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# ALMOST CONVERGENCE OF SEQUENCES IN BANACH SPACES IN WEAK, STRONG, AND ABSOLUTE SENSES

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**Abstract.** We introduce concepts of $\sigma$-$\limsup$ and $\sigma$-$\liminf$ for bounded sequences of real numbers and show a Cauchy criterion for sequences of vectors which converge in the sense of $a\sigma$-limit (i.e., absolute almost convergence). Then a sufficient condition on a bounded sequence $\{\{x_n^{(m)}\}_{n=1}^{\infty}\}_{m=1}^{\infty} \subset \ell^\infty(X)$ is given for the following equality to hold:

$$a\sigma\text{-}\lim_{m\to\infty} \sigma\text{-}\lim_{n\to\infty} x_n^{(m)} = \sigma\text{-}\lim_{n\to\infty} a\sigma\text{-}\lim_{m\to\infty} x_n^{(m)}.$$

Finally, applying this result we show that $\sigma\text{-}\lim_{n\to\infty} f(\sin(n\theta))$ and $\sigma\text{-}\lim_{n\to\infty} f(\cos(n\theta))$ exist whenever $f$ is a weakly continuous function on $[-1,1]$ with values in a reflexive Banach space.

## 1. Introduction

Let $X$ be a real or complex normed linear space. Let $\pi_\sigma$ denote the set of all Banach limits on $\ell^\infty$, the space of all bounded sequences in $\mathbb{C}$ with the sup-norm. Recall that a Banach limit $\phi$ is a positive linear functional on $\ell^\infty$, which satisfies

$$\phi(\{a_{n+k}\}) = \phi(\{a_n\}) \text{ for all } \{a_n\} \text{ and } k = 1, 2, \ldots$$

and maps convergent sequences to their limits. It is known that $\pi_\sigma$ is a weakly$^*$-compact set.

In 1948, Lorentz [5] defined the $\sigma$-limit for a bounded sequence $\{a_n\} \in \ell^\infty$ as

$$\sigma\text{-}\lim a_n := a$$

if $\phi(\{a_n\}) = a$ for all $\phi \in \pi_\sigma$. Some related researches on $\sigma$-limit can be found in [1, 5, 6, 7, 8, 9]. In this paper, for convenience we shall sometimes write $\phi(a_n)$ or

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$\phi_n(a_n)$ instead of $\phi(\{a_n\})$.

In [4], we generalize the definition of $\sigma$-limit from $\ell^\infty$ to $\ell^\infty(X)$, the space of all bounded sequences in a general normed linear space $X$, equipped with the sup norm. A bounded sequence $\{x_n\}$ in $X$ is said to have a *$\sigma$-limit* $x \in X$ (cf. [4]) if $\sigma\text{-}\lim\langle x_n, x^*\rangle = \langle x, x^*\rangle$ for all $x^* \in X^*$. It was shown [4, Theorem 3.2] that a bounded sequence $\{x_n\}$ in $X$ has a $\sigma$-limit $x \in X$ if and only if it is *weakly almost-convergent* to $x$, i.e., for every $x^* \in X^*$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \langle x_{k+m}, x^*\rangle = \langle x, x^*\rangle$$

uniformly on $m \geq 0$. In the same paper, we showed that if $\sigma\text{-}\lim x_n = x$, then $x \in \overline{co}\{x_n; n \geq 0\}$. $\{x_n\}$ is said to be *strongly almost-convergent to* $x$ (cf. [3]) if

$$s\text{-}\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} x_{k+m} = x \ \text{(convergence in norm)}$$

uniformly on $m \geq 0$. If $\sigma\text{-}\lim ||x_n - x|| = 0$, we will say that $\{x_n\}$ is *absolutely almost convergent* or *$a\sigma$-convergent* to $x$, and will use the notation $a\sigma\text{-}\lim x_n = x$ (Note that in [3] we have used the notation $s\sigma\text{-}\lim$. To distinguish absolute almost-convergence from strong almost-convergence, in this paper we adopt the notation $a\sigma\text{-}\lim$ instead of $s\sigma\text{-}\lim$). It is known [3] that

strong convergence $\Rightarrow$ absolute almost-convergence
$\Rightarrow$ strong almost-convergence
$\Rightarrow$ weak almost-convergence.

