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THREE-STEP ITERATIVE CONVERGENCE THEOREMS WITH ERRORS IN BANACH SPACES

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Abstract. Let $q > 1$ and E be a real q-uniformly smooth Banach space, K be a nonempty closed convex subset of E and $T : K \to K$ be a single-valued mapping. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$ be three sequences in K and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$ satisfying some restrictions. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iteration process with errors: $x_{n+1} = (1-\alpha_n)x_n+\alpha_n Ty_n+u_n$, $y_n = (1 - \beta_n)x_n + \beta_n Tz_n + v_n$, $z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n + w_n, n \geq 1$. Sufficient and necessary conditions for the strong convergence $\{x_n\}$ to a fixed point of T is established. We also derive the corresponding new results on the strong convergence of the three-step iterative process.

1. Introduction and Preliminaries

Let E be an arbitrary real Banach space and let $J_q(q > 1)$ denote the generalized duality mapping from E into $2^{E^\star}$ given by

$$J_q(x) = \{f \in E^\star : \langle x, f\rangle = \|x\|^q = \|x\|\|f\|\}$$

where $E^\star$ denote the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and $E^\star$. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. It is known (see e.g. [10]) that $J_q(x) = \|x\|^{q-2}J(x)$ if $x \neq 0$ and that if $E^\star$ is strictly convex, then J_q is single-valued. The single-valued generalized duality mapping will be denoted by j_q in the sequel.

Definition 1.1. Let E be a normed space and K be a nonempty subset of E. Let $T : K \to E$ be a single-valued mapping.

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(i) T is said to be Lipschitzian mapping with constant L if $\forall x,\ y \in K$,

$$\|Tx - Ty\| \leq L\|x - y\|. \tag{1.1}$$

(ii) T is said to be accretive [12] if $\forall x,\ y \in K$, there exists $j_2(x-y) \in J_2(x-y)$ such that

$$\langle Tx - Ty, j_2(x-y)\rangle \geq 0,$$

or equivalently there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \geq 0. \tag{1.2}$$

(iii) T is said to be strongly accretive [12] if $\forall x,\ y \in K$, there exists $j_2(x-y) \in J_2(x-y)$ such that

$$\langle Tx - Ty, j_2(x-y)\rangle \geq k\|x-y\|^2$$

or equivalently there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \geq k\|x-y\|^q \tag{1.3}$$

for some $k > 0$. Without loss of generality, we can assume that $k \in (0,1)$ and such a number k is called the strong accretive constant of T.

(iv) T is said to be (strongly) pseudocontractive [12] if $I - T$ (where I denotes the identity mapping) is a (strongly) accretive mapping. That is $\forall x,\ y \in K$, there exists $j_2(x-y) \in J_2(x-y)$ such that

$$\langle (I-T)x-(I-T)y, j_2(x-y)\rangle \geq 0 \text{ (resp., } \langle (I-T)x-(I-T)y, j_2(x-y)\rangle \geq k\|x-y\|^2)$$

or equivalently there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\langle (I-T)x - (I-T)y, j_q(x-y)\rangle \geq 0 \tag{1.4}$$

$$(\text{resp.}, \langle (I-T)x - (I-T)y, j_q(x-y)\rangle \geq k\|x-y\|^q).$$

The constant k is said to be a strongly accretive constant with respect to $I - T$.

(v) T is said to be strictly pseudocontractive if $\forall x,\ y \in K$, there exists $\lambda > 0$ and $j_2(x-y) \in J_2(x-y)$ such that

$$\langle Tx - Ty, j_2(x-y)\rangle \leq \|x-y\|^2 - \lambda\|(x-Tx)-(x-Ty)\|^2, \tag{1.5}$$

or equivalently there exists $\lambda > 0$ and $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \leq \|x-y\|^q - \lambda\|(x-Tx)-(y-Ty)\|^2\|x-y\|^{q-2}.$$

We note that the strictly pseudocontractive single-valued mapping has been discussed in [4, 11]. Without loss of generality we may assume $\lambda \in (0,1)$. We note that (1.5) can be written in the form

$$\langle (x-Tx)-(y-Ty), j(x-y)\rangle \geq \lambda\|(x-Tx)-(y-Ty)\|^2. \tag{1.6}$$

From (1.6) we have

$$\|x-y\| \geq \lambda\|x-y-(Tx-Ty)\| \geq \lambda\|Tx-Ty\|-\|x-y\|$$

so that

$$\|Tx-Ty\| \leq \frac{(1+\lambda)}{\lambda}\|x-y\|$$

$\forall x,\ y \in K$. Hence a strictly pseudocontractive mapping is also a Lipschitzian mapping with constant greater than 1.

