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THE $(S)_+$-CONDITION FOR VECTOR EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we generalize the $(S)_+$-condition to bifunctions with values in an oredered Hausdorff topological vector space $\mathcal{Z}$, and define a weak $(S)_+$-condition for the bifunctions. These conditions extend naturally to operators from nonempty subsets of a topological vector space X into the set $\mathcal{L}(X, \mathcal{Z})$ of all continuous linear mappings from X into $\mathcal{Z}$. Then we derive some existence results for vector equilibrium problems and vector variational inequalities associated with bifunctions or operators satisfying the weak $(S)_+$-condition.

1. Introduction

Vector equilibrium problems are formulated by considering the associated bifunctions into a topological vector space with a preorder defined by a closed convex cone which has a nonempty interior. All topological vector spaces will be assumed to be real spaces. In this paper, we deal with vector equilibrium problems associated with bifunctions into a Hausdorff topological vector space $\mathcal{Z}$, and fix once for all a closed convex cone $C \subset \mathcal{Z}$ such that $C \neq \mathcal{Z}$ and int$C \neq \emptyset$, where int$C$ is the interior of C in $\mathcal{Z}$.

As well known, the vector equilibrium problem includes the vector variational inequality as a special case. Therefore, we also consider vector variational inequalities associated with operators into the set $\mathcal{L}(X, \mathcal{Z})$ of all continuous linear mappings from a topological vector space X into $\mathcal{Z}$. For $\ell \in \mathcal{L}(X, \mathcal{Z})$ and $x \in X$, we write the value $\ell(x)$ as $\langle \ell, x \rangle$. When $\mathcal{Z} = \mathbb{R}$, $\mathcal{L}(X, \mathcal{Z})$ is the topological dual space X^* of X.

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Let K be a nonempty subset of a topological vector space X. The vector equilibrium problem associated with the bifunction $f : K \times K \longrightarrow \mathcal{Z}$, VEP$(f,\, K)$ for short, is the problem of finding an $\widehat{x} \in K$ such that

$$f(\widehat{x},\, u) \in (-\text{int}C)^c \quad \text{for all } u \in K,$$

where $(-\text{int}C)^c$ is the complement of $-\text{int}C$ in $\mathcal{Z}$. Such an $\widehat{x}$ is called a solution of VEP$(f,\, K)$.

If $T : K \longrightarrow \mathcal{L}(X,\, \mathcal{Z})$ is an operator, then, by considering the bifunction

$$f(x,\, u) = \langle T(x),\, u - x\rangle \quad \text{for } x,\, u \in K,$$

VEP$(f,\, K)$ becomes the vector variational inequality VVI$(T,\, K)$ associated with T. An $\widehat{x} \in K$ is called a solution of VVI$(T,\, K)$ if

$$\langle T(\widehat{x}),\, u - \widehat{x}\rangle \in (-\text{int}C)^c \quad \text{for all } u \in K.$$

When $\mathcal{Z} = \mathbb{R}$ and $C = \{r \in \mathbb{R} : r \geq 0\}$, VEP$(f,\, K)$ becomes the scalar equilibrium problem EP$(f,\, K)$ and VVI$(T,\, K)$ becomes the variational inequality problem VI$(T,\, K)$.

The main work of this paper is to derive existence results for the above problems associated with bifunctions or operators satisfying the $(S)_+$-condition.

The $(S)_+$-condition for an operator T from a subset K of a Banach space B into B^* was introduced by Browder [4], and stated as : for any sequence $\{x_n\}_{n=1}^{\infty}$ in K,

$$x_n \xrightarrow{w} x \in K \ \text{ and } \ \limsup_{n\to\infty} \langle T(x_n),\, x_n - x\rangle \leq 0 \Longrightarrow x_n \longrightarrow x\,,$$

where $x_n \xrightarrow{w} x$ indicates that $\{x_n\}_{n=1}^{\infty}$ weakly converges to x. The $(S)_+$-condition for an operator into $\mathcal{L}(X,\, \mathcal{Z})$ was introduced by Chiang and Yao [9].

Very few existence results for variational inequalities associated with operators satisfying the $(S)_+$-condition were established. One of them is due to Guo and Yao [12, Theorem 2.1]. By a simple argument [12, Theorem 2.1] can be stated as : *Let K be a nonempty weakly compact convex subset of a reflexive Banach space B, and let $T : K \longrightarrow B^*$ be an operator. If T satisfies the $(S)_+$-condition and is demicontinuous, then VI(T, K) has a solution.* See [9] for a discussion. An operator T from a nonempty subset K of a normed space X into X^* is *demicontinuous* if it is continuous from the norm topology of K into the weak* topology of X^* [13, p. 173].

