64(4C_5 + N)(2\rho)^3 R_{3,0}] < \frac{R_3 - R_{3,0}}{2}, \\ &32(4C_5 + N)(2\rho)^3(R_{3,0} + 8\rho R_{4,0}) < \frac{R_4 - R_{4,0}}{2}, \\ &16C_5(2\rho)^3(R_{3,0} + 8\rho R_{4,0}) < \frac{R_5}{2}.\end{aligned}$$

By choosing R_i large enough for $i = 2, 3, 4, 5$ and, under the assumptions that $R_{3,0} = \|\xi\|_{B_1}$, $R_{4,0} = \|\zeta\|_{B_1}$ are both sufficiently small, taking R_1 and the constant C_5 small enough, the x-interval of existence ρ can be greater than 1. ∎

3. Existence of Boundary Controllers

The final goal of this work to prove the existence of the boundary controllers $g(t)$ and $h(t)$ will be achieved in this section. In Theorem 3.2 we acquire the continuously differentiable controllers $g(t)$ and $h(t)$ that lead a prescribed initial data w_0 to zero within a finte time for the problem (1.1)-(1.4). To accomplish our purpose, we need theorem 3.1 beforehand, whose proof is similar to the proof of Theorem 2.1 in the paper of D. Kinderlehrer and L. Nirenberg [8] with some modification. We omit the proof here. See also [10] for more details for a second order parabolic equation.

Theorem 3.1. *Let* $v(x,t) \in C^\infty([0,1]\times[0,1])$ *be a solution of the problem*

$$v_t + v_{xxxx} - f(v,x) = 0 \quad \textit{on} \quad 0 < x < 1, \quad t > 0,$$

$$v(0,t) = 0, \;\; v_x(0,t) = 0 \quad \textit{for} \quad t \geq 0,$$

where $f(v,x)$ *is a Gevrey calss 2 function of* v *and* x *in the range of these two arguments for* $0 \leq x \leq 1$, $t \geq 0$.
Then for each σ, $0 < \sigma < \frac{1}{2}$, $v(x,t)$ *is of Gevrey class 2 in* x *and* t *in*

$$\{(x,t) : 0 \leq x < 1-\sigma, \;\; \sigma < t < 1\};$$

that is, the derivatives of v *satisfy*

$$|\partial_x^k \partial_t^j v| \leq CH^{2k+2j}(2k+2j)!$$

for some positive constants C, H *and for all* $k = 0,1,2,\cdots$, *and* $j = 0,1,2,\cdots$.

Now, we are ready to prove the principal result of this paper.

Theorem 3.2. *Suppose* $f(s,x)$ *defined in* $\mathcal{N}\times[0,1]$, *where* $\mathcal{N}$ *is a neighborhood of the origin, is an analytic function in both arguments in a neighborhood of the origin satisfying*

$$|\, f(s_1,x_1) - f(s_2,x_2) \,| \leq K[\,|\, s_1 - s_2 \,| + |\, x_1 - x_2 \,|^\alpha]$$

for $s_1, s_2 \in \mathcal{N}$, $\;\; x_1, x_2 \in [0,1]$ *for some constants* $K > 0$, $\;\; 0 < \alpha < 1$, *belonging to Gevrey class 2 in* s, *and varying contionouusly with respect to* x. *Let* $f(0,x) = 0$, $\;\; D_1 f(0,x) = 0$ *for all* $x \in [0,1]$ *and let the initial data* $w_0(x)$ *be a continuous sufficiently small function in* $[0,1]$ *and vanish at zero. Then, for any finite time* $T > 0$, *there exist controllers* $g(t)$, $h(t) \in C^\infty((0,\infty) \cap C([0,\infty))$ *such that the solution* $w(x,t)$ *of* (1.1)-(1.4) *satisfies* $w(x,T) \equiv 0$ *for* $x \in [0,1]$.

Proof. We organize the proof in a series of steps.

Step 1. Extend the domain of the initial data $w_0(x)$ to be [0,2] so that $w_0(x)$ remains continuous, $w_0(x) \equiv 0$ in a neighboorhood of 2 and $\|w_0\|_{L^\infty([1,2])} \leq \|w_0\|_{L^\infty([0,1])}$. We also extend the domain of f to be $\mathcal{N} \times [0,2]$ such that all properties of f are retained.

Step 2. With the modified initial condition, we solve the initial boundary-value problem

$$w_t + w_{xxxx} = f(w,x) \quad \text{on} \quad (0,2) \times (0,\infty), \tag{3.1}$$

$$w(0,t) = 0, \quad w_x(0,t) = 0 \quad \text{for} \quad t \geq 0, \tag{3.2}$$

$$w(2,t) = 0, \quad w_x(2,t) = 0 \quad \text{for} \quad t \geq 0, \tag{3.3}$$

$$w(x,0) = w_0(x) \quad \text{for} \quad x \in (0,2). \tag{3.4}$$

It is well-known that the solution $w(x,t)$ exists locally and is bounded, cf. [1, 3]. Let T_1 be any given finite time and $\varepsilon < T_1/2$ be any small positive number so that $f(w,x)$ is analytic in the range of values assumed by w, x for $x \in [0, 2\varepsilon]$. Then clearly $w(x,t)$ belongs to C^∞ for $x \in [0, 2\varepsilon]$ and $t \in [\varepsilon, T_1]$.