In 1967, the concept of a single-valued accretive mapping was introduced by Browder and Kato independently (see e.g. [12]). Browder stated that the following initial value problem

$$\frac{du(t)}{dt} + Tu(t) = 0,\ u(0) = u_0 \tag{1.7}$$

is solvable if T is locally Lipschitzian and accretive on E.

In Hilbert spaces, (1.5) (hence,(1.6)) is equivalent to the following inequality

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|(x-Tx)-(y-Ty)\|^2,\ k=(1-\lambda)<1.$$

Let E be a real $q-$uniformly smooth Banach space with $q>1$, K be a nonempty closed convex subset of E with $K+K \subseteq K$, and $T: K \to K$ be a single-valued mapping with $F(T) \neq \emptyset$. Let $\{u_n\},\ \{v_n\}$ and $\{w_n\}$ be three sequences in K and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $[0,1]$ satisfying certain restrictions. Let $\{x_n\}$ be the sequence generated from $x_1 \in K$ by the three-step iterative process with errors:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n + w_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n + v_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n,\ n \geq 1. \end{cases} \tag{1.8}$$

Especially if $u_n = 0$, $v_n = 0$ and $w_n = 0$ for $n \in N$, then $\{x_n\}$ is called the three-step iterative sequence which was suggested and analyzed by Noor [12]. If $\gamma_n = 0$ and $w_n = 0$ for $n \in N$, then $\{x_n\}$ is called the Ishikawa iterative sequence with error; if $\gamma_n = \beta_n = 0$ and $v_n = w_n = 0$ for $n \in N$, then $\{x_n\}$ is called the Mann iterative sequence with error.

In this paper by using Jensen's inequality and new approximation methods, we construct some simplified conditions to establish the sufficient and necessary conditions for the strong convergence of $\{x_n\}$ to a fixed point of T. The uniqueness of the fixed point of T is also discussed. We note that to compare with [11,Theorem 2 and Corollary 2] our results have the following features: (i) The uniform convexity of E is removed. (ii) The Ishikawa iterative process is replaced by the three-step iterative process with errors. (iii) Our restrictions imposed on $\{\alpha_n\}$ are much weaker than those in [11, Theorem 2 and Corollary 2]. Our results also improve and extend the corresponding results in [1, 8, 12, 13].

Now we give some preliminaries which will be used in the sequel. Let E be a real Banach space. The modulus of smoothness of E is defined as the function $\rho_E : [0,\infty) \to [0,\infty)$:

$$\rho_E(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1, \|y\| \le \tau\}.$$

E is said to be uniformly smooth if and only if $\lim_{\tau\to 0_+}(\rho_E(\tau)/\tau) = 0$. Let $q > 1$. The space E is said to be $q-$uniformly smooth (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \le c\tau^q$. It is well known that Hilbert spaces, L_p and l_p spaces, $1 < p < \infty$ as well as the Sobolev spaces, $W_m^p, 1 < p < \infty$ are $p-$uniformly smooth. Hilbert spaces are 2-uniformly smooth while if $1 < p \le 2$, L_p, l_p and W_m^p are $p-$uniformly smooth. If $p \ge 2$, L_p, l_p and W_m^p are 2-uniformly smooth.

Theorem 1.1. [10]. *Let $q > 1$ and E be a real Banach space. Then the following are equivalent:*

(1) *E is $q-$uniformly smooth.*

(2) *There exists a constant $c_q > 0$ such that for all $x, y \in E$*

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x)\rangle + c_q\|y\|^q. \tag{1.9}$$

Lemma 1.1 [9]. *Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of nonnegative real numbers such that $\sum_{n=1}^\infty b_n < \infty$ and $a_{n+1} \le a_n + b_n, \forall n \ge 1$. Then $\lim_{n\to\infty} a_n$ exists.*

2. Main Results

Throughout this section, L stands for the Lipschitzian constant of T, λ and c_q are the constants appearing in inequalities (1.5), (1.6), (1.9), respectively.

