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SOLUTIONS TO NONAUTONOMOUS ABSTRACT FUNCTIONAL EQUATIONS WITH INFINITE DELAY

Jin Liang and Ti-Jun Xiao

Abstract. In this paper, we obtain two new existence theorems for mild solutions and classical solutions to nonautonomous functional equations with infinite delay in Banach spaces.

1. Introduction

The Cauchy problem for various delay equations in Banach spaces has been receiving more and more attention in recent years (cf, e.g., [3-11,13,15,16,19] and references therein). In this paper, we are concerned with the following nonautonomous abstract functional equation with infinite delay in a Banach space X

$$\begin{cases} u'(t) = A(t)u(t) + f(t, u(t), u_t), \quad \sigma \le t \le T, \\ u_\sigma = \phi, \end{cases} \tag{1.1}$$

where $\phi \in \mathcal{P}$ ($\mathcal{P}$ is an admissible phase space), $0 \le \sigma < T$, $\{A(t)\}_{t\in[\sigma,T]}$ is a family of linear operators in a Banach space X, and $f \in C([\sigma, T] \times X \times \mathcal{P}, X)$ is a given function. We obtain two new existence theorems for mild solutions and classical solutions to nonautonomous functional equations with infinite delay in Banach spaces. One will see that Theorem 2.1 below extends essentially [9, Theorem 3.1], as far as the mild solution of (1.1) is concerned, by dropping the uniform continuity of nonlinear term f from the hypotheses.

For the reader's convenience, we recall here some basic concepts (cf., e.g., [1, 2, 9, 14]).

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Definition 1.1. Let S be a bounded subset of a semi-normed linear space Z. The *Kuratowski measure of noncompactness* of S is defined as

$$\alpha(S) := \inf\{\gamma > 0;\ \ S \text{ has a finite cover by sets of diameter } \leq \gamma\}.$$

Definition 1.2. Let $a < b$. A real nonnegative function $\mathcal{K}(t, \mu, \nu)$ on $[a, b] \times R^+ \times R^+$ is called a Kamke-type function if

(i) it is Lebesgue measurable in t for every $(\mu, \nu) \in R^+ \times R^+$ and continuous in (μ, ν) for a.e. $t \in [a, b]$, and $\mathcal{K}(\cdot, 0, 0) = 0$;

(ii) for all $0 \leq \mu \leq \overline{\mu}$, $0 \leq \nu \leq \overline{\nu}$ and for a.e. $t \in [a, b]$,

$$\mathcal{K}(t, \mu, \nu) \leq \mathcal{K}(t, \overline{\mu}, \overline{\nu}) \leq k_{(\overline{\mu},\overline{\nu})}(t),$$

where $k_{(\overline{\mu},\overline{\nu})}(t)$ is a locally integrable function on (a, b) for each $\overline{\mu}$, $\overline{\nu}$.

Definition 1.3. A linear space $\mathcal{P}$ consisting of functions from R^- into X, with semi-norm $\|\cdot\|_{\mathcal{P}}$, is called an *admissible phase space* if $\mathcal{P}$ has the following properties.

(H1) For any $t_0 \in R$ and $a > 0$, if $x : (-\infty, t_0 + a] \to X$ is continuous on $[t_0, t_0 + a]$ and $x_{t_0} \in \mathcal{P}$, then $x_t \in \mathcal{P}$ and x_t is continuous in $t \in [t_0, t_0 + a]$.

(H2) There exists a continuous function $K(t) > 0$ and a locally bounded function $M(t) \geq 0$ in $t \geq 0$ such that

$$\|x_t\|_{\mathcal{P}} \leq K(t - t_0) \max_{s\in[t_0,t]} \|x(s)\| + M(t - t_0)\|x_{t_0}\|_{\mathcal{P}}$$

for $t \in [t_0, t_0 + a]$ and x as in (H1).

(H3) The quotient space $\mathcal{P}/\|\cdot\|_{\mathcal{P}}$ is a Banach space.

Samples of Kamke-type function and admissible phase space can be found in many papers, for example, in [1, 2, 9].

Definition 1.4. An operator family $\{U(t, s)\}_{0\leq s\leq t\leq T} \subset \mathbf{L}(X)$ is called a (strongly continuous) evolution system if

(1) $U(s, s) = I$, $U(t, r)U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$.