Step 3. Firstly, we claim that $w_{xx}(0,t)$ and $w_{xxx}(0,t)$ are both γ^2 functions in t for $2\varepsilon \leq t \leq T_1$, where $w(x,t)$ is the solution obtained in Step 2. Next, for small $T < T_1$, choosing a cut-off function ψ satisfying $\psi(t) = 1$ for $t \leq T/2$ and $\psi(t) = 0$ for $t \geq T$, we modify $w_{xx}(0,t)$ and $w_{xxx}(0,t)$ to be functions $w_{xx}(0,t)\psi(t)$ and $w_{xxx}(0,t)\psi(t)$ with support in $[0,T]$.

Let $u_0(x) = w(x,\varepsilon)$, where $\varepsilon < T_1/2$ is any small positive number as in Step 2. Since $w(x,t)$ is a $C^\infty([0,1] \times [\varepsilon, T_1])$ solution of the problem

$$w_t + w_{xxxx} = f(w,x) \quad \text{on} \quad (0, 2\varepsilon) \times (\varepsilon, T_1],$$

$$w(0,t) = 0, \quad w_x(0,t) = 0 \quad \text{for} \quad \varepsilon \leq t \leq T_1,$$

$$w(x,\varepsilon) = u_0(x) \quad \text{for} \quad x \in (0, 2\varepsilon),$$

$w(x,t)$ is of Gevrey class 2 in t for $0 \leq x \leq \varepsilon$ and $2\varepsilon \leq t \leq T_1$ by Theorem 3.1. Hence $w_{xx}(0,t)$ and $w_{xxx}(0,t)$ belong to γ^2 in t for $2\varepsilon \leq t \leq T_1$. Moreover, it can be easily seen that small initial data $w_0(x)$ in sup norm implies $\|w_{xx}(0,t)\|_{B_1}$ and $\|w_{xxx}(0,t)\|_{B_1}$ will be sufficiently small for $t \in [2\varepsilon, T]$ when $T < T_1$ is small enough.

Let $\psi(t) \in C^\infty$ on $[0, \infty)$ satisfying

$$
\begin{aligned}
&0 \le \psi(t) \le 1,\\
&\psi(t) = 0 \quad \text{for} \quad t \ge T,\\
&\psi(t) = 1 \quad \text{for} \quad 0 \le t \le \frac{T+2\varepsilon}{2}.
\end{aligned}
$$

We can take $\psi(t)$ to be of Gevrey class 2 with some care (see [7]).

Set

$$
\xi(t) = \begin{cases} w_{xx}(0,t)\psi(t) & \text{for} \quad 2\varepsilon \le t \le T,\\ 0 & \text{for} \quad t \ge T, \end{cases}
$$

and

$$
\zeta(t) = \begin{cases} w_{xxx}(0,t)\psi(t) & \text{for} \quad 2\varepsilon \le t \le T,\\ 0 & \text{for} \quad t \ge T. \end{cases}
$$

Since the Gevrey class of functions forms an algebra which is closed under multiplication, $\xi(t)$, $\zeta(t) \in \gamma^2$ in t for $t \ge 2\varepsilon$ and vanish for $t \ge T$. When T is small, $\xi(t)$, $\zeta(t)$ will be small enough because $\|w_{xx}(0,t)\|_{B_1}$ and $\|w_{xxx}(0,t)\|_{B_1}$ are sufficiently small.

Step 4. Now we solve the Cauchy problem

$$
u_{xxxx} = -u_t + f(u,x) \quad \text{on} \quad (0,2)\times(2\varepsilon,\infty), \tag{3.5}
$$

$$
u(0,t)=0, u_x(0,t)=0, u_{xx}(0,t)=(t), u_{xxx}(0,t)=\zeta(t), \quad \text{for } t \ge 2\varepsilon. \tag{3.6}
$$

Since $\xi(t)$, $\zeta(t)$ are small, according to Theorem 2.2 there exist a constant $\rho > 1$ and a classical solution $u(x,t)$ of (3.5)-(3.6) for $0 < x < \rho$, $t \ge 2\varepsilon$.

Before moving to the final step, we shall derive that $w(x,t)$ and $u(x,t)$ agree in $[2\varepsilon, (T+2\varepsilon)/2]$. Let $z(x,t) = w(x,t) - u(x,t)$. Then it is easy to see that $z \equiv 0$ on $[0,1]\times[2\varepsilon,(T+2\varepsilon)/2]$ by L. Nirenberg's Theorem [11], so $w(x,t)$ and $u(x,t)$ are identical on $[0,1]\times[2\varepsilon,(T+2\varepsilon)/2]$.

Step 5. Consequently, the required boundary controllers $g(t)$ and $h(t)$ can be read off through $w(x,t)$ and $u(x,t)$ by defining $g(t) = w(1,t)$, $h(t) = w_x(1,t)$ for $0 \le t \le 2\varepsilon$ and $g(t) = u(1,t)$, $h(t) = u_x(1,t)$ for $t \ge 2\varepsilon$.

The proof of Theorem 3.2 is complete. ■

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Hsiao-Fang Chou and Yung-Jen Lin Guo
Department of Mathematics,
National Taiwan Normal University,
Taipei, 116, Taiwan.