These implications are strict. Related counter-examples can be found in [3] and [4]. It is known [2] that $\{x_n\}$ strongly converges to $x \in X$ if and only if $\{x_n\}$ is strongly almost-convergent to $x$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Clearly, strong almost-convergence implies $(C, 1)$-convergence. But there is no relation between $(C, 1)$-convergence and weak almost-convergence.

Let $X_\sigma := \{\{x_n\} \in \ell^\infty(X); \sigma\text{-}\lim x_n = x$ for some $x \in X\}$, and $X_{a\sigma} := \{\{x_n\} \in \ell^\infty(X); a\sigma\text{-}\lim x_n = x$ for some $x \in X\}$. These two spaces are closed linear subspaces of $\ell^\infty(X)$. In particular, the space $\mathbb{C}_{a\sigma}$ is a unital Banach subalgebra of $\ell^\infty$ and every Banach limit on $\mathbb{C}_{a\sigma}$ is a multiplicative linear functional on $\mathbb{C}_{a\sigma}$ [3, Corollary 2.9].

Now we define notions of $\limsup$ and $\liminf$ in the sense of $\sigma$-limit and Cauchy sequence in the sense of $a\sigma$-limit.

**Definition 1.**

(a) Let $\{a_n\}$ be a bounded sequence of real numbers. We define $\sigma\text{-}\limsup_{n\to\infty} a_n := \sup_{\phi\in\pi_\sigma} \phi(\{a_n\})$ and $\sigma\text{-}\liminf_{n\to\infty} a_n := \inf_{\phi\in\pi_\sigma} \phi(\{a_n\})$.

(b) A sequence $\{x_n\} \in \ell^\infty(X)$ is said to be a $a\sigma$-Cauchy sequence if

$$\sigma\text{-}\limsup_{n\to\infty} \sigma\text{-}\limsup_{m\to\infty} ||x_n - x_m|| = 0,$$

which, by (a), is equivalent to

$$\psi_n(\phi_m(||x_n - x_m||)) = 0 \text{ for all } \phi, \psi \in \pi_\sigma.$$

It is clear that $a = \sigma\text{-}\lim a_n$ exists if and only if $\sigma\text{-} \lim \limsup_{n\to\infty} a_n = \sigma\text{-}\lim \liminf_{n\to\infty} a_n = a$. In particular, for $a_n \geq 0$, $\sigma\text{-}\lim a_n = 0$ if and only if $\sigma\text{-}\limsup_{n\to\infty} a_n = 0$. Thus, for $\{x_n\} \in \ell^\infty(X)$, where $X$ is a real Banach space, $x = \sigma\text{-}\lim x_n$ exists if and only if

$$\sigma\text{-}\limsup_{n\to\infty} \langle x_n, x^*\rangle = \sigma\text{-}\liminf_{n\to\infty} \langle x_n, x^*\rangle = \langle x, x^*\rangle$$

for all $x^* \in X^*$; and $x = a\sigma\text{-}\lim x_n$ exists if and only if $\sigma\text{-}\limsup_{n\to\infty} \|x_n - x\| = 0$, i.e., $\phi(\{\|x_n - x\|\}) = 0$ for all $\phi \in \pi_\sigma$.

If $\{x_n\}$ is a sequence in $X$, it is easy to see that $\{x_n\}$ is a Cauchy sequence if and only if

$$\limsup_{n\to\infty} \limsup_{m\to\infty} ||x_n - x_m|| = 0.$$

In Theorem 2.3 we prove an analogous Cauchy criterion in the sense of $a\sigma$-limit. In Theorem 2.4, we give a sufficient condition on a sequence $\{\{x_n^{(m)}\}_{n=0}^\infty\}_{m=0}^\infty$ in $\ell^\infty(X)$ for the following equality to hold

$$a\sigma\text{-} \lim_{m\to\infty} \sigma\text{-} \lim_{n\to\infty} x_n^{(m)} = \sigma\text{-} \lim_{n\to\infty} a\sigma\text{-} \lim_{m\to\infty} x_n^{(m)}.$$

