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A NOTE ON UNITARY BASES AND NONCOMMUTATIVE WAVELETS

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Abstract. We use unitary operators in a finite von Neumann algebra as orthonormal basis for certain Hilbert space. Possible applications in non-commutative wavelets and computer graphics are discussed.

1. Introduction

This is a continuation of our study of unitary bases and non commutative wavelets. In this paper, we shall use our unitary bases in an operator algebra to the study of representations of two dimensional functions and discuss certain convergence. Classically, tensor product is used to construct two (or higher) dimensional wavelets from one (or lower) dimensional ones by tensor product. Our construction in this article is based on crossed products. More specifically, we use an irrational rotation action of **Z** (viewed as the group of functions of the form $e^{2\pi in\theta}, n \in \mathbf{Z}$) on functions on the unit circle (may be identified with the unit interval $[0, 1]$). Then we use an approximate representation of functions $f(x, y)$, $(x, y) \in [0, 1] \times [0, 1]$, of two variables into the irrational rotation algebra. Then the unitary basis in the (non-commutative) irrational algebra gives rise to an (approximate) orthonormal basis for functions $f(x, y)$ of two variables. In other words, we may view the irrational rotation algebra as a continuous matrix algebra with continuous indices $(t, \theta) \in [0, 1] \times [0, 1]$. The unitary basis constructed (see our previous paper [4]) here can be described by functions with "thin" supports (in matrix terms, they are certain band matrices). This coincides with the known fact that many of the non-commutative (operator) algebras can be viewed as continuous matrix algebras. The difference between our construction and the classical tensor product construction is that our lower dimensional wavelets are viewed as diagonally supported functions. For applications, we may use some classical wavelets, such as the Haar wavelets,

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for our base unitary supported on the main diagonal. Then the other unitary operator induced by the action can be viewed as a matrix (with values only 0 and 1) with support on a diagonal line (parallel to the main diagonal). In a forth coming paper, we will study the algorithm and convergence properties of our approximation, as well as some applications in cryptography and computer graphics. For some basic topics related to this paper, we refer to [1, 2] and [5].

2. Irrational Rotation Algebras

In our previous paper [4], we have used the embedding of irrational rotation algebras as dense subalgebras of the hyperfinite factor $\mathcal{R}$ of type II_1. In this section we describe more details of this class of algebras and study their representations. First, recall the definition of such algebras. Consider a * algebra generated by two unitary elements U and V such that they satisfy a relation $UV = e^{2\pi i\theta}VU$, where θ is a real number between 0 and 1. We use $\mathcal{A}_\theta$ to denote such a * algebra with generators U and V with the relation described above. When θ is a rational number, the algebra $\mathcal{A}_\theta$ is a finite dimensional algebra. In this case, we write $\theta = \frac{m}{n}$ for $m, n \in \mathbf{N}$ and co-prime, one easily shows that $\mathcal{A}_\theta$ is isomorphic to the full matrix algebra $M_n(\mathbf{C})$. But when θ is irrational, $\mathcal{A}_\theta$ is an infinite-dimensional algebra. Let I be the unit of $\mathcal{A}_\theta$ and define a linear functional τ on $\mathcal{A}_\theta$ by $\tau(I) = 1$ and $\tau(U^mV^n) = 0$ whenever $n, m \in \mathbf{Z}$ and at lease one of them is non zero (since U, V are unitary elements, $U^* = U^{-1}$ and $V^* = V^{-1}$). This τ is a trace on $\mathcal{A}_\theta$, i.e., $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{A}_\theta$, and induces an inner product on $\mathcal{A}_\theta$: $\langle A, B\rangle = \tau(B^*A)$. The completion of $\mathcal{A}_\theta$ with respect to the norm given by this inner product is denoted by $L^2(\mathcal{A}_\theta, \tau)$. Then $\mathcal{A}_\theta$ acts on the Hilbert space $L^2(\mathcal{A}_\theta, \tau)$ by left multiplication. The (operator) norm closure of $\mathcal{A}_\theta$ is the so-called irrational rotation C*-algebra, denoted by $\mathcal{R}_\theta$. The strong operator closure of $\mathcal{A}_\theta$ (or $\mathcal{R}_\theta$) always gives rise to the unique hyperfinite II_1 factor $\mathcal{R}$. The action of $\mathcal{R}$ on the unit vector I gives rise to an embedding of $\mathcal{R}$ into $L^2(\mathcal{A}_\theta, \tau)$. Sometimes, we may use $L^2(\mathcal{R}, \tau)$ instead of $L^2(\mathcal{A}_\theta, \tau)$ to denote the same Hilbert space.

In the rest of this paper, we shall assume that θ is a given irrational number. Then the following lemma is immediate (see, e.g., [4]).

