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ON APPROXIMATE ISOMORPHISMS BETWEEN BANACH $*$-ALGEBRAS OR C^*-ALGEBRAS

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Abstract. In this paper, we study some problems about approximate isomorphisms between Banach $*$-algebras or C^*-algebras.

1. Introduction

The problem of the stability of functional equations has been first studied by Ulam in 1940 (see [7]). He posed the following problem: “Give conditions in order for a linear mapping near an approximately linear mapping to exist”.

In 1941, Hyers [3] showed that:
If $\delta > 0$ and $f : E_1 \to E_2$ is a mapping between Banach spaces such that

$$||f(x + y) - f(x) - f(y)|| \leq \delta, \ \forall\ x, y \in E_1$$

then there exists a unique $T : E_1 \to E_2$ such that $T(x + y) = T(x) + T(y)$ and $||f(x) - T(x)|| \leq \delta$ for all $x, y \in E_1$. In fact, $T(x) = \lim_{n\to\infty} \dfrac{f(2^n x)}{2^n}$. Furthermore, If for any $x \in E_1$, $f(tx)$ is continuous in scalar variable t, then T is a linear mapping.

In 1978, a generalized solution was given by Rassias [5]:
Let $f : E_1 \to E_2$ be a mapping between two Banach spaces E_1 and E_2 such that for any $x \in E_1$, $f(tx)$ is continuous in scalar variable t. If there exists $\theta \geq 0$ and $p \in [0, 1)$ such that $||f(x + y) - f(x) - f(y)|| \leq \theta(||x||^p + ||y||^p)$ for every $x, y \in E_1$, then there exists a unique mapping $T : E_1 \to E_2$ such that $||f(x) - T(x)|| \leq \dfrac{2\theta}{2 - 2^p}||x||^p$, $\forall\ x \in E_1$. Indeed, $T(x) = \lim_{n\to\infty} \dfrac{f(2^n x)}{2^n}$.

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The proof of Rassias [5] is also valid for $p < 0$.

In 1991, Gajda [1] gave a solution for $p > 1$:

Let $f : E_1 \to E_2$ *be a mapping between two Banach spaces* E_1 *and* E_2 *such that for any* $x \in E_1$, $f(tx)$ *is continuous in scalar variable* t. *If there exists* $\theta \geq 0$ *and* $p > 1$ *such that* $||f(x+y) - f(x) - f(y)|| \leq \theta(||x||^p + ||y||^p)$ *for every* $x, y \in E_1$, *then there exists a unique mapping* $T : E_1 \to E_2$ *such that* $||f(x) - T(x)|| \leq \dfrac{2\theta}{2^p - 2}||x||^p$, $\forall\, x \in E_1$. *Indeed,* $T(x) = \lim_{n\to\infty} 2^n f(2^{-n}x)$.

For the case $p = 1$, Rassias and Semrl [6] gave an example of a continuous real-valued function $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x+y) - f(x) - f(y)| \leq |x| + |y|$, $\forall\, x, y \in \mathbb{R}$ such that $\lim_{x\to 0} \dfrac{f(x)}{x} = \infty$. Hence the set $\{\dfrac{|f(x) - T(x)|}{|x|} \mid x \neq 0\}$ is unbounded for any linear mapping $T : \mathbb{R} \to \mathbb{R}$. In other words, an analogue of Rassias's result [5] can not be obtained for $p = 1$.

In 1992, Gavruta [2] genelized the result of Rassias as follows:

Let $(G, +)$ *be an abelian group and* $(X, ||\cdot||)$ *be a Banach space.* $\varphi : G \times G \to [0, \infty)$ *is called an admissible control function if* $\tilde{\varphi}(x, y) := \dfrac{1}{2}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y) < \infty$ *for all* $x, y \in G$. *If* $f : G \to X$ *is a mapping such that* $||f(x+y) - f(x) - f(y)|| \leq \varphi(x, y)$ *for all* $x, y \in G$, *then there exists a unique mapping* $T : G \to X$ *such that* $T(x+y) = T(x) + T(y)$ *and* $||f(x) - T(x)|| \leq \tilde{\varphi}(x, x)$ *for all* $x, y \in G$. *Indeed,* $T(x) = \lim_{n\to\infty} \dfrac{f(2^n x)}{2^n}$.

In 2003, Park [4] establishes the stability of algebra $*$-homomorphisms on a Banach $*$-algebra and the stability of automorphisms on a unital C^*-algebra. His proof actually gave the following two theorems.

Theorem 1.1. [(4)] *Let* A *and* B *be two Banach* $*$*-algebras. Let* $f : A \to B$ *be a mapping such that there exists an admissible control function* $\varphi : B \times B \to [0, \infty)$ *such that*

(i) $||f(\mu x + \mu y) - \mu f(x) - \mu f(y)|| \leq \varphi(x, y)$ *for all scalar* $|\mu| = 1$ *and all* $x, y \in A$.

