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ON CLOSEDNESS IN THE $\mathcal{L}$-TOPOLOGY OF T.V.S.

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Abstract. Let X and Y be Hausdorff and locally convex topological vector spaces. In this paper, we prove that a convex subset of X is closed if and only if it is closed in the topology on X induced by the set of continuous linear mappings from X into Y. As applications, some existence results for vector equilibrium problems and vector variational inequalities associated with discontinuous mappings are given.

1. Introduction

For given topological vector spaces X and Y, let $\mathcal{L}(X, Y)$ denote the set of continuous linear mappings from X into Y. Throughout the paper, all topological vector spaces are assumed to be real spaces. When $Y = \mathbb{R}$, $\mathcal{L}(X, Y)$ is the topological dual space X^*. For $x \in X$ and $\ell \in \mathcal{L}(X, Y)$, we shall alternatively write the value $\ell(x)$ of ℓ at x as $\langle \ell, x \rangle$.

The set $\mathcal{L}(X, Y)$ induces a topology on X generalizing the usual weak topology. This topology is called the $\mathcal{L}$-topology on X induced by $\mathcal{L}(X, Y)$. See Section 2 for more discussion. The $\mathcal{L}$-topology is first considered in [4] for dealing with vector variational inequalities.

As well known, a convex subset of a Hausdorff locally convex topological vector space X is closed if and only if it is weakly closed. The main work of this section is to generalize this result to convex subsets of X in the $\mathcal{L}$-topology; see Theorem 2.1. The work was motivated by an attempt to derive existence results for generalized vector quasi-variational inequalities associated with mappings without continuity assumptions.

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It is worth observing that most of the results in generalized quasi-variational inequalities require the upper semicontinuity assumption, in the sense of multivalued mappings, on the associated mappings. When the associated mappings are single valued, this assumption reduces to the ordinary notion of continuity. There are very few results in the literature without continuity assumption; see [6-8. 12-14] and references there in.

The problem of generalized vector quasi-variational inequalities includes the problem of vector variational inequalities as a special case. As an application of our main result, in Section 3, we shall derive some existence results for vector variational inequalities associated with discontinuous operators. Existence results for generalized vector quasi-variational inequalities associated with discontinuous mappings will appear elsewhere. As vector variational inequality problems closely relate to vector equilibrium problems, our existence results for vector variational inequalities will be obtained from that for vector equilibrium problems associated with discontinuous bifunctions.

Our existence results are established by employing Fan-KKM Theorem [9]. To use this theorem, we need some definitions for multivalued mappings. For any given nonempty set X, let 2^X denote the family of all subsets of X. For given nonempty sets X and Y, a mapping $\Phi : X \longrightarrow 2^Y$ will be also called a multivalued mapping from X into Y. The mapping Φ is said to have nonempty values if $\Phi(x)$ is nonempty for every $x \in X$. When Y is a topological space, we say that Φ has closed values if $\Phi(x)$ is closed in Y for every $x \in X$. Similarly, if Y is a topological vector space, Φ is said to have convex values if every $\Phi(x)$ is convex in Y.

For a given nonempty convex subset K of a topological vector space X, a multivaued mapping Φ from K into X is a KKM mapping if for any nonempty finite set $E \subset K$,

$$\mathrm{co}(E) \subset \bigcup_{x \in E} \Phi(x)\,,$$

where co(E) denotes the convex hull of E in X. Observe that every KKM multivalued mapping has nonempty values. Now, we state Fan-KKM Theorem below.

Theorem 1.1. *Let K be a nonempty convex subset of a Hausdorff topological vector space X, and let $\Phi : K \longrightarrow 2^X$ be a KKM mapping. If Φ has closed values, and if $\Phi(x)$ is compact for some $x \in K$, then $\bigcap_{x \in K} \Phi(x)$ is nonempty.*

2. The $\mathcal{L}$-Topology

For given topological vector spaces X and Y, the $\mathcal{L}$-topology on X induced by $\mathcal{L}(X, Y)$ is the topology having the family

$$\{\ell^{-1}(U) : U \text{ is open in } Y \text{ and } \ell \in \mathcal{L}(X, Y)\}$$

as a subbasis. When $Y = \mathbb{R}$, this topology becomes the usual weak topology. In the rest of this section, we shall consider any fixed topological vector spaces X and Y, and the $\mathcal{L}$-topology on X induced by $\mathcal{L}(X, Y)$ will be simply called the $\mathcal{L}$-topology on X. The corresponding topological space will be denoted by $X_{\mathcal{L}}$. Note that $X_{\mathcal{L}}$ is a topological vector space, and that $X_{\mathcal{L}}$ is Hausdorff if and only if X is Hausdorff and locally convex [4, Theorem 3.1].