The $(S)_+$-condition for real valued bifunctions was first considered by Chadli, Wong and Yao [6]. They considered real valued bifunctions defined on subsets of normed spaces. For any nonempty subset K of a normed space X, a bifunction

$f : K \times K \longrightarrow \mathbb{R}$ is said to satisfy the $(S)_+$-condition if for any sequence $\{x_n\}_{n=1}^{\infty}$ in K,

$$x_n \xrightarrow{w} x \in K \text{ and } \liminf_{n\to\infty} f(x_n, x) \geq 0 \Longrightarrow x_n \longrightarrow x\,.$$

With the above definition, Chadli, Wong and Yao proved an existence result [6, Theorem 2.1] for the problem EP(f, K) with f satisfying the $(S)_+$-condition.

For the vector equilibrium problems associated with bifunctions satisfying the $(S)_+$-condition, Fang and Huang established an existence result [111, Theorem 3.1]fah for bifunctions from nonempty subsets of a reflexive Banach space into a Banach space.

In Section 2, by using the vectorial limit inferiors defined in [5], the $(S)_+$-condition for bifunctions with values in $\mathcal{Z}$ is formulated analogously to that given in [6]. After a minor modification, we also define a *weak* $(S)_+$-condition for bifunctions into $\mathcal{Z}$ so that a bifunction will satisfy the weak $(S)_+$-condition if it satisfies the $(S)_+$-condition. These conditions extend naturally to operators from subsets of a topological vector space X into $\mathcal{L}(X, \mathcal{Z})$, and the $(S)_+$-condition for such operators coincides with that given in [9].

In Section 3, we derive some existence results for vector equilibrium problems associated with bifunctions satisfying the weak $(S)_+$-condition. One of our results generalizes [6, Theorem 2.1] in some sense; see Corollary 3.4 and Remark 3.6. Corollary 3.4 also generalizes Fang and Huang's result [11, Theorem 3.1].

By using Corollary 3.4 and taking account of upper semicontinuous operators introduced in [7], in Section 4, we establish some existence results for vector variational inequalities associated with upper semicontinuous operators satisfying the weak $(S)_+$-condition. One of our results can be regarded as a vector version of [12, Theorem 2.1]; see Corollary 4.7.

Our existence results are established by using Fan-KKM Theorem [10]. To employ the theorem, we need some basic definitions and notations. For any given nonempty set X, let 2^X denote the family of all subsets of X, and let $\mathcal{F}(X)$ denote the family of all nonempty finite subsets of X. When X is a topological vector space, we denote co(E) by the convex hull of $E \subset X$, and E^c by the complement of E in X.

For given nonempty sets X and Y, a mapping $\Phi : X \longrightarrow 2^Y$ will be also called a multivalued mapping from X into Y. The image of Φ is defined by $\Phi(X) = \bigcup_{x\in X} \Phi(x)$. When Y is a topological space, Φ is said to have *closed values* if $\Phi(x)$ is closed in Y for every $x \in X$. For any given nonempty convex subset K of a topological vector space X, a multivalued mapping $\Phi : K \longrightarrow 2^X$ is called a KKM mapping if

$$\text{co}(E) \subset \bigcup_{x\in E} \Phi(x) \quad \text{ for every } E \in \mathcal{F}(K).$$

Now, Fan-KKM Theorem is stated as follows. Let Φ be a multivalued mapping from a nonempty convex subset K of a Hausdorff topological vector space X into X. Assume that Φ is a KKM mapping, and that Φ has closed values. If there is a nonempty compact and convex subset D of K such that $\bigcap_{x\in D} \Phi(x)$ is compact in X, then $\bigcap_{x\in K} \Phi(x) \neq \emptyset$.

2. The $(S)_+$-Condition

The $(S)_+$-condition for bifunctions into $\mathcal{Z}$ is formulated analogously to that given in [6]. To state the $(S)_+$-condition, we need the $\mathcal{L}$-topology on a topological vector space defined in [9] which generalizes the weak topology. Also, we need limit inferiors of nets in $\mathcal{Z}$ which was introduced in [5] for defining vector topological pseudomonotonicity.

The $\mathcal{L}$-topology on a topological vector space X is the topology having the sets $\ell^{-1}(U)$ as subbasis, where U is open in $\mathcal{Z}$ and $\ell \in \mathcal{L}(X, \mathcal{Z})$. When $\mathcal{Z} = \mathbb{R}$, the $\mathcal{L}$-topology on X coincides with the weak topology. Let $X_{\mathcal{L}}$ denote the space X equipped with the $\mathcal{L}$-topology. Note that $X_{\mathcal{L}}$ is a topological vector space, and that if X is Hausdorff and locally convex then $X_{\mathcal{L}}$ is Hausdorff [9, Theorem 3.1].

For any subset E of X, the closure of E in $X_{\mathcal{L}}$ will be called the $\mathcal{L}$-closure of E. The set E will be called $\mathcal{L}$-closed (respectively, $\mathcal{L}$-open) if E is closed (respectively, open) in $X_{\mathcal{L}}$. Similarly, E is called $\mathcal{L}$-compact if it is compact in $X_{\mathcal{L}}$. When $\mathcal{Z} = \mathbb{R}$, the notion of $\mathcal{L}$-compactness reduces to the notion of weak compactness.