(ii) $||f(x^*) - f(x)^*|| \leq \varphi(x, x)$ *for all* $x \in A$.

(iii) $||f(zw) - f(z)f(w)|| \leq \varphi(z, w)$ *for all self-adjoint* $z, w \in A$.

Then there exists a unique algebra $*$*-homomorphism* $T : A \to B$ *such that* $||f(x) - T(x)|| \leq \varphi(x, x)$ *for all* $x \in A$.

Theorem 1.2. [(4)] *Let A and B be two unital C^*-algebra and $\varphi : A \times A \to [0, \infty)$ be an admissible control function. If $f : A \to B$ be a bijective mapping with $f(xy) = f(x)f(y)$, and satisfying condition*

(i) *of Theorem 1.1 and*

(ii') $||f(u^*) - f(u)^*|| \leq \varphi(u, u)$ *for all unitary elements u of A.*

Assume that $\lim_{n\to\infty} \frac{f(2^n 1_A)}{2^n}$ is invertible where 1_A is the identity of A. Then f is actually an automorphism.

In this paper, we explore further variations of the above results.

2. Main Results

We use the following notations through out this paper.

- Let A and B denote Banach $*$-algebras or C^*-algebras.
- Let $\mathbb{T}$ denote the unit circle.
- Let 1_A denote the identity of the corresponding algebra if it exists.
- Let A_{sa} denote the set of self-adjoint elements in A.
- Let $\mathcal{U}(A)$ denote the group of unitary elements in A.

We will first apply similar techniques as in [4] to get the following lemma. Then we will use the lemma and other things to have our results.

Lemma 2.1. *Let $f : A \to B$ be a mapping between two C^*-algebras A and B. If there exists an admissible control function $\varphi : A \times A \to [0, \infty)$ such that*

(i) $||f(\mu x + \mu y) - \mu f(x) - \mu f(y)|| \leq \varphi(x, y)$, $\forall\ \mu \in \mathbb{T}$, $x, y \in A$

(ii) $||f(x^*) - f(x)^*|| \leq \varphi(x, x)$, $\forall\ x \in A$

(iii) $||f(\alpha\beta uv) - f(\alpha u)f(\beta v)|| \leq \varphi(\alpha u, \beta v)$, $\forall\ \alpha, \beta \in \mathbb{R}$, $u, v \in \mathcal{U}(A)$

then $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ defines the unique $*$-homomorphism such that

$$||f(x) - T(x)|| \leq \tilde{\varphi}(x, x),\ \forall\ x \in A.$$

Proof. Let $\mu = 1$ in (i), by Gavruta's result, there exists a unique additive function $T : A \to B$ such that $||f(x) - T(x)|| \leq \tilde{\varphi}(x, x)$, $\forall\ x \in A$. Indeed, $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$.

Substitute the x, y in (i) by $2^{n-1}x$, then

$$||f(2^n \mu x) - 2\mu f(2^{n-1}x)|| \leq \varphi(2^{n-1}x, 2^{n-1}x),\ \forall\ \mu \in \mathbb{T},\ x \in A.$$

Therefore,

$$||\mu f(2^n x) - 2\mu f(2^{n-1}x)|| \le |\mu| ||f(2^n x) - 2f(2^{n-1}x)|| \le \varphi(2^{n-1}x, 2^{n-1}x).$$

We have

$$\begin{aligned} ||f(2^n \mu x) - \mu f(2^n x)|| &\le ||f(2^n \mu x) - 2\mu f(2^{n-1}x)|| + ||2\mu f(2^{n-1}x) - \mu f(2^n x)|| \\ &\le 2\varphi(2^{n-1}x, 2^{n-1}x). \end{aligned}$$

Hence

$$2^{-n}||f(2^n \mu x) - \mu f(2^n x)|| \le 2^{-(n-1)}\varphi(2^{n-1}x, 2^{n-1}x) \to 0.$$

Thus we have

$$\forall\ \mu \in \mathbb{T}, x \in A,\ T(\mu x) = \lim_{n\to\infty} \frac{f(2^n \mu x)}{2^n} = \lim_{n\to\infty} \frac{\mu f(2^n x)}{2^n} = \mu T(x).$$