Lemma 2.1. *The set $\{U^mV^n : m, n \in \mathbf{Z}\}$ forms an orthonormal basis of the Hilbert space $L^2(\mathcal{A}_\theta, \tau)$.*

We shall identify elements of $\mathcal{R}$ with vectors in $L^2(\mathcal{R}, \tau)$, which, in turn, are ℓ^2-sums with respect to the above orthonormal basis. But not every element in $L^2(\mathcal{R}, \tau)$ gives rise to a bounded linear operator by the left multiplication on $\mathcal{A}_\theta$ (viewed as a dense subspace of $L^2(\mathcal{R}, \tau)$).

The following two results are of some interests to the structure of the factor $\mathcal{R}$ in relation to its generators U and V.

Proposition 2.2. *If $A(I+U^{-1}V)=0$ for some $A\in\mathcal{R}$, then $A=0$.*

Proof. Since $A\in L^2(\mathcal{R},\tau)$, we can write $A=\sum_{m,n}\lambda_{m,n}U^mV^n$, for $\lambda_{m,n}\in$ **C** and $\sum_{m,n}|\lambda_{m,n}|^2<\infty$. Then

$$\begin{aligned} A(I+U^{-1}V) &= \sum_{m,n}\lambda_{m,n}U^mV^n+\sum_{m,n}\lambda_{m,n}U^mV^nU^{-1}V \\ &= \sum_{m,n}\lambda_{m,n}U^mV^n+\sum_{m,n}\lambda_{m,n}e^{2\pi in\theta}U^{m-1}V^{n+1} \\ &= 0, \end{aligned}$$

so that $\lambda_{m,n}=-\lambda_{m+1,n-1}e^{2\pi i(n-1)\theta}$ for all integers m,n. Since $|e^{2\pi in\theta}|=1$, one of $\lambda_{m,n}$ being non-zero would yield infinitely many non-zero $\lambda_{m,n}$ with the same modulus. Then A would not be in $L^2(\mathcal{R},\tau)$, so that $\lambda_{m,n}$ must be 0 for all integers m,n. ∎

Theorem 2.3. *The element $U+V$ generates $\mathcal{R}$ as a von Neumann algebra.*

Proof. We will show that U and V are in the weak-operator closure of the algebra generated by $U+V$ and its adjoint $U^{-1}+V^{-1}$. Let $\mathcal{B}$ be the abelian von Neumann subalgebra of $\mathcal{R}$ generated by $U^{-1}V$, and $\mathcal{M}$ the von Neumann subalgebra of $\mathcal{R}$ generated by $U+V$ and its adjoint $U^{-1}+V^{-1}$.

First we show that $\mathcal{M}$ contains $\mathcal{B}$. It is not hard to see that $\mathcal{M}$ and $\mathcal{B}$ both contain the identity I of $\mathcal{R}$. Since $(U+V)(U^{-1}+V^{-1})=2I+VU^{-1}+UV^{-1}$ and $(U^{-1}+V^{-1})(U+V)=2I+U^{-1}V+V^{-1}U$, we have that $VU^{-1}+UV^{-1}$ and $U^{-1}V+V^{-1}U$ are contained in $\mathcal{M}$. But we have the equalities $VU^{-1}+UV^{-1}=e^{2\pi i\theta}U^{-1}V+UV^{-1}$ and $U^{-1}V+V^{-1}U=U^{-1}V+e^{2\pi i\theta}UV^{-1}$. The linear combinations of these two elements give us that $U^{-1}V$ is in $\mathcal{M}$, so that $\mathcal{M}$ contains $\mathcal{B}$.

Using function calculus in $\mathcal{B}$ ($\mathcal{B}$ is $*$-isomorphic to $L^\infty(0,1)$), we know that $I+U^{-1}V$ generates $\mathcal{B}$ as a von Neumann algebra. Then there are elements A_n in $\mathcal{B}$ such that $(I+U^{-1}V)A_n$ tends to I in strong operator topology. Since $U+V\in\mathcal{M}$ by our assumption, we have that $(U+V)A_n=U(I+U^{-1}V)A_n$ tends to U strongly. This implies that U lies in $\mathcal{M}$. The same method yields that V lies in $\mathcal{M}$. This completes the proof of our result. ∎

It is well known (see, e.g., [4]) that $\mathcal{R}$ contains many other dense subalgebras such as the class of UHF C*-algebras. Thus the ascending union of certain finite dimensional matrix subalgebras of $\mathcal{R}$ is dense in $\mathcal{R}$. In the following section, we shall study the approximations of elements in $\mathcal{R}$ by matrices and representations of functions of two variables by elements in $\mathcal{R}$.