A subset E of X will be called $\mathcal{L}$-open ($\mathcal{L}$-closed or $\mathcal{L}$-compact) if E is open (closed or compact) in $X_{\mathcal{L}}$. It is clear that every $\mathcal{L}$-open (or $\mathcal{L}$-closed) subset of X is originally open (or closed) in X. Similarly, compact (or bounded) subsets of X are $\mathcal{L}$-compact (or $\mathcal{L}$-bounded).

In general, closed subsets of X are not necessarily $\mathcal{L}$-closed in X. It is well known that a closed subset of a Hausdorff locally convex topological vector space is weakly closed if it is convex; see e.g., [11, Theorem 3.12]. We prove in Theorem 2.1 to generalize this result to the notion of $\mathcal{L}$-closedness. It is worth to mention that if X is a normed space and Y is a Banach space, then the notion of $\mathcal{L}$-boundedness coincides with that of boundedness; see [5, Proposition 2.2].

To state our main theorem, we need some notations. For any given subset E of X, we shall use $\overline{E}^{\mathcal{L}}$ for the closure of E in $X_{\mathcal{L}}$. The original closure of E is denoted by $\overline{E}$ as usual.

Theorem 2.1. *Assume that X and Y are Hausdorff locally convex topological vector spaces. If E is a nonempty convex subset of X, then $\overline{E}^{\mathcal{L}} = \overline{E}$.*

As immediate consequences of Theorem 2.1, we obtain:

Theorem 2.2. *Assume that X and Y are Hausdorff locally convex topological vector spaces.*

(i) *A convex subset of X is $\mathcal{L}$-closed if and only if it is original closed.*

(ii) *A vector subspace of X is originally closed if and only if it is $\mathcal{L}$-closed.*

(iii) *A convex subset of X is originally dense if and only if it is dense in $X_{\mathcal{L}}$.*

As another consequence of Theorem 2.1, we consider the convergence of sequences in $X_{\mathcal{L}}$. A sequence $\{x_n\}_{n=1}^{\infty}$ is called $\mathcal{L}$-convergent in X if it converges in $X_{\mathcal{L}}$ to some $x \in X$, written by $x_n \xrightarrow{\mathcal{L}} x$. We shall write $x_n \longrightarrow x$ when $\{x_n\}_{n=1}^{\infty}$ converges to x originally in X. Note that

$$x_n \xrightarrow{\mathcal{L}} x \Longleftrightarrow \ell(x_n) \longrightarrow \ell(x) \quad \text{for all } \ell \in \mathcal{L}(X, Y).$$

Now, by a similar argument as in [11, Theorem 3.13, p. 65], the following theorem is obtained from Theorem 2.1.

Theorem 2.3. *Assume that X is a metrizable locally convex topological vector space. If $\{x_n\}_{n=1}^{\infty}$ is a sequence in X with $x_n \xrightarrow{\mathcal{L}} x \in X$, then there is a sequence $\{\widehat{x}_n\}_{n=1}^{\infty}$ in the convex hull of $\{x_n\}_{n=1}^{\infty}$ such that $\widehat{x}_n \longrightarrow x$.*

Theorem 2.1 will be proved by two lemmas. First, we recall some preliminary definitions. A subset $\mathcal{P}$ of Y is called a *hyperplane* if it is of the form $y+V$ for some $y \in Y$ and some linear subspace V of Y of codimension 1, i.e., $\dim(Y/V) = 1$, or equivalently, $V = \ker(\varphi)$ for some non-zero linear mapping $\varphi : Y \longrightarrow \mathbb{R}$. Note that $\mathcal{P}$ is closed if and only if φ is continuous, i.e., $\varphi \in Y^*$; see e.g., [11, Theorem 1.18, p. 14]. When $\mathcal{P}$ is closed, we associate to $\mathcal{P}$ two disjoint open subsets of Y given by

$$\mathcal{P}_+ = y + \{\zeta \in Y : \varphi(\zeta) > 0\} \quad \text{and} \quad \mathcal{P}_- = y + \{\zeta \in Y : \varphi(\zeta) < 0\}\,.$$