(2) $(t, s) \to U(t, s)$ is strongly continuous for $0 \leq s \leq t \leq T$.

"Evolution system" is also called evolution family, evolution operator, evolution process, propagator, or fundamental solution (cf., e.g., [4, 12, 14, 18, 19]).

Parabolicity Assumption (cf., e.g., [14, 19]):

(A1) For all $t \in [0, T]$, $\mathcal{D}(A(t)) = D$ being dense in X.

(A2) For every $t \in [0,T]$ and complex number λ with $\mathrm{Re}\lambda \leq 0$, $(\lambda + A(t))^{-1}$ exists and satisfies

$$\left\|(\lambda + A(t))^{-1}\right\| \leq \frac{W}{1+|\lambda|}, \quad \mathrm{Re}\lambda \leq 0,\ t \in [0,T],$$

for a constant W.

(A3) There are constants $\alpha \in (0,1]$ and $\overline{W}$ such that

$$\left\|(A(t) - A(s))A(r)^{-1}\right\| \leq \overline{W}\ |t-s|^{\alpha}, \quad t,s,r \in [0,T].$$

2. Main Results

Theorem 2.1. *Let* $0 \leq \sigma < T$, $\{U(t,s)\}_{0\leq s\leq t\leq T} \subset \mathbf{L}(X)$ *being an evolution system,* $\mathcal{P}$ *an admissible phase space, and* $f \in C([\sigma,T] \times X \times \mathcal{P}, X)$*. Suppose that there is a Kamke function* $\mathcal{K}(\cdot,\cdot,\cdot)$ *on* $[\sigma,T] \times R^{+} \times R^{+}$ *such that*

(i) *for every* $t \in [\sigma,T]$ *and for every bounded set* $B \subset X$ *and* $\Omega \subset \mathcal{P}$*,*

$$\alpha(F(\{t\} \times \{s\} \times B \times \Omega)) \leq \mathcal{K}(s, \alpha(B), \alpha(\Omega)), \quad a.e.\ s \in [\sigma,t],$$

where $F(t,s,\cdot,\cdot) = U(t,s)f(s,\cdot,\cdot)$.

(ii) $\varpi(t) \equiv 0$ *is the unique nonnegative absolutely continuous solution to the differential equation*

$$\varpi'(t) = 2M\mathcal{K}(t, \varpi(t), K(t)\varpi(t)), \quad a.e.\ t \in (\sigma,T]$$

satisfying

$$\lim_{t\to\sigma^{+}} \tfrac{\varpi(t)}{t-\sigma} = \varpi(\sigma) = 0; \tag{2.1}$$

(iii)

$$\lim_{t\to\sigma^{+}} \mathcal{K}(t, \varsigma(t), K(t-\sigma)\varsigma(t)) = 0, \tag{2.2}$$

for every nonnegative absolutely continuous function $\varsigma(t)$ *on* $[\sigma,T]$ *satisfying* (2.1)*; where* $M = \sup_{\sigma\leq s\leq t\leq T} \|U(t,s)\|$ *and* $K(\cdot)$ *is the function as in* (H2)*. Then for each* $\phi \in \mathcal{P}$*, there exists a real number* $T_{\sup}(\phi, U(\cdot,\cdot), f)$ *and a*

$$u : (-\infty, T_{\sup}(\phi, U(\cdot,\cdot), f)) \to X$$

such that

$$(2.3)\quad u(t)=\begin{cases} U(t,\sigma)\phi(0)+\displaystyle\int_\sigma^t U(t,s)f(s,u(s),u_s)ds, & t\in[\sigma,T_{\sup}(\phi,U(\cdot,\cdot),f)),\\ \phi(t-\sigma), \quad t\in(-\infty,\sigma]. \end{cases}$$