Lemma 2.1. *Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$ and $T : K \to K$ be a*

Lipschitzian mapping with Lipschitzian constant L and the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K, and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0,1]$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then

$$\|x_{n+1} - x^\star\|^q \leq (1+\delta_n)\|x_n - x^\star\|^q + \theta_n, \forall n \geq 1, \forall x^\star \in F(T), \tag{2.1}$$

where

$$\delta_n = -\alpha_n + \alpha_n(1-\beta_n)L^q + \alpha_n\beta_n(1-\gamma_n)L^{2q} + \alpha_n\beta_n\gamma_n L^{3q},$$

and

$$\begin{aligned}\theta_n \; = \; & q\alpha_n\beta_n L^{2q}\|w_n\|\|z_n - x^\star - w_n\|^{q-1} + \alpha_n\beta_n L^{2q}c_q\|w_n\|^q \\ & +q\alpha_n L^q\|v_n\|\|y_n - v_n - x^\star\|^{q-1} + \alpha_n L^q c_q\|v_n\|^q + q\|u_n\| \\ & \|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q.\end{aligned}$$

Proof. Let $x^\star$ be an arbitrary element in $F(T)$. Then it follows from (1.8) and (1.9) that

$$\begin{aligned}\|x_{n+1} - x^\star\|^q \; &= \; \|(1-\alpha_n)x_n + \alpha_n T y_n + u_n - x^\star\|^q \\ &\leq \; \|(1-\alpha_n)x_n + \alpha_n T y_n - x^\star\|^q \\ &\quad +q\langle u_n, j_q(x_{n+1} - u_n - x^\star)\rangle + c_q\|u_n\|^q \\ &\leq \; \|(1-\alpha_n)x_n + \alpha_n T y_n - x^\star\|^q \\ &\quad +q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q.\end{aligned} \tag{2.2}$$

By Jensen's inequality, we have

$$\begin{aligned}\|(1-\alpha_n)x_n + \alpha_n T y_n - x^\star\|^q \; &= \; \|(1-\alpha_n)(x_n - x^\star) + \alpha_n(T y_n - x^\star)\|^q \\ &\leq \; (1-\alpha_n)\|x_n - x^\star\|^q + \alpha_n\|T y_n - x^\star\|^q \\ &\leq \; (1-\alpha_n)\|x_n - x^\star\|^q + \alpha_n L^q\|y_n - x^\star\|^q\end{aligned} \tag{2.3}$$

and by (1.9) and Jensen's inequality, we have

$$
\begin{aligned}
\|y_n - x^\star\|^q &= \|(1-\beta_n)x_n + \beta_n T z_n + v_n - x^\star\|^q \\
&= \|(1-\beta_n)(x_n - x^\star) + \beta_n(Tz_n - x^\star) + v_n\|^q \\
&\le \|(1-\beta_n)(x_n - x^\star) + \beta_n(Tz_n - x^\star)\|^q \\
&\quad + q\langle v_n, j_q(y_n - v_n - x^\star)\rangle + c_q\|v_n\|^q \\
&\le (1-\beta_n)\|x_n - x^\star\|^q + \beta_n L^q \|x_n - x^\star\|^q \\
&\quad + q\|v_n\|\|y_n - v_n - x^\star\|^{q-1} \\
&\quad + c_q\|v_n\|^q,
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
\|z_n - x^\star\|^q &= \|(1-\gamma_n)x_n + \gamma_n T x_n + w_n - x^\star\|^q \\
&= \|(1-\gamma_n)(x_n - x^\star) + \gamma_n(Tx_n - x^\star) + w_n\|^q \\
&\le \|(1-\gamma_n)(x_n - x^\star) + \gamma_n(Tx_n - Tx^\star)\|^q \\
&\le +q\langle w_n, j_q(z_n - w_n - x^\star)\rangle + c_q\|w_n\|^q \\
&\le (1-\gamma_n)\|x_n - x^\star\|^q + \gamma_n L^q \|x_n - x^\star\|^q \\
&\quad + q\|w_n\|\|z_n - w_n - x^\star\|^{q-1} \\
&\quad + c_q\|w_n\|^q.
\end{aligned} \tag{2.5}
$$