Two disjoint subsets A and B of X are called *strictly separated* by $\mathcal{L}(X, Y)$ if there exist an $\ell \in \mathcal{L}(X, Y)$ and a closed hyperplane $\mathcal{P} \subset Y$ such that

$$A \subset \ell^{-1}(\mathcal{P}_+) \quad \text{and} \quad B \subset \ell^{-1}(\mathcal{P}_-)\,.$$

Lemma 2.4. *Assume that Y is a Hausdorff locally convex topological vector space. If y_0 is a non-zero vector in Y, then there exists $\varphi \in Y^*$ such that $\varphi(y_0) = 1$ and $Y = Y_0 \oplus V$, where $Y_0 = \{ty_0 : t \in \mathbb{R}\}$ and $V = \ker(\varphi)$.*

Proof. Let Y_0 be equipped with the subspace topology from Y. Since Y is Hausdorff, the mapping $t \longmapsto ty_0$ is a linear homeomorphism from $\mathbb{R}$ onto Y_0 [10, Theorem (5.9.1), p. 89]. Consider the linear functional $f : Y_0 \longrightarrow \mathbb{R}$ defined by

$$f(ty_0) = t \quad \text{for } t \in \mathbb{R}.$$

Since Y is Hausdorff and locally convex, there exists $g \in Y^*$ such that

$$g(y) = f(y) \quad \text{for } y \in Y_0;$$

see [10, Theorem (8.4.6), p. 156]. It follows from [10, Theorems (5.8.1), p. 86, and (8.4.8), p. 157]nab that there exists $\pi \in \mathcal{L}(Y, Y_0)$ such that

$$Y = Y_0 \oplus V \quad \text{and} \quad \pi(y) = y \quad \text{for } y \in Y_0,$$

where $V = \ker(\pi)$. Clearly, the composite $\varphi = g \circ \pi$ is a continuous linear functional on Y. For $(t, v) \in \mathbb{R} \times V$, $\varphi(ty_0 + v) = f(ty_0) = t$. This completes the proof. ∎

Lemma 2.5. *Assume that X and Y are Hausdorff locally convex topological vector spaces. Let A and K be disjoint nonempty convex subsets of X. If A is closed and K is compact, then A and K are strictly separated by $\mathcal{L}(X, Y)$.*

Proof. By [11, Theorem 3.4, p. c58], there exist a $g \in X^*$ and an $r \in \mathbb{R}$ such that

$$g(x) > r \text{ for all } x \in A \quad \text{and} \quad g(x) < r \text{ for all } x \in K.$$

Choose any non-zero vector $y_0 \in Y$, and let $Y_0 = \{ty_0 : t \in \mathbb{R}\}$. By Lemma 2.4, there exists $\varphi \in Y^*$ such that

$$\varphi(y_0) = 1 \quad \text{and} \quad Y = Y_0 \oplus V\,,$$

where $V = \ker(\varphi)$. Consider the closed hyperplane $\mathcal{P} = ry_0 + V$. Note that

$$\mathcal{P}_+ = \{ty_0 + v : t > r \text{ and } v \in V\} \quad \text{and} \quad \mathcal{P}_- = \{ty_0 + v : t < r \text{ and } v \in V\}\,.$$

Let $h : \mathbb{R} \longrightarrow Y$ be the continuous linear mapping defined by $h(t) = ty_0$ for all $t \in \mathbb{R}$. Clearly, the composite $\ell = h \circ g$ lies in $\mathcal{L}(X,\, Y)$ satisfying $A \subset \ell^{-1}(\mathcal{P}_+)$ and $K \subset \ell^{-1}(\mathcal{P}_-)$. This completes the proof. ∎

Proof of Theorem 2.1. Clearly, $\overline{E} \subset \overline{E}^{\mathcal{L}}$. To prove the opposite inclusion, we consider any fixed point $x_0 \in X \setminus \overline{E}$. By Lemma 2.5, there exist an $\ell \in \mathcal{L}(X,\, Y)$ and a closed hyperplane $\mathcal{P}$ in Y such that

$$x_0 \in \ell^{-1}(\mathcal{P}_-) = U \quad \text{and} \quad \overline{E} \subset \ell^{-1}(\mathcal{P}_+)\,.$$