Proof. For $\phi\in\mathcal{P}$, write

$$\overline{M}:=M\left[\max_{s\in[\sigma,T]}\|f(s,\phi(0),\phi)\|+1\right].$$

From the proof of [9, Theorem 3.1], we know that for each $\phi\in\mathcal{P}$, there exists a $\delta\in(0,\frac{1}{2})$, a $\tau>\sigma$ and a sequence $u[n](\cdot)_{n\in N}$ in the following set

$$\begin{aligned}\mathcal{P}_{\phi,\delta}^{[\sigma,\tau]}:=\Big\{&u:(-\infty,\tau]\to X;\ \ u\Big|_{[\sigma,\tau]}\in C([\sigma,\tau],X),\ u_\sigma=\phi,\\ &\max_{s\in[\sigma,t]}\|u(s)-U(s,\sigma)\phi(0)\|\le(t-\sigma)\overline{M}\quad\text{and}\\ &\max\{\|u(t)-\phi(0)\|,\|u_t-\phi\|_{\mathcal{P}}\}\le\delta(\phi)\ \ \text{for all } t\in[\sigma,\tau]\Big\},\end{aligned}$$

such that

$$(2.4)\qquad \overline{u}[n](t):=\begin{cases} U(t,\sigma)\phi(0)+\displaystyle\int_\sigma^t U(t,s)f(s,u[n](s),u[n]_s)ds, & t\in[\sigma,\tau],\\ \phi(t-\sigma), \quad t\in(-\infty,\sigma] \end{cases}$$

is in $\mathcal{P}_{\phi,\delta}^{[\sigma,\tau]}$,

$$\|f(s,u[n](s),u[n]_s)-f(s,\phi(0),\phi)\|\le 1,\quad s\in[\sigma,\tau],$$

and

$$\lim_{n\to\infty}\|\overline{u}[n](t)-u[n](t)\|=0\quad\text{uniformly for}\ \ t\in[\sigma,\tau].$$

So,

$$(2.5)\qquad \alpha\left(\{\overline{u}[n](s)\}_{n\in N}\right)=\alpha\left(\{u[n](s)\}_{n\in N}\right),\qquad\text{for each } s\in[\sigma,\tau],$$

$$(2.6)\quad \alpha\left(\left\{\overline{u}[n](\cdot)\Big|_{[\sigma,t]}\right\}_{n\in N}\right)=\alpha\left(\left\{u[n](\cdot)\Big|_{[\sigma,t]}\right\}_{n\in N}\right),\ \ \text{for every } t\in[\sigma,\tau].$$

Since for every $t\in[\sigma,\tau)$ and $\varepsilon>0$, there are subsets $C_1(t)$, $\ldots$, $C_l(t)$ ($l\in N$ depends on t) of $\left\{\int_\sigma^t U(t,s)f(s,u[n](s),u[n]_s)ds\right\}_{n\in N}$ such that

$$\begin{aligned} & diam(C_k(t)) \\ \leq \ & \alpha\left(\left\{\int_\sigma^t U(t,s)f(s,u[n](s),u[n]_s)ds\right\}_{n\in N}\right)+\varepsilon, \quad k=1,\dots,l, \\ & \left\{\int_\sigma^t U(t,s)f(s,u[n](s),u[n]_s)ds\right\}_{n\in N} = \bigcup_{k=1}^{l} C_k(t), \end{aligned}$$

where $diam(C_k(t))$ means the diameter of the set $C_k(t)$. Thus, if we define for every $\eta \in [t,\tau]$,

$$\begin{aligned} \widetilde{C}_k(t,\eta) := \Bigg\{ & \int_\sigma^{\cdot} U(\cdot,s)f(s,u[n](s),u[n]_s)ds\Big|_{[t,\eta]}; \\ & \int_\sigma^t U(t,s)f(s,u[n](s),u[n]_s)ds \in C_k(\eta)\Bigg\}, k=1,\dots,l, \end{aligned}$$

then

$$\left\{\int_\sigma^{\cdot} U(\cdot,s)f(s,u[n](s),u[n]_s)ds\Big|_{[t,\eta]}\right\}_{n\in N} = \bigcup_{k=1}^{l} \widetilde{C}_k(t,\eta).$$