In Section 3, we first give two examples showing the existence of $\sigma\text{-} \lim_{n\to\infty} \sin^m (n\theta)$, $\sigma\text{-} \lim_{n\to\infty} \cos^m(n\theta)$, and $\sigma\text{-} \lim_{n\to\infty} e^{\imath n\theta}$ for all $\theta \in \mathbb{R}$ and $m = 0, 1, 2, \ldots$. Using these facts and applying Theorem 2.4, we show (Theorem 3.3) that for any weakly continuous function $f : [-1, 1] \to X$ both $\sigma\text{-} \lim_{n\to\infty} f(\sin(n\theta))$ and $\sigma\text{-} \lim_{n\to\infty} f(\cos(n\theta))$ exists. It is also shown that if a function $f : \Delta \to \mathbb{C}$ is continuous on the closed disc $\Delta$ of $\mathbb{C}$ and is analytic in the interior of $\Delta$, then $\sigma\text{-} \lim_{n\to\infty} f(e^{\imath n\theta})$ exists.

## 2. Main Result

Recall that the canonical mapping $J : X \to X^{**}$ is defined by $\langle x^*, J_x\rangle := \langle x, x^*\rangle \equiv x^*(x)$ for all $x \in X$ and $x^* \in X^*$.

**Lemma 2.1.** *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two bounded sequences in* $X$*. Suppose there is a* $\phi \in \pi_\sigma$ *such that*

$$\psi_n(\phi_m(||x_n - y_m||)) = 0 \text{ for all } \psi \in \pi_\sigma.$$

*Then* $a\sigma\text{-}\lim x_n = x$ *for some* $x \in X$ *and* $\langle x, x^* \rangle = \phi_m(\langle y_m, x^* \rangle)$*.*

*Proof.* Define $h(x^*) := \phi_m(\langle y_m, x^* \rangle)$ for $x^* \in X^*$. It is clear that $h \in X^{**}$. Then we have for every $m = 1, 2, \ldots$ and $x^* \in X^*$

$$\begin{aligned} |\langle x^*, J_{x_n} - h \rangle| &= |\langle x_n, x^* \rangle - \phi_m(\langle y_m, x^* \rangle)| \\ &\leq \phi_m(|\langle x_n - y_m, x^* \rangle|) \\ &\leq \phi_m(||x_n - y_m||)||x^*||. \end{aligned}$$

This implies that $||J_{x_n} - h|| \leq \phi_m(||y_m - x_n||)$. By the assumption, we have for every $\psi \in \pi_\sigma$

$$\psi_n(||J_{x_n} - h||) \leq \psi_n(\phi_m(||y_m - x_n||)) = 0.$$

Therefore we have $a\sigma\text{-}\lim J_{x_n} = h$ and hence $\{J_{x_n}\}$ is strongly almost-convergent to $h$. This shows that $h \in J(X)$. Hence $h = J_x$ for some $x \in X$, which implies that

$$\phi_m(\langle y_m, x^* \rangle) = h(x^*) = \langle x^*, J_x \rangle = \langle x, x^* \rangle.$$

Since $||J_{x_n} - h|| = ||J_{x_n} - J_x|| = ||x_n - x||$ for all $n \geq 1$, we must have

$$\sigma\text{-}\lim ||x_n - x|| = \sigma\text{-}\lim ||J_{x_n} - J_x|| = 0.$$

This proves that $a\sigma\text{-}\lim x_n = x$ and the proof is complete.

If we take $x_n = y_n$ for all $n$ in Lemma 2.1, we obtain the following Cauchy criterion for the existence of the $a\sigma$-limit.