Since U is an $\mathcal{L}$-open subset of X with $U \cap E = \emptyset$, we have $\overline{E}^{\mathcal{L}} \subset X \setminus U$ and $x_0 \notin \overline{E}^{\mathcal{L}}$. This proves $\overline{E}^{\mathcal{L}} \subset \overline{E}$, and completes the proof. ∎

2. VEP and VVI Associated with Discontinuous Mappings

This section is devoted to deriving some existence results for vector equilibrium problems and vector variational inequalities associated with discontinuous bifunctions or operators. These problems are formulated by considering mappings with values in an ordered topological vector space. We shall fix once for all a Hausdorff topological vector space $\mathcal{Z}$ with a preorder defined by a closed convex cone $C \subset \mathcal{Z}$ such that $C \neq \mathcal{Z}$ and Int$C \neq \emptyset$, where Int$C$ is the interior of C in $\mathcal{Z}$. In this section, the $\mathcal{L}$-topology on a given topological vector space X will be referred to as the $\mathcal{L}$-topology induced by $\mathcal{L}(X,\, \mathcal{Z})$.

To state our results, we need some preliminary definitions and notations. For any subset A of a topological vector space X, we denote A^c by the complement of A in X, and when A is nonempty, we denote $\mathcal{F}(A)$ by the family of all nonempty finite subsets of A.

Let K be a nonempty subset of a topological vector space X. The vector equilibrium problem VEP$(f,\, K)$ associated with a bifunction $f : K \times K \longrightarrow \mathcal{Z}$

is the problem of finding an $\widehat{x} \in K$ such that $f(\widehat{x}, u) \in (-\text{Int}C)^c$ for all $u \in K$. Such an $\widehat{x}$ is called a solution of the problem VEP(f, K).

If $T : K \longrightarrow \mathcal{L}(X, \mathcal{Z})$ is an operator, then by considering the bifunction

$$f(x, u) = \langle T(x), u - x\rangle \quad \text{for } x, u \in K,$$

VEP(f, K) becomes the vector variational inequality VVI(T, K) associated with T. An $\widehat{x} \in K$ is called a solution of VVI(T, K) if $\langle T(\widehat{x}), u - \widehat{x}\rangle \in (-\text{Int}C)^c$ for all $u \in K$.

When $\mathcal{Z} = \mathbb{R}$ and C is the set of all non-negative real numbers, the problem VEP(f, K) becomes the equilibrium problem EP(f, K) associated with $f : K \times K \longrightarrow \mathbb{R}$, while VVI$(T, K)$ becomes the variational inequality VI(T, K) associated with $T : K \longrightarrow X^*$.

Recall that a bifunction f as given above is called vector 0-diagonally convexc [2] if for any finite set $\{u_1, ..., u_n\} \subset K$,

$$x = \sum_{j=1}^{n} \lambda_j u_j \text{ with } \lambda_j \geq 0 \text{ for all } j \text{ and } \sum_{j=1}^{n} \lambda_j = 1 \Longrightarrow \sum_{j=1}^{n} \lambda_j f(x, u_j) \in (-\text{Int}C)^c\,,$$

and that f is called C-quasiconvex-like [1] if for all $x, y_1, y_2 \in K$ and for $0 \leq t \leq 1$,

$$f(x, ty_1 + (1-t)y_2) \in f(x, y_1) - C \quad \text{or} \quad f(x, ty_1 + (1-t)y_2) \in f(x, y_2) - C\,.$$

When $\mathcal{Z} = \mathbb{R}$ and C is the set of non-negative real numbers, vector 0-diagonally convexity reduces to 0-diagonally convexity introduced by Zhow and Chen [15], and a real bifunction is called *quasiconvex-like* if it is C-quasiconvex-like.

A function $g : K \longrightarrow \mathcal{Z}$ is called *C-quasiconcave* [3] if for any $x_0, x_1 \in K$,

$$g(x_0) \in g(x_t) - C \quad \text{or} \quad g(x_1) \in g(x_t) - C\,,$$

where $x_t = (1-t)x_0 + tx_1$ for $0 \leq t \leq 1$. When $\mathcal{Z} = \mathbb{R}$ and C is the set of non-negative real numbers, a function $g : K \longrightarrow \mathbb{R}$ is simply called *quasiconcave* when it is C-quasiconcave, i.e., $g(x_0) \leq g(x_t)$ or $g(x_1) \leq g(x_t)$ for any $x_0, x_1 \in K$, where x_t is given above.