For each $k=1,\dots,l$, we choose arbitrarily two elements

$$\int_\sigma^{\cdot} U(\cdot,s)f(s,u[i_k](s),u[i_k]_s)ds\Big|_{[t,\eta]}, \quad \int_\sigma^{\cdot} U(\cdot,s)f(s,u[j_k](s),u[j_k]_s)ds\Big|_{[t,\eta]}$$

from $\widetilde{C}_k(t,\eta)$. Then

$$\begin{aligned} & \left\|\int_\sigma^{\cdot} U(\cdot,s)f(s,u[i_k](s),u[i_k]_s)ds\Big|_{[t,\eta]} - \int_\sigma^{\cdot} U(\cdot,s)f(s,u[j_k](s),u[j_k]_s)ds\Big|_{[t,\eta]}\right\|_{C[t,\eta]} \\ = \ & \max_{\nu\in[t,\eta]}\left\|\int_\sigma^{\nu} U(\nu,s)f(s,u[i_k](s),u[i_k]_s)ds - \int_\sigma^{\nu} U(\nu,s)f(s,u[j_k](s),u[j_k]_s)ds\right\| \\ \leq \ & \max_{\nu\in[t,\eta]}\left\|\int_\sigma^{t} U(\nu,s)f(s,u[i_k](s),u[i_k]_s)ds - \int_\sigma^{t} U(\nu,s)f(s,u[j_k](s),u[j_k]_s)ds\right\| \\ & + \max_{\nu\in[t,\eta]}\left\|\int_t^{\nu} U(\nu,s)f(s,u[i_k](s),u[i_k]_s)ds - \int_t^{\nu} U(\nu,s)f(s,u[j_k](s),u[j_k]_s)ds\right\| \\ \leq \ & \max_{\nu\in[t,\eta]}\|U(\nu,t)\| \left\|\int_\sigma^{t} U(t,s)\left[f(s,u[i_k](s),u[i_k]_s)ds - f(s,u[j_k](s),u[j_k]_s)\right]ds\right\| \\ & +2(\eta-t)\overline{M}, \quad k=1,\dots,l, \ i,j\in N. \end{aligned}$$

Therefore, if $\eta - t$ is small enough, then

$$\alpha\left(\left\{\int_\sigma^{\cdot} U(\cdot,s)f(s,u[n](s),u[n]_s)ds\Big|_{[t,\eta]}\right\}_{n\in N}\right)$$

$$\le\ diam(\widetilde{C}_k(t,\eta))$$

$$\le\ M\alpha\left(\left\{\int_\sigma^t U(t,s)f(s,u[n](s),u[n]_s)ds\right\}_{n\in N}\right)+2\varepsilon.$$

Thus, by the properties of Kuratowski measure of noncompactness, we obtain, for each $\bar{t}\in[\sigma,\tau)$,

$$\begin{aligned}
&\alpha\left(\left\{\int_\sigma^{\cdot} U(\cdot,s)f(s,u[n](s),u[n]_s)ds\Big|_{[\sigma,\bar{t}]}\right\}_{n\in N}\right)\\
&\le \sup_{t\in[\sigma,\bar{t}]}\lim_{\eta\uparrow t}\alpha\left(\left\{\int_\sigma^{\cdot} U(\cdot,s)f(s,u[n](s),u[n]_s)ds\Big|_{[t,\eta]}\right\}_{n\in N}\right)\\
&\le M\sup_{t\in[\sigma,\bar{t}]}\alpha\left(\left\{\int_\sigma^t U(t,s)f(s,u[n](s),u[n]_s)ds\right\}_{n\in N}\right)\\
&\le 2M^2\int_\sigma^{\bar{t}}\alpha\left(\{f(s,u[n](s),u[n]_s)\}_{n\in N}\right)ds.
\end{aligned}\tag{2.7}$$

Following the arguments in the proof of [9 Theorem 3.1], we can get the desired conclusion. For the reader's convenience, we give here a sketch of this proof.

Actually, by (H2) we have

$$\max_{\nu\in[\sigma,s]}\|u[n](\nu)-u[0](\nu)\| = \max_{\nu\in[\sigma,s]}\|u[n](\nu)-U(t,\sigma)z\|\le(s-\sigma)\overline{M},$$
$$s\in[\sigma,\tau],\ n\in N,$$

$$\max_{\nu\in[\sigma,s]}\|u[n]_\nu-u[0]_\nu\|_{\mathcal{P}}\le(s-\sigma)\overline{M}\max_{t\in[0,T-\sigma]}K(t),\quad s\in[\sigma,\tau],\ n\in N.$$

So the continuity of f implies that for each $\varepsilon>0$, there is $0<\eta\le\tau-\sigma$ such that

$$\alpha\left(\{f(s,u[n](s),u[n]_s)\}_{s\in[\sigma,\sigma+\eta],\ n\in N}\right)\le\frac{\varepsilon}{2}.$$