Now for any $\lambda \in \mathbb{C}$, there exists an $M \in \mathbb{N}$ such that $|\frac{\lambda}{M}| < \frac{1}{3}$. Therefore, there exist $\mu_1, \mu_2, \mu_3 \in \mathbb{T}$ such that $\frac{3\lambda}{M} = \mu_1 + \mu_2 + \mu_3$ (by considering the case $\frac{3\lambda}{M} = r \in [0, 1)$ with $\mu_1 = 1$ and $\overline{\mu_2} = \mu_3$). Also, from additivity, $T(x) = T(3 \cdot \frac{1}{3}x) = 3T(\frac{1}{3}x)$, we have $T(\frac{1}{3}x) = \frac{1}{3}T(x)$. Hence, by the above,

$$\begin{aligned} T(\lambda x) &= T(\frac{M}{3} \cdot \frac{3}{M}\lambda x) \\ &= MT(\frac{1}{3} \cdot \frac{3\lambda}{M}x) \\ &= \frac{M}{3}T(\mu_1 x + \mu_2 x + \mu_3 x) \\ &= \frac{M}{3}(\mu_1 T(x) + \mu_2 T(x) + \mu_3 T(x)) \\ &= \frac{M}{3} \cdot \frac{3\lambda}{M}T(x) \\ &= \lambda T(x). \end{aligned}$$

That is, T is $\mathbb{C}$ linear.

Similaryly, by (ii), $\forall x \in A$, $||f(2^n x^*) - f(2^n x)^*|| \le \varphi(2^n x, 2^n x)$. Therefore, $2^{-n}||f(2^n x^*) - f(2^n x)^*|| \le 2^{-n}\varphi(2^n x, 2^n x)$. Hence,

$$\forall\ x \in A,\ T(x^*) = \lim_{n\to\infty} \frac{f(2^n x^*)}{2^n} = \lim_{n\to\infty} \frac{f(2^n x)^*}{2^n} = T(x)^*.$$

In (iii), take $\alpha = \beta = 2^n$, we have

$$||f(4^n uv) - f(2^n u)f(2^n v)|| \le \varphi(2^n u, 2^n v).$$

Therefore,

$$||\frac{f(4^n uv)}{4^n} - \frac{f(2^n u)}{2^n}\frac{f(2^n v)}{2^n}|| \leq 4^{-n}\varphi(2^n u, 2^n v) \to 0, \text{ as } n \to \infty.$$

Thus

$$\begin{aligned} T(uv) &= \lim_{n\to\infty} \frac{f(4^n uv)}{4^n} = \lim_{n\to\infty} \frac{f(2^n u)}{2^n}\frac{f(2^n u)}{2^n} \\ &= \lim_{n\to\infty} \frac{f(2^n u)}{2^n} \lim_{n\to\infty} \frac{f(2^n u)}{2^n} = T(u)T(v). \end{aligned}$$

Since every element in C^*-algebra A can be expressed as a linear combination of elements in $\mathcal{U}(A)$, $\forall\ x, y \in A$, we may assume $x = \sum_{i=1}^{n} \alpha_i u_i$ and $y = \sum_{j=1}^{m} \beta_j v_j$ for some $u_i, v_j \in \mathcal{U}(A)$ and $\alpha, \beta \in \mathbb{C}$.

$$\begin{aligned} T(xy) &= T(\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j u_i v_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j T(u_i v_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i\beta_j T(u_i)T(v_j) \\ &= T(\sum_{i=1}^{n} \alpha_i u_i) T(\sum_{j=1}^{m} \beta_j v_j) \\ &= T(x)T(y). \end{aligned}$$

Therefore, T is indeed a $*$-homomorphism. ∎

Our first result is as follows.

Theorem 2.2. *Let $f : A \to B$ be a mapping between two C^*-algebras A, B such that*

(i) $||f(\mu x + \mu y) - \mu f(x) - \mu f(y)|| \leq \theta(||x||^p + ||y||^p)$, $\forall\ \mu \in \mathbb{T}$, $x, y \in A$

(ii) $||f(x^*) - f(x)^*|| \leq \theta(||x||^p + ||y||^p)$, $\forall\ x \in A$

(iii') $||f(xy) - f(x)f(y)|| \leq \theta(||x||^p + ||y||^p)$, $\forall\ x, y \in A$, *where* $\theta \geq 0$ *and* $p \in [0, 1)$

then $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *defines the unique $*$-homomorphism such that*

$$||f(x) - T(x)|| \leq \frac{2\theta}{2 - 2^p}||x||^p,\ \forall\ x \in A.$$

Moreover, we have

1 *If A is unital, then $T(1_A)$ is a projection satisfing $T(x) = T(1_A)f(x) = f(x)T(1_A)$, $\forall\ x \in A$.*

2 *If $\mathcal{U}(B) \subset f(\mathcal{U}(A))$, then $T(1_A)$ is a central projection in B, and $T(A)$ is an ideal of B. In particular, if B is simple then T is a $*$-epimorphism.*