Hence form (2.4) and (2.5), we have

$$
\begin{aligned}
\|y_n - x^\star\|^q &\le (1-\beta_n)\|x_n - x^\star\|^q + \beta_n L^q \{(1-\gamma_n)\|x_n - x^\star\|^q \\
&\quad + \gamma_n L^q \|x_n - x^\star\|^q \\
&\quad + q\|w_n\|\|z_n - w_n - x^\star\|^{q-1} + c_q\|w_n\|^q\} \\
&\quad + q\|v_n\|\|y_n - v_n - x^\star\|^{q-1} + c_q\|v_n\|^q \\
&= \{1 - \beta_n + \beta_n L^q(1-\gamma_n) + \beta_n\gamma_n L^{2q}\}\|x_n - x^\star\|^q \\
&\quad + q\beta_n L^q \|w_n\|\|z_n - x^\star - w_n\|^{q-1} + \beta_n L^q c_q \|w_n\|^q \\
&\quad + q\|v_n\|\|y_n - v_n - x^\star\|^{q-1} + c_q\|v_n\|^q,
\end{aligned} \tag{2.6}
$$

From (2.2)–(2.6), we derive that

$$\|x_{n+1} - x^\star\|^q \le (1+\delta_n)\|x_n - x^\star\|^q + \theta_n$$

where

$$\delta_n = -\alpha_n + \alpha_n(1-\beta_n)L^q + \alpha_n\beta_n(1-\gamma_n)L^{2q} + \alpha_n\beta_n\gamma_n L^{3q}$$

and

$$\begin{aligned}\theta_n &= q\alpha_n\beta_n L^{2q}\|w_n\|\|z_n - x^\star - w_n\|^{q-1} + \alpha_n\beta_n L^{2q}c_q\|w_n\|^q \\ &\quad + q\alpha_n L^q\|v_n\|\|y_n - v_n - x^\star\|^{q-1} + \alpha_n L^q c_q\|v_n\|^q \\ &\quad + q\|u_n\|\|x_{n+1} - u_n - x^\star\|^{q-1} + c_q\|u_n\|^q. \qquad \blacksquare\end{aligned}$$

Lemma 2.2. *Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$ and $T : K \to K$ be a Lipschitzian mapping with Lipschitzian constant L and the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^\infty$, $\{v_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ be sequences in K, and $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ be real sequences in $[0, 1]$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Furthermore, if $\sum_{n=1}^\infty \alpha_n < \infty$, then there is a constant $M > 0$ $($e.g. $M = e^{\sum_{n=1}^\infty \delta_n})$ such that*

$$\|x_{n+m} - x^\star\|^q \le M\|x_n - x^\star\|^q + M(\sum_{k=n}^{n+m-1} \theta_n) \tag{2.7}$$

$\forall m,\ n \in N$, $\forall x^\star \in F(T)$. *In particular,*

$$\|x_{n+1} - x^\star\|^q \le M\|x_1 - x^\star\|^q + M\sum_{k=1}^{n} \theta_k \tag{2.8}$$

$\forall n \in N$, $\forall x^\star \in F(T)$.

Proof. Since $\sum_{n=1}^\infty \alpha_n < \infty$, by Lemma 2.1, $\sum_{n=1}^\infty \delta_n < \infty$ and $\forall n \in N$,

$$\begin{aligned}\|x_{n+1} - x^\star\|^q &\le (1 + \delta_n)\|x_n - x^\star\|^q + \theta_n \\ &\le e^{\delta_n}\|x_n - x^\star\|^q + \theta_n.\end{aligned}$$

Hence by induction, we have

$$\begin{aligned}\|x_{n+1} - x^\star\|^q &\le e^{\delta_n}\|x_n - x^\star\|^q + \theta_n \\ &\le e^{\delta_n}[e^{\delta_{n-1}}\|x_{n-1} - x^\star\|^q + \theta_{n-1}] + \theta_n \\ &\le \cdots \\ &\le e^{\sum_{k=1}^n \delta_k}\|x_1 - x^\star\|^q + e^{\sum_{k=1}^n \delta_k}(\textstyle\sum_{k=1}^n \theta_k) \\ &\le M\|x_1 - x^\star\|^q + M(\textstyle\sum_{k=1}^n \theta_k)\end{aligned}$$

for all $n \in N$ and

$$\|x_{n+m} - x^\star\|^q \le M\|x_n - x^\star\|^q + M\sum_{k=n}^{n+m-1}\theta_k$$

for all $m,\ n \in N$. ■

Theorem 2.1. *Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$ and $T : K \to K$ be a Lipschitzian mapping with Lipschitzian constant L and the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^\infty$, $\{v_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ be three sequences in K and $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ be real sequences in $[0, 1]$. Also suppose $\sum_{n=1}^\infty \alpha_n < \infty$, $\sum_{n=1}^\infty \|u_n\| < \infty$, $\sum_{n=1}^\infty \|v_n\| < \infty$ and $\sum_{n=1}^\infty \|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then the sequence $\{x_n\}$ is bounded and*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0$$

if and only if the sequence $\{x_n\}$ converges strongly to a fixed point of T where $d(x_n, F(T))$ is the distance of x_n to set $F(T)$, i.e., $d(x_n, F(T)) = \inf_{u^\star\in F(T)}\|x_n - u^\star\|$.