**Corollary 2.2.** *Let* $\{x_n\}$ *be a bounded sequence in* $X$*. Suppose there is a* $\phi \in \pi_\sigma$ *such that*

$$\psi_n(\phi_m(||x_n - x_m||)) = 0 \text{ for all } \psi \in \pi_\sigma.$$

*Then* $a\sigma\text{-}\lim x_n = x$ *for some* $x \in X$*.*

If the sequence $\{x_n\} \in X_{a\sigma}$ has the $a\sigma$-limit $x$, then $\phi(\{\|x_n - x\|\}) = 0$ for all $\phi \in \pi_\sigma$, so that $\sigma\text{-}\limsup_{n\to\infty} \|x_n - x\| = 0$. Hence

$$\begin{aligned} &\sigma\text{-}\limsup_{n\to\infty} \sigma\text{-}\limsup_{m\to\infty} ||x_n - x_m|| \\ &\quad \leq \sigma\text{-}\limsup_{n\to\infty} \sigma\text{-}\limsup_{m\to\infty} ||x_n - x|| + \sigma\text{-}\limsup_{n\to\infty} \sigma\text{-}\limsup_{m\to\infty} ||x - x_m|| \\ &\quad = \sigma\text{-}\limsup_{n\to\infty} ||x_n - x|| + \sigma\text{-}\limsup_{m\to\infty} ||x - x_m|| = 0. \end{aligned}$$

So, a $a\sigma$-convergent sequence $\{x_n\}$ must be a $a\sigma$-Cauchy sequence. Combining this fact and Corollary 2.2, we have the following theorem.

**Theorem 2.3.** *A sequence $\{x_n\} \in \ell^\infty(X)$ is $a\sigma$-convergent if and only if it is a $a\sigma$-Cauchy sequence in $X$.*

**Theorem 2.4.** *Suppose $X$ is a Banach space. If $\{\mathbf{w}^{(m)}\}_{m=1}^\infty$ ($\mathbf{w}^{(m)} := \{x_n^{(m)}\}_{n=1}^\infty \in X_\sigma$) is a sequence in $X_\sigma$ such that*

$$a\sigma\text{-}\lim_{m\to\infty} \mathbf{w}^{(m)} = \mathbf{w} \textit{ for some } \mathbf{w} = \{x_n\} \in \ell^\infty(X).$$

*For each $m \in \mathbb{N}$ let $y_m := \sigma\text{-}\lim_{n\to\infty} x_n^{(m)}$. Then $a\sigma\text{-}\lim_{m\to\infty} x_n^{(m)} = x_n$ for all $n \in \mathbb{N}$, $\mathbf{w} \in X_\sigma$, and $a\sigma\text{-}\lim_{m\to\infty} y_m = \sigma\text{-}\lim_{n\to\infty} x_n$, that is,*

$$a\sigma\text{-}\lim_{m\to\infty} \sigma\text{-}\lim_{n\to\infty} x_n^{(m)} = \sigma\text{-}\lim_{n\to\infty} a\sigma\text{-}\lim_{m\to\infty} x_n^{(m)}. \tag{2.1}$$

*In particular, if $\{\mathbf{w}^{(m)}\}_{m=1}^\infty$ is a sequence in $X_\sigma$ converging to a bounded sequence $\mathbf{w} = \{x_n\} \in \ell^\infty(X)$ in sup-norm, then $\mathbf{w} \in X_\sigma$ and*

$$s\text{-}\lim_{m\to\infty} \sigma\text{-}\lim_{n\to\infty} x_n^{(m)} = \sigma\text{-}\lim_{n\to\infty} x_n. \tag{2.2}$$

*Proof.* Since $||x_n^{(m)} - x_n|| \leq ||\mathbf{w}^{(m)} - \mathbf{w}||_\infty$ for all $m, k = 1, 2, \ldots$ and $a\sigma\text{-}\lim_{m\to\infty} \mathbf{w}^{(m)} = \mathbf{w}$, we have $a\sigma\text{-}\lim_{m\to\infty} x_n^{(m)} = x_n$ for all $n = 1, 2, \ldots$. It follows from the closedness of $X_\sigma$ (cf. [3, Theorem 2.6]) that $\mathbf{w} = a\sigma\text{-}\lim_{m\to\infty} \mathbf{w}^{(m)} =\in X_\sigma$. Hence $x := \sigma\text{-}\lim_{n\to\infty} x_n$ exists. By Theorem 2.3, $\{\mathbf{w}^{(m)}\}$ is a $a\sigma$-Cauchy sequence. Therefore we have for all $x^* \in X^*$, $m, n, l = 1, 2, \ldots$