3 *If the range of T contains an invertible element in B, then $f = T$.*

Proof. Let $\varphi(x,y) = \theta(||x||^p + ||y||^p)$, then φ is an admissible control function. Conditions (i) and (ii) are exactly the conditions (i) and (ii) as in Lemma 2.1. $\forall \alpha, \beta \in \mathbb{R}$, $u, v \in \mathcal{U}(A)$, let $x = \alpha u$, $y = \beta v$, then (iii') becomes (iii) as in Lemma 2.1. Therefore, by Lemma 2.1, $T(x) = \lim_{n\to\infty} \dfrac{f(2^n x)}{2^n}$ defines the unique $*$-homomorphism such that

$$||f(x) - T(x)|| \leq \tilde{\varphi}(x,x) = \frac{2\theta}{2-2^p}||x||^p,\ \forall\ x \in A.$$

If A is unital, since $T(1_A^2) = T(1_A)$ and $T(1_A)^* = T(1_A^*) = T(1_A)$, $T(1_A)$ is a projection. By substituting $y = 2^n 1_A$ in (iii'), since $n \in \mathbb{N}$, $p \geq 0$, we have

$$||f(2^n x) - f(x)f(2^n 1_A)|| \leq \theta(||x||^p + ||2^n 1_A||^p) \leq \theta(||2^n x||^p + ||2^n 1_A||^p).$$

Hence, by the convergence of $\tilde{\varphi}(x, 1_A) = \dfrac{1}{2}\displaystyle\sum_{k=0}^{\infty} 2^{-k}\theta(||2^k x||^p + ||2^k 1_A||^p)$,

$$||\frac{f(2^n x)}{2^n} - f(x)\frac{f(2^n 1_A)}{2^n}|| \leq 2^{-n}\theta(||2^n x||^p + ||2^n 1_A||^p) \to 0,\ \text{as } n \to \infty.$$

We have

$$T(x) = f(x)T(1_A),\ \forall\ x \in A.$$

Similarly, we have $T(x) = T(1_A)f(x)$.

For any $y \in B$, y can be written as a linear combination of elements in $\mathcal{U}(B)$, i.e., $y = \displaystyle\sum_{i=1}^{k} \alpha_i v_i$, $\exists\ \alpha_i \in \mathbb{C}$, $v_i \in \mathcal{U}(B)$. If $\mathcal{U}(B) \subset f(\mathcal{U}(A))$, then $y = \displaystyle\sum_{i=1}^{k} \alpha_i f(u_i)$, $\exists\ \alpha_i \in \mathbb{C}$, $u_i \in \mathcal{U}(A)$. Therefore,

$$T(1_A)y = T(1_A)\sum_{i=1}^{n}\alpha_i f(u_i) = \sum_{i=1}^{n}\alpha_i T(1_A)f(u_i) = \sum_{i=1}^{n}\alpha_i T(u_i) = T(\sum_{i=1}^{n}\alpha_i u_i),$$
$$yT(1_A) = \sum_{i=1}^{n}\alpha_i f(u_i)T(1_A) = \sum_{i=1}^{n}\alpha_i f(u_i)T(1_A) = \sum_{i=1}^{n}\alpha_i T(u_i) = T(\sum_{i=1}^{n}\alpha_i u_i).$$

Hence $T(1_A)$ is central in B and $\forall\ y \in B$, $T(1_A)y \subset T(A)$ and $yT(1_A) \subset T(A)$. Thus $yT(A) = yT(1_A \cdot A) = yT(1_A)T(A) \subset T(A)T(A) \subset T(A)$. Similarly, $T(A)y \subset T(A)$.

Similarly, by substituting x by $2^n x$ in (iii'), since $n \in \mathbb{N}$, $p \in [0,1)$, we have

$$||f(2^n xy) - f(2^n x)f(y)|| \le \theta(||2^n x||^p + ||y||^p) \le \theta(||2^n x||^p + ||2^n y||^p)$$

Hence, by the convergence of $\tilde{\varphi}(x,y) = \dfrac{1}{2}\sum_{k=0}^{\infty} 2^{-k}\theta(||2^k x||^p + ||2^k y||^p)$,

$$||\frac{f(2^n xy)}{2^n} - \frac{f(2^n x)f(y)}{2^n}|| \le 2^{-n}\theta(||2^n x||^p + ||2^n y||^p) \to 0, \text{ as } n \to \infty.$$

We have

$$T(xy) = T(x)f(y),\ \forall\ x, y \in A.$$

If $T(A)$ contains an invertible element $T(x_0)$ in B, then from $T(x_0)T(x) = T(x_0 x) = T(x_0)f(x)$, $\forall\ x \in A$, we have $T(x) = f(x)$, $\forall\ x \in A$. ■

Actually, the argument above can be modified to prove the following lemma.