Proof. If the sequence $\{x_n\}$ converges strongly to a fixed point of T, say, $y^\star \in F(T)$, it is easy to deduce that the sequence $\{x_n\}$ is bounded. Note that

$$d(x_n, F(T)) = \inf_{u^\star\in F(T)} \|x_n - u^\star\| \le \|x_n - y^\star\| \to 0 \text{ as } n \to \infty.$$

Therefore $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Suppose that the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Since the sequence $\{x_n\}$ is bounded and the series $\sum_{n=1}^\infty \|u_n\|$, $\sum_{n=1}^\infty \|v_n\|$ and $\sum_{n=1}^\infty \|w_n\|$ are finite, from (2.8) there is a $\tilde{M} > 0$ such that $\|x_n - x^\star\| < \tilde{M}$, $\|x_{n+1} - u_n - x^\star\| < \tilde{M}$, $\|u_n\| < \tilde{M}$ and $\|v_n\| < \tilde{M}$. Then

$$\sum_{n=1}^\infty \theta_n \le \sum_{n=1}^\infty \alpha_n L^q(2 + L)^{q-1}\tilde{M}^q + q\sum_{n=1}^\infty \|u_n\|\tilde{M}^{q-1} + c_q\sum_{n=1}^\infty \|u_n\|^q < \infty.$$

Hence the sequence $\{\|x_{n+1} - x^\star\|^q\}$ is bounded, so is $\{\|x_{n+1} - x^\star\|\}$. Also from (2.1) and Lemma 1.1, we know that $\lim_{n\to\infty} \|x_n - x^\star\|$ exists. Furthermore, from (2.1) we have

$$(d(x_{n+1}, F(T)))^q \le (d(x_n, F(T)))^q + \delta_n\tilde{M}^q + \theta_n.$$

By Lemma 1.1, we have $\lim_{n\to\infty}(d(x_n, F(T)))^q$ exists. Since $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, $\lim_{n\to\infty} d(x_n, F(T)) = 0$. By using the same argument in [2, Theorem 2.1], we have the sequence $\{x_n\}$ converges strongly to a fixed point of T. ■

As mentioned in (1.6), a strictly pseudocontractive mapping is also a Lipschitzian mapping, we have the following corollary.

Corollary 2.1. *Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$ and $T : K \to K$ be a strictly pseudocontractive mapping and the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K, $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$. Also suppose $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$ and $\sum_{n=1}^{\infty} \|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then the sequence $\{x_n\}$ is bounded and*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0$$

if and only if the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. The conclusion of Corollary 2.1 follows immediately from Theorem 2.1 and the fact that a strictly pseudocontractive mapping is also a Lipschitzian mapping. ■

Theorem 2.2. *Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$ and $T : K \to K$ be a Lipschitzian strongly pseudocontraction mapping with Lipschitzian constant L and strongly accrective constant $k \in (0, 1)$ with respect to $I - T$. Assume that the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$. Also suppose $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$ and $\sum_{n=1}^{\infty} \|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then the sequence $\{x_n\}$ is bounded and*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0$$

if and only if the sequence $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. Since all conditions of Theorem 2.1 hold, from Theorem 2.1 we have that the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ if and only if the sequence $\{x_n\}$ converges strongly to a fixed point, say $x^\star$, of T. Actually the fixed point $x^\star$ is unique. Indeed if there is another fixed point $\bar{x}$, we have

$$\bar{x} = T\bar{x} \text{ and } x^\star = Tx^\star.$$

If we choose that $\bar{x} = T\bar{x}$ and $x^\star = Tx^\star$ from the strongly pseudocontraction of T, there is $j_q(\bar{x} - x^\star) \in J_q(\bar{x} - x^\star)$, such that

$$0 = \langle 0 - 0, j_q(\bar{x} - x^\star)\rangle = \langle (I - T)\bar{x} - (I - T)x^\star, j_q(\bar{x} - x^\star)\rangle \geq k\|\bar{x} - x^\star\|^q.$$

This implies that $\bar{x} = x^\star$ and the proof is complete. ∎

We can deduce similar conclusion of Theorem 2.2 for a Lipschitzian strongly accrective mapping as follows whose proof will omitted.