Set

$$\varsigma(t):=\begin{cases}2M^2\int_\sigma^\tau\alpha\left(\{f(s,u[n](s),u[n]_s)\}_{n\in N}\right)ds, & t\in[\tau,T],\\ 2M^2\int_\sigma^t\alpha\left(\{f(s,u[n](s),u[n]_s)\}_{n\in N}\right)ds, & t\in[\sigma,\tau].\end{cases}$$

Then $\varsigma(t)$ is a nonnegative absolutely continuous function $\varsigma(t)$ on $[\sigma, T]$, $\varsigma(\sigma) = 0$, and

$$\lim_{t\to\sigma^+} \frac{\varsigma(t)}{t-\sigma} = 0. \tag{2.8}$$

On the other hand, it follows from (2.4)-(2.7) and (H2) that

$$\begin{aligned}
\varsigma(t) &\leq 2\int_\sigma^t \mathcal{K}\left(s, \alpha\left(\{u[n](s)\}_{n\in N}\right), \alpha\left(\{u[n]_s\}_{n\in N}\right)\right) ds \\
&\leq 2\int_\sigma^t \mathcal{K}\left(s, \alpha\left(\{\overline{u}[n](s)\}_{n\in N}\right), K(s-\sigma)\alpha\left(\left\{u[n](\cdot)\Big|_{[\sigma,s]}\right\}_{n\in N}\right)\right) ds \\
&\leq 2\int_\sigma^t \mathcal{K}\left(s, \varsigma(s), K(s-\sigma)\varsigma(s)\right) ds, \quad \text{a.e. } t\in(\sigma,\tau].
\end{aligned}$$

Put

$$\overline{\varsigma}(t) := \begin{cases} \displaystyle\int_\sigma^\tau \mathcal{K}\left(s, \varsigma(s), K(s-\sigma)\varsigma(s)\right) ds, & t\in[\tau, T], \\ \displaystyle\int_\sigma^t \mathcal{K}\left(s, \varsigma(s), K(s-\sigma)\varsigma(s)\right) ds, & t\in[\sigma,\tau]. \end{cases}$$

Then, $\overline{\varsigma}(t)$ is also a nonnegative absolutely continuous function on $[\sigma, T]$, $\overline{\varsigma}(\sigma) = 0$, and

$$\lim_{t\to\sigma^+} \frac{\overline{\varsigma}(t)}{t-\sigma} = 0$$

by (2.2) and (2.8). Clearly,

$$\overline{\varsigma}'(t) \leq 2\mathcal{K}\left(t, \overline{\varsigma}(t), K(t-\sigma)\overline{\varsigma}(t)\right), \quad \text{a.e. } t\in[\sigma, T].$$

This implies that $\overline{\varsigma}(t)\equiv 0$, and so $\varsigma(t)\equiv 0$. Therefore, by (2.4) and (2.6),

$$\alpha\left(\left\{u[n](\cdot)\Big|_{[\sigma,\tau]}\right\}_{n\in N}\right) = 0,$$

i.e., there exists an increasing subsequence $\{n_k\}_{k\in N} \subset N$ and a function $u : (-\infty,\tau] \to X$ with $u(\cdot)\Big|_{[\sigma,\tau]} \in C([\sigma,\tau], X)$ and $u_\sigma = \phi$ such that

$$\lim_{k\to\infty} \max_{t\in[\sigma,\tau]} \|u[n_k](t) - u(t)\| = 0.$$

From (H2), it follows that $\lim_{k\to\infty} \max_{t\in[\sigma,\tau]} \|u[n_k]_t - u_t\|_{\mathcal{P}} = 0$. Consequently, $u(t)$ is a mild solution to (1.1) on $[\sigma,\tau]$.