Now, we are ready to establish some existence results. The following lemma is an immediate consequence of Fan-KKM Theorem whose proof is omitted.

Lemma 3.1. *Let K be a nonempty compact and convex subset of a Hausdorff topological vector space. For a given bifunction $f : K \times K \longrightarrow \mathcal{Z}$, let $\Phi_f : K \longrightarrow 2^K$ be the multivalued mapping defined by*

$$\Phi_f(u) = \{x \in K : f(x, u) \in (-\text{Int}C)^c\}\,.$$

If Φ_f is a KKM mapping and has closed values, then VEP$(f,\, K)$ has a solution.

Remark 3.2. Let K be a nonempty convex subset of a topological vector space. For a given bifunction $f : K \times K \longrightarrow \mathcal{Z}$, the multivalued mapping Φ_f given in Lemma 3.1 is a KKM mapping if either f is vector 0-diagonally convex [2, proof of Lemma 3.6] or f is C-quasiconvex-like with $f(x,\, x) \in (-\text{Int}C)^c$ for every $x \in K$ [2, proof of Lemma 3.9].

Lemma 3.3. *Let K be a nonempty closed and convex subset of a Hausdorff topological vector space. For any given bifunction $f : K \times K \longrightarrow \mathcal{Z}$, let Φ_f be the multivalued mapping given in Lemma 3.1. Then VEP$(f,\, K)$ has a solution if*

(i) *Φ_f is a KKM mapping and has closed values, and*

(ii) *(Coercivity) There exist nonempty compact subsets A and B of K with B convex such that if $x \in K \cap A^c$, then $f(x,\, u_x) \in (-\text{Int}C)$ for some $u_x \in B$.*

Proof. Consider any $E \in \mathcal{F}(K)$. By Lemma 3.1, there is an $x_E \in \text{co}(E \cup B)$ such that

$$f(x_E,\, u) \in (-\text{Int}C)^c \quad \text{for all } u \in \text{co}(E \cup B).$$

Since $B \subset \text{co}(E \cup B)$, the coercivity condition implies $x_E \in A$. This proves that

$$S_E = \{x \in A : f(x,\, u) \in (-\text{Int}C)^c \text{ for all } u \in \text{co}(E \cup B)\} \neq \emptyset\,.$$

Let $\overline{S}_E$ be the closure of S_E in A. It is easy to see that the family $\{\overline{S}_E : E \in \mathcal{F}(K)\}$ has the finite intersection property. It follows from the compactness of A that

$$S = \bigcap_{E \in \mathcal{F}(K)} \overline{S}_E \neq \emptyset\,.$$

We claim that any $\widehat{x} \in S$ is a solution of VEP$(f,\, K)$.

For an arbitrary $u \in K$, let $U = \{\widehat{x},\, u\}$. Since $\widehat{x} \in \overline{S}_U$, there is a net $\{x_\alpha\}$ in S_U such that $x_\alpha \longrightarrow \widehat{x}$. Note that $f(x_\alpha,\, u) \in (-\text{Int}C)^c$ for every α, i.e., $x_\alpha \in \Phi_f(u)$. The closedness of $\Phi_f(u)$ implies $\widehat{x} \in \Phi_f(u)$. The proof is complete. ■

Corollary 3.4. *Let K be a nonempty closed and convex subset of a Hausdorff topological vector space, and let $f : K \times K \longrightarrow \mathcal{Z}$ be a bifunction. Assume that for every $u \in K$ the set*

$$\{x \in K : f(x,\, u) \in (-\text{Int}C)^c\}$$

is closed in K, and that the coercivity condition given in Lemma 3.3 is satisfied. Then VEP$(f,\, K)$ has a solution if either

(a) *f is vector 0-diagonally convex, or*

(b) *f is C-quasiconvex-like with $f(x, x) \in (-\text{Int}C)^c$ for every $x \in K$.*

Theorem 3.5. *Assume that $\mathcal{Z}$ is locally convex and X is a Hausdorff locally convex topological vector space. Let K be a nonempty closed and convex subset of X. For any given bifunction $f : K \times K \longrightarrow \mathcal{Z}$, let Φ_f be the multivalued mapping given in Lemma 3.1. Then VEP(f, K) has a solution if the following conditions are satisfied.*