$$\langle y_m - x, x^*\rangle = \langle y_m - x_n^{(m)}, x^*\rangle + \langle x_n^{(m)} - x_n^{(l)}, x^*\rangle + \langle x_n^{(l)} - x_n, x^*\rangle + \langle x_n - x, x^*\rangle.$$

This implies

$$\begin{aligned}
&\text{Re}\langle y_m - x, x^*\rangle \\
\leq\ &\text{Re}\langle y_m - x_n^{(m)}, x^*\rangle + ||x_n^{(m)} - x_n^{(l)}|| \cdot ||x^*|| \\
&+\text{Re}\langle x_n^{(l)} - x_n, x^*\rangle + \text{Re}\langle x_n - x, x^*\rangle \\
\leq\ &\text{Re}\langle y_m - x_n^{(m)}, x^*\rangle + ||\mathbf{w}^{(m)} - \mathbf{w}^{(l)}||_\infty \cdot ||x^*|| \\
&+\text{Re}\langle x_n^{(l)} - x_n, x^*\rangle + \text{Re}\langle x_n - x, x^*\rangle.
\end{aligned}$$

Therefore we have for every $\phi, \psi \in \pi_\sigma$

$$
\begin{aligned}
&\mathrm{Re}\langle y_m - x, x^* \rangle \\
&\le \psi_n(\mathrm{Re}\langle y_m - x_n^{(m)}, x^* \rangle) + \sigma\text{-}\lim_{l\to\infty} ||\mathbf{w}^{(m)} - \mathbf{w}^{(l)}||_\infty \cdot ||x^*|| \\
&\quad + \psi_n(\mathrm{Re}\phi_l(\langle x_n^{(l)} - x_n, x^* \rangle)) + \psi_n(\mathrm{Re}\langle x_n - x, x^* \rangle) \\
&= \mathrm{Re}\psi_n(\langle y_m - x_n^{(m)}, x^* \rangle) + ||\mathbf{w}^{(m)} - \mathbf{w}||_\infty \cdot ||x^*|| \\
&\quad + \mathrm{Re}\psi_n(\phi_l(\langle x_n^{(l)} - x_n, x^* \rangle)) + \mathrm{Re}\psi_n(\langle x_n - x, x^* \rangle) \\
&= 0 + ||\mathbf{w}^{(m)} - \mathbf{w}||_\infty \cdot ||x^*|| + 0 + 0.
\end{aligned}
\tag{2.3}
$$

Since $x^* \in X^*$ is arbitrary, it follows from the Hahn-Banach theorem that (2.3) implies

$$
||y_m - x|| \le ||\mathbf{w}^{(m)} - \mathbf{w}||_\infty \text{ for all } m \ge 1. \tag{2.4}
$$

By the assumption $a\sigma\text{-}\lim_{m\to\infty} \mathbf{w}^{(m)} = \mathbf{w}$, we have that

$$
\sigma\text{-}\limsup_{m\to\infty} ||y_m - x|| \le \sigma\text{-}\limsup_{m\to\infty} ||\mathbf{w}^{(m)} - \mathbf{w}||_\infty = 0.
$$

Therefore $a\sigma\text{-}\lim_{m\to\infty} y_m = x$. This proves (2.1). If the sequence $\{\mathbf{w}^{(m)}\}$ converges to $\mathbf{w} = \{x_n\}$ in sup-norm, then $s\text{-}\lim_{m\to\infty} x_n^{(m)} = x_n$ and (2.4) implies $s\text{-}\lim_{m\to\infty} y_m = x$, i.e., (2.2) holds. This completes the proof.