Theorem 2.3. *Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$ and $T : K \to K$ be a Lipschitzian strongly accrective mapping with Lipschitzian constant L and strongly accrective constant $k \in (0, 1)$. Assume that the set $F(T)$ of fixed points of T is nonempty. Let $\{u_n\}_{n=1}^\infty$, $\{v_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ be three sequences in K and $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ be real sequences in $[0, 1]$. Also suppose $\sum_{n=1}^\infty \alpha_n < \infty$, $\sum_{n=1}^\infty \|u_n\| < \infty$, $\sum_{n=1}^\infty \|v_n\| < \infty$ and $\sum_{n=1}^\infty \|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (1.8) with errors. Then the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ if and only if the sequence $\{x_n\}$ converges strongly to an unique fixed point of T.*

We note that if we take $u_n = 0,\ v_n = 0$ and $w_n = 0\ \forall n \geq 1$ in Theorem 2.1 and 2.2, then we can obtain the corresponding new results on the strong convergence of the three-step iterative process:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n,\ n \geq 1. \end{cases} \tag{2.9}$$

Corollary 2.2. *Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E and $T : K \to K$ be a Lipschitzian mapping with Lipschitzian constant L and the set $F(T)$ of fixed points of T is nonempty. Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ be real sequences in $[0, 1]$. Also suppose, $\sum_{n=1}^\infty \alpha_n < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process (2.9). Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T. In addition, if T is also a strongly pseudocontractive mapping, then the sequence $\{x_n\}$ converges strongly to an unique fixed point of T.*

Proof. It follows from Lemma 2.1 that

$$\|x_{n+1}-x^\star\|^q \leq (1+\delta_n)\|x_n-x^\star\|^q \leq e^{\sum_{j=1}^n \delta_j}\|x_1-x^\star\|^q \leq e^{\sum_{j=1}^\infty \delta_j}\|x_1-x^\star\|^q < \infty.$$

This shows that $\{x_n\}$ is bounded. The conclusion of the corollary follows from Theorem 2.1 and Theorem 2.2. ∎

From Theorem 2.3 we have a sufficient and necessary condition for three-step iterative approximation of solutions to equation $Tx = f$ in K as follows.

Theorem 2.4. *Let E be a real $q-$uniformly smooth Banach space with $q > 1$ and K be a nonempty convex subset of E with $K + K \subseteq K$ and $T : K \to K$ be a strongly pseudocontractive mapping such that $I - T : K \to K$ is Lipchitzian with Lipschitzian constant L and strongly accrective constant $k \in (0, 1)$ with respect to T. For any given $f \in K$, define $S : K \to K$ by*

$$Sx = f - Tx + x, \ \forall x \in K.$$

Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be three sequences in K and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be real sequences in $[0, 1]$. Also suppose $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$ and $\sum_{n=1}^{\infty} \|w_n\| < \infty$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_1 \in K$ by the three-step iterative process with errors:

$$\begin{cases} z_n = (1 - \gamma_n)x_n + \gamma_n S x_n + w_n, \\ y_n = (1 - \beta_n)x_n + \beta_n S z_n + v_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + u_n, \ n \geq 1. \end{cases} \tag{2.10}$$

If the set $F(S)$ of fixed points of S is nonempty, then the sequence $\{x_n\}$ is bounded and

$$\liminf_{n\to\infty} d(x_n, F(S)) = 0$$

if and only if the sequence $\{x_n\}$ converges strongly to an unique solution of the equation $Tx = f$ in K.

Proof. Since $T : K \to K$ is a strongly pseudocontractive mapping with strongly accrective constant $k \in (0, 1)$ such that $I - T : K \to K$ is Lipchitzian with Lipschitzian constant L, S is a Lipchitzian strongly accrective mapping with constant $k \in (0, 1)$ and with Lipschitzian constant L. From Theorem 2.3, the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(S)) = 0$ if and only if the sequence $\{x_n\}$ converges strongly to an unique fixed point, say $\hat{x}$, of S. For this fixed point $\hat{x}$ of S, we have $\hat{x} = S\hat{x} = f - T\hat{x} + \hat{x}$, that is, $f = T\hat{x}$. Hence the sequence $\{x_n\}$ is bounded and $\liminf_{n\to\infty} d(x_n, F(S)) = 0$ if and only if the sequence $\{x_n\}$ converges strongly to an unique solution of the equation $Tx = f$ in K. ∎

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