Lemma 2.3. *Let $f : A \to B$ be a mapping between two Banach $*$-algebras A and B. If there exists an admissible control function $\varphi : A \times A \to [0, \infty)$ such that*

(i) $||f(\mu x + \mu y) - \mu f(x) - \mu f(y)|| \le \varphi(x,y)$, $\forall\ \mu \in \mathbb{T}$, $x, y \in A$

(ii) $||f(x^*) - f(x)^*|| \le \varphi(x,x)$, $\forall\ x \in A$

(iii') $||f(xy) - f(x)f(y)|| \le \varphi(xy, xy)$, $\forall\ x, y \in A$

then $T(x) = \lim_{n\to\infty} \dfrac{f(2^n x)}{2^n}$ defines the unique $*$-homomorphism such that

$$||f(x) - T(x)|| \le \tilde{\varphi}(x,x),\ \forall\ x \in A.$$

Proof. Since conditions (i) and (ii) are exactly the conditions (i) and (ii) as in Lemma 2.1, the proof there shows that $T(x) = \lim_{n\to\infty} \dfrac{f(2^n x)}{2^n}$ defines the unique additive $*$-preserving function such that $||f(x) - T(x)|| \le \tilde{\varphi}(x,x)$, $\forall\ x \in A$. We only have to prove that T is also multiplicative.

Substituting x, y in (iii'') by $2^n x$, $2^n y$, we have

$$||f(4^n xy) - f(2^n x)f(2^n y)|| \le \varphi(4^n xy, 4^n xy), \forall\ x, y \in A.$$

Then, $\forall\ x, y \in A$, by the convergence of $\tilde{\varphi}(xy, xy) = \dfrac{1}{2}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k xy, 2^k xy)$, we have

$$||\frac{f(4^n xy)}{4^n} - \frac{f(2^n x)}{2^n}\frac{f(2^n y)}{2^n}|| \leq 4^{-n}\varphi(4^n xy, 4^n xy) \to 0, \text{ as } n \to \infty.$$

Therefore, $T(xy) = T(x)T(y)$, $\forall\ x, y \in A$. ■

Example 2.4. Let $A = \mathbb{C} \times \mathbb{C} = B$ with norm $||(a, b)|| = |a| + |b|$, involution $(a, b)^* = (\overline{a}, \overline{b})$, and multiplication $(a, b)(c, d) = (ac, bd)$, then A, B are both Banach $*$-algebras. Let $f : A \to B$ be $f(a, b) = (a, 1 - e^{|b|})$. Let $\varphi : A \times A \to [0, \infty)$, $\varphi(x, y) \equiv c$. Then the corresponding

$$\tilde{\varphi} = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k}\varphi(x, y) = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k}c \equiv c.$$

If $c \geq 3$, then we have as in the above lemma

(i) $\forall\ x = (a, b), y = (c, d) \in A$, $\mu \in \mathbb{T}$,

$$\begin{aligned}
&||f(\mu x + \mu y) - \mu f(x) - \mu f(y)||\\
=\ &||f(\mu a + \mu c, \mu b + \mu d) - \mu f(a, b) - \mu f(c, d)||\\
=\ &||(\mu a + \mu c, 1 - e^{-|\mu b + \mu d|}) - \mu(a, 1 - e^{-|b|}) - \mu(c, 1 - e^{-|d|})||\\
=\ &||(0, 1 - e^{-|\mu b + \mu d|} - 2\mu + \mu e^{-|b|} + \mu^{-|d|})||\\
\leq\ &\left|1 - e^{-|\mu b + \mu d|}\right| + |2 - e^{-|b|} - e^{-|d|}|\\
\leq\ &3 \leq c = \varphi(x, y)
\end{aligned}$$

(ii) $\forall\ x = (a, b) \in A$,

$$\begin{aligned}
||f(x^*) - f(x)^*|| &= ||f(\overline{a}, \overline{b}) - (\overline{a}, 1 - e^{-|b|})||\\
&= ||(\overline{a}, 1 - e^{-|\overline{b}|}) - (\overline{a}, 1 - e^{-|b|})||\\
&= 1 \leq c = \varphi(x, x)
\end{aligned}$$

(iii''') $\forall\ x = (a, b), y = (c, d) \in A$,

$$\begin{aligned}
&||f(xy) - f(x)f(y)||\\
=\ &||f(ac, bd) - f(a, b)f(c, d)||\\
=\ &||(ac, 1 - e^{-|bd|}) - (a, 1 - e^{-|b|})(c, 1 - e^{-|d|})||\\
=\ &||(ac, 1 - e^{-|bd|}) - (ac, 1 - e^{-|b|} - e^{-|d|} + e^{-|b| - |d|})||
\end{aligned}$$

$$= |e^{-|b|} + e^{-|d|} - e^{-|bd|} - e^{-|b|-|d|}|$$
$$\leq 2 \leq c = \varphi(xy, xy)$$

Therefore, $\forall\ x = (a, b) \in A$,

$$T(x) = T(a,b) = \lim_{n\to\infty} \frac{f(2^n a, 2^n b)}{2^n} = \lim_{n\to\infty} (a, \frac{1 - e^{-|2^n b|}}{2^n}) = (a, 0)$$

is the unique $*$-homomorphism such that $\forall\ x = (a, b) \in A$,

$$||f(x) - T(x)|| = ||(a, 1 - e^{-|b|}) - (a, 0)|| = |1 - e^{-|b|}| \leq 1 \leq \tilde{\varphi}(x, x).$$

Similarly, we can get sufficient conditions when the $*$-homomorphism is actually an inner automorphism.