## 3. Applications

In this section, for a nonempty compact subset $\Omega$ of $\mathbb{C}$, we shall denote by $C(\Omega)$ the Banach space consisting of all continuous complex-valued functions and $C_{\mathbb{R}}(\Omega) := \{f \in C(\Omega) | f \text{ is real-valued } \}$ equipped with the sup-norm $||\cdot||_\infty$.

**Example 1.** (a) If $\theta \in 2\pi\mathbb{Z}$, then $e^{\imath n\theta} = 1$ for all $n \in \mathbb{Z}$, so $\sigma\text{-}\lim_{n\to\infty} e^{\imath n\theta} = 1$.
(b) If $\theta \notin 2\pi\mathbb{Z}$, then $e^{\imath\theta} \ne 1$ and we have for every $\phi \in \pi_\sigma$

$$
e^{\imath\theta}\phi_n(e^{\imath n\theta}) = \phi_n(e^{\imath(n+1)\theta}) = \phi_n(e^{\imath n\theta}).
$$

This implies $\phi_n(e^{\imath n\theta}) = 0$ for $\phi \in \pi_\sigma$ and hence $\sigma\text{-}\lim_{n\to\infty} e^{\imath n\theta} = 0$.

**Example 2.** For every $m = 0, 1, 2, \ldots$ and for every $\theta \in \mathbb{R}$, both $\sigma\text{-}\lim_{n\to\infty} \sin^m(n\theta)$ and $\sigma\text{-}\lim_{n\to\infty} \cos^m(n\theta)$ exist.

It is obvious for the case $m = 0$. So, we may assume $m = 1, 2, \ldots$. By Example 1, we obtain that

$$\begin{aligned} &\sigma\text{-}\lim_{n\to\infty} \sin^m(n\theta) \\ &= \sigma\text{-}\lim_{n\to\infty} (\frac{e^{\imath n\theta} - e^{-\imath n\theta}}{2\imath})^m \\ &= \sigma\text{-}\lim_{n\to\infty} \frac{1}{(2\imath)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m+j} e^{\imath nj\theta} e^{-\imath n(m-j)\theta} \\ &= \sigma\text{-}\lim_{n\to\infty} \frac{1}{(2\imath)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m+j} e^{\imath n(2j-m)\theta} \\ &= \frac{1}{(2\imath)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m+j} \sigma\text{-}\lim_{n\to\infty} e^{\imath n(2j-m)\theta} \end{aligned}$$

exists. Similarly,

$$\sigma\text{-}\lim_{n\to\infty} \cos^m(n\theta) = \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} \sigma\text{-}\lim_{n\to\infty} e^{\imath n(2j-m)\theta}$$

exists.

Now, we consider the case that $\theta \in \mathbb{R}$ is such that $k\theta \notin 2\pi\mathbb{Z}$ for every nonzero integer $k$. If $m$ is a positive odd integer, then

$$\sigma\text{-}\lim_{n\to\infty} \sin^m(n\theta) = \frac{1}{(2\imath)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m+j} \sigma\text{-}\lim_{n\to\infty} e^{\imath n(2j-m)\theta} = 0;$$

if $m$ is a nonnegative even integer and $m = 2k$, then

$$\sigma\text{-}\lim_{n\to\infty} \sin^m(n\theta) = \frac{1}{2^{2k}(-1)^k} \binom{2k}{k} (-1)^k = \frac{1}{2^{2k}} \binom{2k}{k}.$$

Similarly, we have

$$\sigma\text{-}\lim_{n\to\infty} \cos^m(n\theta) = \begin{matrix} 0 & \text{if } m \text{ is a positive odd integer} \\ \frac{1}{2^{2k}}\binom{2k}{k} & \text{if } m = 2k \text{ is a nonnegative even integer.} \end{matrix}$$

**Theorem 3.1.** *For every $\theta \in \mathbb{R}$, both $\sigma\text{-}\lim_{n\to\infty} f(\sin(n\theta))$ and $\sigma\text{-}\lim_{n\to\infty} f(\cos(n\theta))$ exist for all $f \in C[-1, 1]$. In particular, $\sigma\text{-}\lim_{n\to\infty} |\sin(n\theta)|$ and $\sigma\text{-}\lim_{n\to\infty} |\cos(n\theta)|$ exist.*