Theorem 2.2. *Let the "Parabolicity Assumption" hold and* $\{U(t,s)\}_{\ 0\le s\le t\le T}$ *be the evolution system associated with the family* $\{A(t)\}_{\ t\in[0,T]}$*. Let* $\mathcal{P}$ *be an admissible phase space,* $f \in C([0,T]\times X\times\mathcal{P}, X)$ *satisfying the hypotheses of Theorem* 2.1*, and for all*

$$u(\cdot)\in\mathcal{P}^{[0,T]} := \left\{ u:(-\infty,T]\to X;\ \ u\Big|_{[0,T]}\in C([0,T],X),\ u_0\in\mathcal{P}\right\},$$

$f(s,u(s),u_s)\in D\ (s\in[0,T])$ *and*

$$\int_0^T \|A(t_0)f(s,u(s),u_s)\|ds<\infty \tag{2.9}$$

for some $t_0\in[0,T]$*. Then for each* $\phi\in\mathcal{P}$*, there exists a real number* $T_{\sup}(\phi,U(\cdot,\cdot),f)$ *such that* (1.1) *has a classical solution* $u(t)$ *on* $[0,T_{\sup}(\phi,\,U(\cdot,\cdot),f))$.

Proof. From [14, 19] we know that the evolution system $\{U(t,s)\}_{0\le s\le t\le T}$ satisfies

(i)′ For all $0\le s<t\le T$, $U(t,s): X\to D:=\mathcal{D}(A(t))$, $t\to U(t,s)$ is strongly differentiable, and $\frac{\partial}{\partial t}U(t,s)\in\mathbf{L}(X)$ being strongly continuous on $0\le s<t\le T$.

(ii)′ For all $0\le s<t\le T$,

$$\begin{cases} \dfrac{\partial}{\partial t}U(t,s)=A(t)U(t,s), \\ \|A(t)U(t,s)A^{-1}(s)\|\le\widetilde{M}, \end{cases} \tag{2.10}$$

where $\widetilde{M}$ is a constant.

(iii)′ For every $z\in D$ and $t\in(0,T]$, $U(t,s)z$ is differentiable in s on $0\le s\le t\le T$, and

$$\frac{\partial}{\partial s}U(t,s)z=-U(t,s)A(s)z. \tag{2.11}$$

Thus from Theorem 2.1 it follows that for each $\phi\in\mathcal{P}$, there exists a real number $T_{\sup}(\phi,U(\cdot,\cdot),f)$ such that (2.3) has a solution $u(t)$ on $[0,T_{\sup}(\phi,\,U(\cdot,\cdot),f))$. We have by (2.10),

$$\frac{\partial}{\partial t}U(t,s)f(s,u(s),u_s)\ =\ A(t)U(t,s)A(s)^{-1}A(s)A(t_0)^{-1}A(t_0)f(s,u(s),u_s),$$

$$0\le s<t\le T_{\sup}(\phi,U(\cdot,\cdot),f),$$

and by "Parabolic Assumption", there is a constant $\overline{M}$ such that

$$\|A(t)A(t_0)^{-1}\| \leq \overline{M}, \qquad \text{for each } t \in [0, T_{\sup}(\phi, U(\cdot,\cdot), f)).$$

Therefore, by (2.9), we get

$$\begin{aligned}\int_0^t \left\| \frac{\partial}{\partial t} U(t,s) f(s, u(s), u_s) \right\| ds \ &\leq \ \int_0^T \left\| \frac{\partial}{\partial t} U(t,s) f(s, \overline{u}^t(s), \overline{u}^t_s) \right\| ds \\ &< \ \infty, \qquad t \in [0, T_{\sup}(\phi, U(\cdot,\cdot), f)),\end{aligned}$$

where

$$\overline{u}^t(s) = \begin{cases} u(t), & s \in [t, T], \\ u(s), & s \in (-\infty, t]. \end{cases}$$

Hence,

$$\begin{aligned} u'(t) \ &= \ A(t)U(t,0)\phi(0) + f(t, u(t), u_t) + \int_0^t A(t)U(t,s)f(s, u(s), u_s) ds \\ &= \ A(t)u(t) + f(t, u(t), u_t), \quad t \in [0, T_{\sup}(\phi, U(\cdot,\cdot), f)), \end{aligned}$$

i.e., $u(t)$ is a classical solution of (1.1) on $[0, T_{\sup}(\phi, U(\cdot,\cdot), f))$.

Moreover, (i)′ and (2.11) imply that a classical solution of (1.1) is also a mild solution of (2.3). This means (1.1) has a unique classical solution for each $\phi \in \mathcal{P}$.■

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Jin Liang and Ti-Jun Xiao
Department of Mathematics,
University of Science and Technology of China,
Hefei, Anhui 230026,
People's Republic of China.
E-mail: jliang@ustc.edu.cn, xiaotj@ustc.edu.cn