Theorem 2.5. *Let $f : A \to A$ be a mapping on a Banach $*$-algebra A. Suppose there is an invertible element $f(x_0)$ in A. If there exists an admissible control function $\varphi : A \times A \to [0, \infty)$ such that*

(i) $||f(\mu x + \mu y) - \mu f(x) - \mu f(y)|| \leq \varphi(x, y)$, $\forall\ \mu \in \mathbb{T}$, $x, y \in A$

(ii) $||f(x^*) - f(x)^*|| \leq \varphi(x, x)$, $\forall\ x \in A$

(iii) $||f(x) - f(x_0)xf(x_0)^{-1}|| \leq \varphi(x, x)$, $\forall\ x \in A$

then $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = f(x_0)xf(x_0)^{-1}$ defines the unique $*$-homomorphism such that

$$||f(x) - T(x)|| \leq \tilde{\varphi}(x, x),\ \forall\ x \in A.$$

Proof. From conditions (i) and (ii), we know $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ defines the unique additive $*$-preserving function such that $||f(x) - T(x)|| \leq \tilde{\varphi}(x, x)$, $\forall\ x \in A$. We only have to prove that $T(x) = f(x_0)xf(x_0)^{-1}$, $\forall\ x \in A$. That is T is an inner automorphism. Thus T is multiplicative since inner automorphisms must be multiplicative. (To see this, $T(xy) = f(x_0)xyf(x_0)^{-1} = f(x_0)xf(x_0)^{-1}f(x_0)yf(x_0)^{-1} = T(x)T(y)$.)

Now, by substituting x by $2^n x$ in (iii'''), we have

$$||f(2^n x - f(x_0)2^n xf(x_0)^{-1}|| \leq \varphi(2^n x, 2^n x),\ \forall\ x \in A.$$

Therefore, $\forall\ x \in A$, by the convergence of $\tilde{\varphi}(x, x) = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k x)$, we have

$$||\frac{f(2^n x)}{2^n} - f(x_0)xf(x_0)^{-1}|| \leq 2^{-n}\varphi(2^n x, 2^n x) \to 0, \text{ as } n \to \infty.$$

Hence $T(x) = f(x_0)xf(x_0)^{-1}$, $\forall\, x \in A$. ∎

Example 2.6. Let $A = \mathbb{C}$, $f : A \to A$, $f(x) = x+1$. Let $\varphi : A \times A \to [0, \infty)$, $\varphi(x, y) \equiv c$, a constant > 1. Then the corresponding $\tilde{\varphi}(x, y) \equiv c$, and $f(1) = 2$ is invertible. Then, as in the above theorem,

(i) $\forall\, x, y \in A$, $\mu \in \mathbb{T}$,

$$|f(\mu x+\mu y)-\mu f(x)-\mu f(y)|=|(\mu x+\mu y+1)-\mu x-1-\mu y-1|=1 \leq c=\varphi(x, y).$$

(ii) $\forall\, x \in A$,

$$|f(x^*) - f(x)^*| = |\overline{\overline{x}} + 1 - \overline{(\overline{x} + 1)}| = |x + 1 - x - 1| = 0 \leq c = \varphi(x, x).$$

(iii) Fix $f(1) = 2$. $\forall\, x \in A$,

$$|f(x) - 2x2^{-1}| = |x + 1 - x| = 1 \leq c = \varphi(x, x)$$

Therefore, $\forall\, x = (a, b) \in A$,

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \lim_{n\to\infty} \frac{2^n x + 1}{2^n} = x = 2x2^{-1}$$

is the unique $*$-homomorphism such that $\forall\, x \in A$,

$$|f(x) - T(x)| = |x + 1 - x| = 1 \leq c = \tilde{\varphi}(x, x).$$

On the other hand, we may relax the condition (iii) in Lemma 2.1 a little bit and consider further consider some sufficient condition for isometry and $*$-automorphism as in the following theorem.