*Proof.* Since the $\sigma$-limit is linear, we may assume that $f$ is a real-valued function. Define $h(\theta) := \sin(\theta)$ or $\cos(\theta)$ for $\theta \in \mathbb{R}$. Let $E := \{f \in C_{\mathbb{R}}[-1, 1] | \sigma\text{-}\lim_{n\to\infty} f(h(n\theta)) \text{ exists } \}$. By last two examples, $E$ contains all polynomials and $E$

is a linear subspace of $C_{\mathbb{R}}[-1,1]$. Since the set of all polynomials is dense in $C_{\mathbb{R}}[-1,1]$ by the famous Weierstrass theorem, it suffices to show that $E$ is closed. Let $\{f_m\}$ be a sequence in $E$ convergent to some element $f \in C_{\mathbb{R}}[-1,1]$. Then $\{f_m(h(n\theta))\}_{n=1}^{\infty}$, $m = 1, 2, \ldots$, is a sequence in $\mathbb{R}_\sigma$ convergent to $\{f(h(n\theta))\}$ in sup-norm. It follows from Theorem 2.4 that $\{f(h(n\theta))\} \in \mathbb{R}_\sigma$ and

$$\lim_{m\to\infty} \sigma\text{-}\lim_{n\to\infty} f_m(h(n\theta)) = \sigma\text{-}\lim_{n\to\infty} \lim_{m\to\infty} f_m(h(n\theta)) = \sigma\text{-}\lim_{n\to\infty} f(h(n\theta)).$$

This completes the proof.

**Theorem 3.2.** *Let $\Delta$ be the closed disc $\{\lambda \in \mathbb{C}; |\lambda| \le 1\}$ and let $A(\Delta)$ be the algebra of all continuous functions $f : \Delta \to \mathbb{C}$ that can be approximated uniformly by polynomials on $\Delta$* (cf. [10, p. 410]). *Then $\sigma\text{-}\lim_{n\to\infty} f(e^{\imath n\theta})$ exists for all $\theta \in \mathbb{R}$. Furthermore, if, in addition, $k\theta \notin 2\pi\mathbb{Z}$ for every nonzero integer $k$, then $\sigma\text{-}\lim_{n\to\infty} f(e^{\imath n\theta}) = f(0)$.*

*Proof.* Let $\theta \in \mathbb{R}$ be arbitrary. If $f$ is a polynomial, it follows from Example 1 that $\sigma\text{-}\lim_{n\to\infty} f(e^{\imath n\theta})$ exists. Suppose $f$ is continuous on $\Delta$ and is analytic in the interior of $\Delta$. Then there is a sequence $\{f_m\}$ of polynomials such that $f_m \to f$ uniformly on $\Delta$. Therefore for every $m \ge 1$ $\{f_m(e^{\imath n\theta})\}_{n=1}^{\infty} \in \mathbb{C}_\sigma$ and $\{f_m(e^{\imath n\theta})\}_{n=1}^{\infty}$ converges to $\{f(e^{\imath n\theta})\}_{n=1}^{\infty}$ uniformly as $m \to \infty$. Since $\mathbb{C}_\sigma$ is a Banach space, this implies $\{f(e^{\imath n\theta})\}_{n=1}^{\infty} \in \mathbb{C}_\sigma$. Therefore $\sigma\text{-}\lim_{n\to\infty} f(e^{\imath n\theta})$ exists. Now, we suppose $k\theta \notin 2\pi\mathbb{Z}$ for every nonzero integer $k$. By Example 1, we have we have $\sigma\text{-}\lim_{n\to\infty} f_m(e^{\imath n\theta}) = f_m(0)$ for all $m \ge 1$. It follows from Theorem 2.4 that

$$\begin{aligned} \sigma\text{-}\lim_{n\to\infty} f(e^{\imath n\theta}) &= \sigma\text{-}\lim_{n\to\infty} \lim_{m\to\infty} f_m(e^{\imath n\theta}) \\ &= \lim_{m\to\infty} \sigma\text{-}\lim_{n\to\infty} f_m(e^{\imath n\theta}) \\ &= \lim_{m\to\infty} f_m(0) = f(0). \end{aligned}$$

This completes the proof.