(i) *Φ_f is a KKM mapping and has closed values.*

(ii) *For every $u \in K$, the function $x \longmapsto f(x, u)$ is C-quasiconcave.*

(iii) *(Coercivity) There exist nonempty $\mathcal{L}$-compact subsets A and B of K with B convex such that if $x \in K \cap A^c$, then $f(x, u_x) \in (-\text{Int}C)$ for some $u_x \in B$.*

Proof. Let $K_{\mathcal{L}}$ denote the subspace topology on K from $X_{\mathcal{L}}$. By Theorem 2.1, $K_{\mathcal{L}}$ is closed in $X_{\mathcal{L}}$. The sets A and B given in (iii) are compact subsets of $K_{\mathcal{L}}$. By Lemma 3.3, it remains to show that $\Phi_f : K_{\mathcal{L}} \longrightarrow 2^{K_{\mathcal{L}}}$ has closed values, or equivalently, $\Phi_f(u)$ is $\mathcal{L}$-closed in X for every $u \in K$. By Theorem 2.1 again, this will be done if Φ_f has convex values.

Suppose on the contrary that $\Phi_f(u_0)$ is not convex for some $u_0 \in K$. There exist distinct $x_0, x_1 \in \Phi_f(u_0)$ such that $x_t \notin \Phi_f(u_0)$ for some t with $0 < t < 1$, where $x_t = (1-t)x_0 + tx_1$. The condition (ii) implies that

$$f(x_0, u_0) \in f(x_t, u_0) - C \quad \text{or} \quad f(x_1, u_0) \in f(x_t, u_0) - C\,.$$

By assumption, we have $f(x_t, u_0) \in (-\text{Int}C)$. Thus,

$$f(x_0, u_0) \in (-\text{Int}C) - C = -\text{Int}C \quad \text{or} \quad f(x_1, u_0) \in (-\text{Int}C) - C = -\text{Int}C\,.$$

This is a contradiction, and completes the proof. ■

Theorem 3.6. *Assume that $\mathcal{Z}$ is locally convex and X is a Hausdorff locally convex topological vector space. Let K be a nonempty closed and convex subset of X, and let $f : K \times K \longrightarrow \mathcal{Z}$ be a bifunction. Assume that for every $u \in K$ the set $\{x \in K : f(x, u) \in (-\text{Int}C)^c\}$ is closed in K, and that the conditions (ii) and (iii) of Theorem 3.5 are satisfied. Then VEP(f, K) has a solution if*

(a) *f is vector 0-diagonally convex, or*

(b) *f is C-quasiconvex-like with $f(x, x) \in (-\text{Int}C)^c$ for every $x \in K$.*

Corollary 3.7. *Assume that X is a Hausdorff locally convex topological vector space. Let K be a nonempty closed and convex subset of X, and let $f : K \times K \longrightarrow \mathbb{R}$ be a real bifunction. Then* EP(f, K) *has a solution if the following conditions are satisfied.*

(i) *For every* $u \in K$ *the set* $\{x \in K : f(x, u) \geq 0\}$ *is closed in* K.

(ii) f *is* 0*-diagonally convex, or is quasiconvex-like with* $f(x, x) \geq 0$ *for every* $x \in K$.

(iii) *For every* $u \in K$, *the function* $x \longmapsto f(x, u)$ *is quasiconcave.*

(iv) *(Coercivity) There exist nonempty weakly compact subsets* A *and* B *of* K *with* B *convex such that if* $x \in K \cap A^c$, *then* $f(x, u_x) < 0$ *for some* $u_x \in B$.

At the end of the paper, we give some existence results for vector variational inequalities. Let K be a nonempty convex subset of a topological vector space X. For any given operator $T : K \longrightarrow \mathcal{L}(X, \mathcal{Z})$, consider the bifunction $f : K \times K \longrightarrow \mathcal{Z}$ defined by

$$f(x, u) = \langle T(x), u - x\rangle \quad \text{for } x, u \in K.$$

We claim that f is vector 0-diagonally convex. Indeed, for any finite set $\{u_1, u_2, ..., u_n\} \subset K$, if $x = \sum_{j=1}^{n} \lambda_j u_j$ with $\lambda_j \geq 0$ for all j and $\sum_{j=1}^{n} \lambda_j = 1$, then

$$\sum_{j=1}^{n} \lambda_j f(x, u_j) = \langle T(x), \sum_{j=1}^{n} t_j(u_j - x)\rangle = \langle T(x), x - x\rangle = 0 \in (-\text{Int}C)^c .$$

From Corollary 3.4, we obtain the following theorem which extends [12, Theorem 3.1] to vector case for infinite dimensional spaces.