Theorem 2.7. *Let $f : A \to B$ be a mapping between two C^*-algebras A and B. Let $\varepsilon : A \to B$ be a function such that $\forall\, x \in A$, $2^{-n}||\varepsilon(2^n x)|| \to 0$ as $n \to \infty$. If there exists an admissible control function $\varphi : A \times A \to [0, \infty)$ such that*

(i) $||f(\mu x + \mu y) - \mu f(x) - \mu f(y)|| \leq \varphi(x, y)$, $\forall\, \mu \in \mathbb{T}$, $x, y \in A$

(ii) $||f(x^*) - f(x)^*|| \leq \varphi(x, x)$, $\forall\, x \in A$

(iii) $||f(\alpha\beta uv) - [f(\alpha u) + \varepsilon(\alpha u)][f(\beta v) + \varepsilon(\beta v)]|| \leq \varphi(\alpha u, \beta v)$, $\forall\, \alpha, \beta \in \mathbb{R}$, $u, v \in \mathcal{U}(A)$

then $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *defines the unique* $*$*-homomorphism such that*

$$||f(x) - T(x)|| \leq \tilde{\varphi}(x, x), \ \forall\ x \in A.$$

Furthermore, we have

1. *If* $|\ ||f(x) - f(y)|| - ||x - y||\ | \leq \varphi(x, y), \forall\ x, y \in A$*, then* T *is an isometry.*
2. *If, in addition,* $A = B$ *and* $\forall\ v \in \mathcal{U}(A)$*,* $\exists\ u \in \mathcal{U}(A)$ *such that* $||f(2^n u) - 2^n v|| \leq \varphi(2^n u, 2^n v)$*,* $\forall\ n \in \mathbb{N}$*, then* T *is an automorphism.*

Proof. From conditions (i) and (ii), we know $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ defines the unique additive $*$-preserving function such that $||f(x)-T(x)|| \leq \tilde{\varphi}(x, x), \forall\ x \in A$.

To prove T is multiplicative, substituting $\alpha = 2^n = \beta$, we have

$$||f(4^n uv) - [f(2^n u) + \varepsilon(2^n u)][f(2^n v) + \varepsilon(2^n v)]|| \leq \varphi(2^n u, 2^n v), \ \forall\ u, v \in \mathcal{U}(A).$$

Then, $\forall\ u, v \in \mathcal{U}(A)$, by the convergence of $\tilde{\varphi}(u, v) = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k u, 2^k v)$, we have

$$||\frac{f(4^n uv)}{4^n} - \frac{f(2^n u)+\varepsilon(2^n u)}{2^n}\frac{f(2^n v)+\varepsilon(2^n v)}{2^n}|| \leq 4^{-n}\varphi(2^n u, 2^n v) \to 0, \text{ as } n \to \infty.$$

Since $2^{-n}||\varepsilon(2^n u)|| \to 0$ as $n \to \infty$, and $2^{-n}||\varepsilon(2^n v)|| \to 0$ as $n \to \infty$, it follows that $T(uv) = T(u)T(v), \forall\ u, v \in \mathcal{U}(A)$. Since every element in C^*-algebra A can be expressed as a linear combination of elements in $\mathcal{U}(A)$, as in the proof of Lemma 2.1, T is multiplicative. Hence T is a $*$-homomorphism.

If $|\ ||f(x) - f(y)|| - ||x - y||\ | \leq \varphi(x, y), \forall\ x, y \in A$, then substitute x, y by $2^n x, 2^n y$, we have

$$|\ ||f(2^n x) - f(2^n y)|| - ||2^n x - 2^n y||\ | \leq \varphi(2^n x, 2^n y).$$

Therefore, $\forall\ x, y \in A$, by the convergence of $\tilde{\varphi}(x, y) = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x, 2^k y)$, we have

$$\left| || \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n} || - ||x - y|| \right| \leq 2^{-n}\varphi(2^n x, 2^n y) \to 0, \text{ as } n \to \infty.$$

Hence, $||T(x) - T(y)|| = ||x - y||, \forall\ x, y \in A$. That is, T is an isometry.

If, in addition, $A = B$ and $\forall\, v \in \mathcal{U}(A)$, $\exists\, u \in \mathcal{U}(A)$ such that $||f(2^n u) - 2^n v|| \leq \varphi(2^n u, 2^n v)$, $\forall\, n \in \mathbb{N}$, then by the convergence of $\tilde{\varphi}(u,v) = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k}\varphi(2^k u, 2^k v)$, we have

$$||\frac{f(2^n u)}{2^n} - v|| \leq 2^{-n}\varphi(2^n u) \to, \text{ as } n \to \infty.$$

Therefore, $T(u) = v$. That is, $\mathcal{U}(A) \subset T(\mathcal{U}(A))$. Since every element in C^*-algebra A can be expressed as a linear combination of elements in $\mathcal{U}(A)$. We have T is onto, hence a $*$-automorphism. ∎

Example 2.8. Let $A = \mathbb{C} = B$, $f : A \to B$, $f(x) = x - |x|e^{-|x|}$. Let $\varphi : A \times A \to [0, \infty)$, $\varphi(x,y) \equiv c$. Then the corresponding

$$\tilde{\varphi} = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k}\varphi(x,y) = \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k} c \equiv c.$$