**Remark.** Indeed, $A(\Delta) \equiv \{f : \Delta \to \mathbb{C} |\ f$ is continuous on $\Delta$ and is analytic in the interior of $\Delta\}$. For, if $f : \Delta \to \mathbb{C}$ is continuous on $\Delta$ and is analytic in the interior of $\Delta$, and if $0 < r < 1$, then the function $f_r(z) := f(rz)$ is analytic on $\{z \in \mathbb{C}; |z| < \frac{1}{r}\}$. Therefore $f_r$ can be approximated (uniformly on $\Delta$) by a sequence of polynomials. Since $f_r \to f$ uniformly on $\Delta$ as $r \nearrow 1$, we must have that $f$ can be approximated by a sequence of polynomials uniformly on $\Delta$. In Theorems 3.1 and 3.2, we have $|\sigma\text{-}\lim_{n\to\infty} f(h(n\theta))| \le ||f||_\infty$, the sup-norm of $f$ (see [4, Theorem 3.2]). This fact is used in the proof of the next theorem.

**Theorem 3.3.** *Suppose $X$ is a reflexive Banach space and $\theta \in \mathbb{R}$.*

$(i)$ *If* $f : [-1, 1] \to X$ *is weakly continuous, then both* $\sigma\text{-}\lim_{n\to\infty} f(\sin(n\theta))$ *and* $\sigma\text{-}\lim_{n\to\infty} f(\cos(n\theta))$ *exist.*

$(ii)$ *If* $f : \Delta \to X$ *is weakly continuous on* $\Delta$ *and* $f$ *is analytic in the interior of* $\Delta$*, then*

$$\sigma\text{-}\lim_{n\to\infty} f(e^{\imath n\theta}) \text{ exists.}$$

*Furthermore, if, in addition,* $k\theta \notin 2\pi\mathbb{Z}$ *for every nonzero integer* $k$*, then* $\sigma\text{-}\lim_{n\to\infty} f(e^{\imath n\theta}) = f(0)$.

*Proof.* Fix a $\theta \in \mathbb{R}$. Suppose a function $f$ is as mentioned in part (i) (resp. (ii)) and suppose $h(t) := \sin(t)$ or $\cos(t)$ (resp. $h(t) := e^{\imath t}$), $t \in \mathbb{R}$. For every $x^* \in X^*$, we define

$$F(x^*) := \sigma\text{-}\lim_{n\to\infty} \langle f(h(n\theta)), x^*\rangle.$$

By Theorems 3.1 and 3.2, $F$ is well-defined. Since the $\sigma$-limit is linear, so is $F$.

On the other hand, we have

$$|F(x^*)| = |\sigma\text{-}\lim_{n\to\infty} \langle f(h(n\theta)), x^*\rangle| \leq ||f||_\infty \cdot ||x^*||$$

for all $x^* \in X^*$. Since $X$ is reflexive, this implies that $F = J_x$ for some $x \in X$. Therefore we have for every $x^* \in X^*$

$$\sigma\text{-}\lim_{n\to\infty} \langle f(h(n\theta)), x^*\rangle = \langle x^*, J_x\rangle = \langle x, x^*\rangle.$$

This proves that

$$\sigma\text{-}\lim_{n\to\infty} f(h(n\theta)) = x.$$

Now, we suppose $k\theta \notin 2\pi\mathbb{Z}$ for every nonzero integer $k$. It follows from Theorem 3.2 that for every $x^* \in X^*$

$$\sigma\text{-}\lim_{n\to\infty} \langle f(e^{\imath n\theta}), x^*\rangle = \langle f(0), x^*\rangle.$$

Therefore $\sigma\text{-}\lim_{n\to\infty} f(h(n\theta)) = f(0)$. This completes the proof.

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