3. Representations of Functions of Two Variables in $\mathcal{R}$

Suppose that the hyperfinite II_1 factor $\mathcal{R}$ has a dense subalgebra given by the ascending union of full matrix algebras $M_{n_j}(\mathbf{C})$ with $j = 1, 2, \ldots$ and $n_j | n_{j+1}$. We have used this fact to construct unitary basis for $\mathcal{R}$ with nice properties (see [4]). In the first part of this section, we shall construct an approximate embedding of the functions of two variables into the hyperfinite II_1 factor $\mathcal{R}$. In the second part, we discuss the convergence of this approximation.

For simplicity, we shall consider continuous functions on the domain $[0, 1] \times [0, 1]$. We define a matrix-like multiplication, denoted by $\odot$, for two such functions $f(s, t)$ and $g(s, t)$ as follows:

$$(f \odot g)(s, t) = \int_0^1 f(s, x) g(x, t) dx.$$

This type of multiplication is studied in integral operator theory. When f is given and g varies, the integral operator induced by f is trace-class (and thus a compact operator) in the usual Hilbert space structure for functions on $[0, 1] \times [0, 1]$ (with respect to Lebesgue measure). Let $\mathcal{H} = L^2([0, 1] \times [0, 1])$, Tr denote the usual trace for operators acting on $\mathcal{H}$. Let $\mathcal{T}(\mathcal{H})$ be the algebra of all trace class operators on $\mathcal{H}$. Then $\langle A, B \rangle = \mathrm{Tr}(B^* A)$ defines an inner product on $\mathcal{T}(\mathcal{H})$. Suppose $\{e_i\}_{i=1}^\infty$ is an orthonormal basis for $\mathcal{H}$. Then $\{e_i \otimes e_j\}_{i,j}$ form an orthonormal basis for $\mathcal{T}(\mathcal{H})$, where, for any x in $\mathcal{H}$, $e_i \otimes e_j(x) = \langle x, e_i \rangle e_j$. Here we shall not study, in details, the decomposition of functions on $[0, 1] \times [0, 1]$ with respect to these types of bases in $\mathcal{T}(\mathcal{H})$. In stead, we shall use the convergence given by the normalized trace on $\mathcal{R}$. For any large n (here we assume that $n = n_j$ for some j), we partition the square $[0, 1] \times [0, 1]$ into n^2 small squares each of which has side length $1/n$. We may approximate a continuous function f by a local constant function f_0 such that f_0 is constant, denoted by $a_{j,k}$, in each small square $(\frac{j}{n}, \frac{j+1}{n}) \times (\frac{k}{n}, \frac{k+1}{n})$ for all $0 \leq j, k \leq n - 1$. Now we may identify f_0 with the matrix $A = (a_{j,k})_{j,k}$. For the given n, $A \in M_n(\mathbf{C}) \subseteq \mathcal{R}$. From Lemma 2.1,

$$A = \sum_{j,k \in \mathbf{Z}} \lambda_{j,k} U^j V^k,$$

where U, V are the unitary generators of $\mathcal{R}$ given above and $\lambda_{j,k} = \tau(A V^{-k} U^{-j})$. The convergence of the above sum is in strong-operator topology and thus in trace-norm.

Classically, the convergence in wavelet decomposition of functions is in usual Lebesque measure of the underlying domain. Our above representation of functions on the unit square converge, in the approximation by finite linear combinations of the base elements, with respect to a singular measure on the unit square. More precisely the unit square has a uniform singular measure. Here we describe more details of the convergence.

We shall consider a local constant function f, where, in each small square $(\frac{j}{n}, \frac{j+1}{n}) \times (\frac{k}{n}, \frac{k+1}{n})$ for all $0 \le j, k \le n-1$, f take a constant value $a_{j,k}$. Again, we use a matrix $A = (a_{j,k})_{j,k}$ to denote this function. Conversely, any matrix can be viewed as a local constant function in a similar way. Then we view A as an element in $\mathcal{R}$. Thus $A = \sum_{l,m \in \mathbf{Z}} \lambda_{l,m} U^l V^m$, where U, V are irrational unitary generators of $\mathcal{R}$ with the given irrational number θ. In applications, especially in numerical computations, we use rational numbers $\frac{p}{q}$ to approximate θ, where p, q are positive integers. In general, q is very large. We may choose q an integer multiple of n (or $q = n$ when n is a large number). Now we let U be the $q \times q$ diagonal unitary with diagonal entries given by $e^{2\pi i/q}, e^{2\cdot 2\pi i/q}, \ldots, e^{q\cdot 2\pi i/q}$, V be the unitary matrix with $(j, j-p)$-entry 1 and 0 else where (here $j = 1, \ldots, q$ and $j - p$ is identified with $j - p + q$ when $j - p < 1$). Since A is an $n \times n$ matrix and q is a multiple of n, we may view A as a $q \times q$ matrix where each constant $a_{j,k}$ is identified with $\frac{q}{n} \times \frac{q}{n}$ scalar matrix. Let τ be the normalized trace on all $q \times q$ matrices. Then $A = \sum_{l,m=1}^{q} \lambda_{l,m} U^l V^m$, where $\lambda_{l,m} = \tau(AV^{-m}U^{-l})$. In applications, when A is an approximation to a (continuous) function f, now we may use the finite sum $\sum_{l,m=1}^{q} \lambda_{l,m} U^l V^m$ to replace such an approximation. When f is continuous, we may use the matrix $(f(j/q, k/q))_{j,k}$ instead of A. More computation will be given in a forthcoming paper.