Let $\varepsilon : A \to A$, $\varepsilon(x) = |x|e^{-|x|}$, then $\forall\, a \in A$, we have $2^{-n}|\varepsilon(2^n a)| \to 0$ as $n \to \infty$. From calculus, $|te^{-t}| \leq e^{-1}$, $\forall\, t \in [0,\infty)$. If $c \geq 3e^{-1}$, then as in the above theorem,

(i) $\forall\, x, y \in A$, $\mu \in \mathbb{T}$,

$$\begin{aligned}
&|f(\mu x + \mu y) - \mu f(x) - \mu f(y)| \\
=\ & \left|\mu x + \mu y - |\mu x + \mu y|e^{-|\mu x + \mu y|} - \mu x + |\mu x|e^{-|\mu x|} - \mu y + |\mu y|e^{-|\mu y|}\right| \\
=\ & \left|-|x+y|e^{-|x+y|} + |x|e^{-|x|} + |y|e^{-|y|}\right| \\
=\ & |x+y|e^{-|x+y|} + |x|e^{-|x|} + |y|e^{-|y|} \leq 3e^{-1} \leq c = \varphi(x,y)
\end{aligned}$$

(ii) $\forall\, x \in A$,

$$\begin{aligned}
|f(x^*) - f(x)^*| &= |f(\overline{x}) - \overline{(x - |x|e^{-|x|})}| \\
&= \left|(\overline{x} - |\overline{x}|e^{-|\overline{x}|}) - \overline{x} + |x|e^{-|x|}\right| \\
&= 0 \leq c = \varphi(x,x)
\end{aligned}$$

(iii''') $\forall\, \alpha, \beta \in \mathbb{R}$, $\forall\, u, v \in \mathcal{U}(A)$,

$$\begin{aligned}
&|f(\alpha u \beta v) - [f(\alpha u) - \varepsilon(\alpha u)][f(\beta v) - \varepsilon(\beta v)]| \\
=\ & \big|(\alpha u \beta v - |\alpha u \beta v|e^{-|\alpha u \beta v|}) \\
& - [(\alpha u - |\alpha u|e^{-|\alpha u|}) - |\alpha u|e^{-|\alpha u|}][(\beta v - |\beta v|e^{-|\beta v|}) - |\beta v|e^{-|\beta v|}]\big| \\
=\ & \left|(\alpha u \beta v - |\alpha u \beta v|e^{-|\alpha u \beta v|}) - \alpha u \beta v\right| \\
=\ & |\alpha u \beta v|e^{-|\alpha u \beta v|} \leq e^{-1} \leq c = \varphi(\alpha u, \beta v).
\end{aligned}$$

Therefore, $\forall\ x \in A$,

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = \lim_{n\to\infty} \frac{2^n x - |2^n x| e^{-|2^n x|}}{2^n} = x$$

is the unique $*$-homomorphism such that $\forall\ x \in A$,

$$|f(x) - T(x)| = \Big|(x - |x|e^{-|x|}) - x\Big| = |x|e^{-|x|} \le e^{-1} \le c \le \tilde{\varphi}(x, x).$$

Moreover, T is an isometry, and we check $\forall\ x, y \in A$,

$$\begin{aligned} \big|\,|f(x) - f(y)| - |x - y|\,\big| &= \big|\,|(x - |x|e^{-|x|}) - (y - |y|e^{-|y|})| - |x - y|\,\big| \\ &= \big|\,|(x - y) - (|x|e^{-|x|} + |y|e^{-|y|})| - |(x - y)|\,\big| \\ &\le |x|e^{-|x|} + |y|e^{-|y|} \le 2e^{-1} \le c = \varphi(x, y). \end{aligned}$$

Finally, T is automorphism. We check $A = B$ and $\forall\ v \in \mathcal{U}(B) = \mathbb{T} = \mathcal{U}(A)$, let $u = v$, then $|f(2^n u) - 2^n v| = 0 \le c = \varphi(2^n u, 2^n v)$, $\forall\ n \in \mathbb{N}$.

References

1. Z. Gajda, On stability of additive mappings, *Internat. J. Math. Sci.*, **14** (1991), 431-434.
2. P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), 431-436.
3. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. U.S.A.*, **27** (1941), 222-224.
4. Chun-Gil Park, On an approximate automorphism on a C^*-algebra, *Proc. Amer. Math. Soc.*, **132** (2004), 1739-1745.
5. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.
6. Th. M. Rassias and P. Semrl, On the behavior of mappings which do not satisfy Hyers-Ulam stability, *Proc. Amer. Math. Soc.*, **173** (1993), 325-338.
7. S. M. Ulam, *Problems in modern mathematics*, Chap. VI, Wiley, New York, 1960.

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