Theorem 3.8. *Let* K *be a nonempty closed and convex subset of a Hausdorff topological vector space, and let* $T : K \longrightarrow \mathcal{L}(X, \mathcal{Z})$ *be an operator. Then VVI*(T, K) *has a solution if the following conditions are satisfied.*

(i) *For every* $u \in K$ *the set* $\{x \in K : \langle T(x), u - x\rangle \in (-\text{Int}C)^c\}$ *is closed in* K.

(ii) *(Coercivity) There exist nonempty compact subsets* A *and* B *of* K *with* B *convex such that if* $x \in K \cap A^c$, *then* $\langle T(x), u_x - x\rangle \in (-\text{Int}C)$ *for some* $u_x \in B$.

The following theorem is obtained from Theorem 3.5.

Theorem 3.9. *Assume that* $\mathcal{Z}$ *is locally convex and* X *is a Hausdorff locally convex topological vector space. Let* $T : K \longrightarrow \mathcal{L}(X, \mathcal{Z})$ *be an operator, where* K *is a nonempty closed and convex subset of* X. *Then VVI*(T, K) *has a solution if*

(i) $\{x \in K : \langle T(x), u - x\rangle \in (-\text{Int}C)^c\}$ *is closed and convex for every* $u \in K$, *and*

(ii) *(Coercivity) There exist nonempty* $\mathcal{L}$*-compact subsets* A *and* B *of* K *with* B *convex such that if* $x \in K \cap A^c$, *then* $\langle T(x), u_x - x\rangle \in (-\text{Int}C)$ *for some* $u_x \in B$.

References

1. Q. H. Ansari and J. C. Yao, An existence result for the generalized vector equilibrium problem, *Appl. Math. Lett.*, **12** (1999), 53-56.
2. O. Chadli, Y. Chiang and S. Huang, Topological pseudomonotonicity and vector equilibrium problems, *J. Math. Anal. Appl.*, **270** (2002), 435-450.
3. Y. Chiang, O. Chadli and J. C. Yao, Existence of Solutions to Implicit Vector Variational Inequalities, *J. Optim. Theory Appl.*, **116** (2003), 251-264.
4. Y. Chiang and J. C. Yao, Vector Variational Inequalities and the $(S)_+$-Condition, *J. Optim. Theory Appl.*, **123** (2004), 271-290.
5. Y. Chiang, Semicontinuous Mappings into T.V.S. with Applications to Mixed Vector Variational-Like Inequalities, *J. Global Optim.*, to appear.
6. P. Cubiotti, Finite-Dimensional Quasivariational Inequalities Associated with Discontinuous Functions, *J. Optim. Theory Appl.*, **72** (1992), 577-582.
7. P. Cubiotti, Generalized Quasivariational Inequalities without Continuities, *J. Optim. Theory Appl.*, **92** (1997), 477-495.
8. P. Cubiotti, On the Discontinuous Infinite-Dimensional Generalized Quasivariational Inequality Problem, *J. Optim. Theory Appl.*, **115** (2002), 97-111.
9. K. Fan, A Generalization of Tychonoff's Fixed-Point Theorem, *Math. Ann.*, **142** (1961), 305-310.
10. L. Narici and E. Beckenstein, *Topological Vector Spaces*, Marcel Dekker, Inc., 1985.
11. W. Rudin, *Functional Analysis*, McGraw-Hill, 1973.
12. J. C. Yao and J. S. Guo, Variational and Genrealized Variational Inequalities with Discontinuous Mappings, *J. Math. Anal. Appl.*, **182** (1994), 371-392.
13. J. C. Yao, Generalized Quasi-Variational Inequality Problems with Discontinuous Mappings, *Math. Oper. Res.*, **20** (1995), 465-478.
14. N. D. Yen, On a Class of Discontinuous Vector-Valued Functions and the Associated Quasivariational Inequalities, *Optimization*, **30** (1994), 197-203.
15. J. X. Zhow and G. Chen, Diagonal convexity conditions for problems in convex analysis and quasi-variational inequalities, *J. Math. Anal. Appl.*, **132** (1988), 213-225.

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