Note that the convergence given by the normalized trace τ on $q \times q$ matrices is not equivalent to Lebesque measure on the unit square, but comparable to Lebesque measure multiplied by the constant q. Thus our method works the best for small valued functions f so that $\sum_{j,k=1}^{q} |f(j/q, k/q)|^2$ is comparable to (or smaller than) q.

Numerically, we do not need to obtain the local constant approximation for continuous functions. We can obtain the coefficients $\lambda_{j,k}$ directly from the function f and the unitary elements U and V.

4. Continuous Matrix Representation

In this section, we initiate another representation. We shall use ideas of simple maximal abelian subalgebras of $\mathcal{R}$ to construct another embedding of the functions into $\mathcal{R}$. The idea comes from the continuous matrix representation for factors of type II_1 (due to Ambrose and Singer). This works when the factor has a "simple" maximal abelian subalgebra (or, *masa*). A masa $\mathcal{A}$ in a II_1 factor $\mathcal{M}$ is called *simple* if there is a vector x in $L^2(\mathcal{M},\tau)$ such that $\mathcal{A}x\mathcal{A}$ span a dense subspace of $L^2(\mathcal{M},\tau)$. One easily shows that $\mathcal{R}$ has a simple masa. In fact, we have the following theorem (see [3]):

Theorem 4.1. *Let $\mathcal{R}$ and U, V be given as above. Suppose $\mathcal{U}$ and $\mathcal{V}$ are the maximal abelian subalgebras of $\mathcal{R}$ generated by U and V respectively. Choose any unitary element W in $\mathcal{V}$ so that all the coefficients λ_n in $W = \sum_{n\in\mathbf{Z}} \lambda_n V^n$ are non zero. Then $\mathcal{R}$ is the closed linear span of $\mathcal{U}W\mathcal{U}$. Thus the linear span of $\mathcal{U}W\mathcal{U}$ is dense in $L^2(\mathcal{R},\tau)$ and $\mathcal{U}$ is a simple masa.*

If we identify $\mathcal{U}$ with $L^\infty[0,1]$ and $\mathcal{U}\otimes\mathcal{U}$ with $L^\infty[0,1]\otimes L^\infty[0,1]$ with variables given by (s,t), then we may define a map $\Psi : f(s)g(t) \mapsto f(s)Wg(t)$, which extends linearly to $L^\infty[0,1]\otimes L^\infty[0,1]$. Again we realize all functions given by finite sums of simple tensor products as elements in $\mathcal{R}$. Using linear basis $\{U^nWU^m : n,m \in \mathbf{Z}\}$, we can approximate functions of two variables by finite linear combinations of these base elements. Note that Ψ is only a one-to-one correspondence between $L^\infty[0,1]\otimes L^\infty[0,1]$ and certain elements in $\mathcal{R}$. The elements in $\mathcal{R}$ are not viewed as "local constant" functions directly. This may give us ideas to develop certain methods in cryptography. The convergence of this approximation is unknown (for a large class of functions, it might be comparable to the convergence on the unit square given by the usual Lebesgue measure).

The advantage of this representation is that only one unitary varies with its powers and the other can be chosen fixed. The disadvantage is that we do not have an orthonormal basis in general. Thus it is hard to compute the linear coefficients in the approximation. We do not know if one can choose certain nice vectors to replace W so the coefficients can be easily computed in the linear approximation. The continuity property of Ψ is another interesting problem. We end this article with the following question: *Can one choose an element X in $\mathcal{R}$ or $L^2(\mathcal{R},\tau)$ (replacing the above W) so that the map Ψ extends to $C([0,1]\times[0,1])$*?

References

1. C. K. Chui, *An Introduction to Wavelets,* Acad. Press, New York, 1992.
2. X. Dai and D. Larson, Wandering vectors for unitary systems and orthogonal wavelets, *Mem. Amer. Math. Soc.,* **134** (1998), No. 640.
3. L. Ge, S. Popa, On some decomposition properties for factors of type II_1, *Duke Math. J.,* **94** (1998), 79-101.
4. W. Guo, Unitary Bases and Noncommutative Wavelets, *Acta Mathematica Sinica (English Series),* **19(3)** (2003), 1-11.

5. R. Kadison and J. Ringrose, *'Fundamentals of the Operator Algebras,* Vols. I and II, Academic Press, Orlando, 1983 and 1986.

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