It is clear that every $\mathcal{L}$-open (respectively, $\mathcal{L}$-closed) subset of X is originally open (respectively, closed) in X. Similarly, compact subsets of X are $\mathcal{L}$-compact.

For any given net $\{x_\alpha\}$ in X, we shall write $x_\alpha \longrightarrow x \in X$ when $\{x_\alpha\}$ converges to x in the original topology on X. The net $\{x_\alpha\}$ will be called *$\mathcal{L}$-convergent* to x, written by $x_\alpha \xrightarrow{\mathcal{L}} x$, if $\{x_\alpha\}$ converges to x in $X_{\mathcal{L}}$, i.e., $\langle \ell, x_\alpha\rangle \longrightarrow \langle \ell, x\rangle$ in $\mathcal{Z}$ for every $\ell \in \mathcal{L}(X, \mathcal{Z})$. The notion of $\mathcal{L}$-convergence coincides with the notion of weak convergence when $\mathcal{Z} = \mathbb{R}$.

As in the scalar case, limit inferiors of nets in $\mathcal{Z}$ are defined by using vector superiors and inferiors introduced in [3]. For a subset E of $\mathcal{Z}$, let $\overline{E}$ denote the closure of E in $\mathcal{Z}$. The superior of E with respect to C is defined by

$$\operatorname{Sup}(E, C) = \{z \in \overline{E} : (z + \operatorname{int} C) \cap E = \emptyset\},$$

and the inferior of E with respect to C is defined by

$$\operatorname{Inf}(E, C) = \{z \in \overline{E} : (z - \operatorname{int} C) \cap E = \emptyset\}.$$

As C is fixed, we shall simply write Sup$\,(E,\,C) = $ Sup$\,E$ and Inf$\,(E,\,C) = $ Inf$\,E$. A standard argument shows that Inf$\,(-E) = -$Sup$\,E$. See [8] for more discussion on vector superiors and inferiors.

For a given net $\{z_\alpha\}_{\alpha\in\mathcal{I}}$ in $\mathcal{Z}$, let $A_\alpha = \{z_\beta \,:\, \beta \succeq \alpha\}$ for every $\alpha \in \mathcal{I}$. The limit inferior of $\{z_\alpha\}_{\alpha\in\mathcal{I}}$ is defined by

$$\text{Liminf}\, z_\alpha = \text{Sup}\,(\bigcup_{\alpha\in\mathcal{I}} \text{Inf}\, A_\alpha)\,.$$

Now, for any given nonempty subset K of a topological vector space X, a bifunction $f \,:\, K \times K \longrightarrow \mathcal{Z}$ is said to satisfy the $(S)_+$-condition if for any net $\{x_\alpha\}$ in K,

$$x_\alpha \xrightarrow{\mathcal{L}} x \in K \quad \text{and} \quad \text{Liminf}\, f(x_\alpha,\, x) \subset (-\text{int}C)^c \Longrightarrow x_\alpha \longrightarrow x\,.$$

Note that an operator $T \,:\, K \longrightarrow \mathcal{L}(X,\,\mathcal{Z})$ satisfies the $(S)_+$-condition given in [9] if and only if the bifunction function $(x,\,u) \longrightarrow \langle T(x),\, u - x\rangle$ satisfies the $(S)_+$-condition.

Our existence results will be established by replacing the above $(S)_+$-condition by a weak one given below which we shall call the *weak* $(S)_+$-condition. A bifunction f as given above is said to satisfy the weak $(S)_+$-condition if any net $\{x_\alpha\}$ in K with

$$x_\alpha \xrightarrow{\mathcal{L}} x \in K \quad \text{and} \quad \text{Liminf}\, f(x_\alpha,\, x) \subset (-\text{int}C)^c\,,$$

has a subnet $\{x_\lambda\}$ such that $x_\lambda \longrightarrow x$.

Similarly, an operator $T \,:\, K \longrightarrow \mathcal{L}(X,\,\mathcal{Z})$ is said to satisfy the weak $(S)_+$-condition if the bifunction $(x,\,u) \longrightarrow \langle T(x),\, u - x\rangle$ satisfies the weak $(S)_+$-condition. Clearly, a bifunction or an operator satisfies the weak $(S)_+$-condition if it satisfies the $(S)_+$-condition.

2. Existence Results for Vector Equilibrium Problems

This section is devoted to deriving some existence results for vector equilibrium problems associated with bifunctions satisfying the weak $(S)_+$-condition. To state our existence results, we need some basic definitions.

A function f from a topological space Ω into $\mathcal{Z}$ is called *C-upper semicontinuous* [17] if $f^{-1}(z - \text{int}C)$ is open in Ω for every $z \in \mathcal{Z}$.

The (L)-condition. Let K be a nonempty convex subset of a topological vector space. A bifunction $f : K \times K \longrightarrow \mathcal{Z}$ is said to satisfy the (L)-*condition* if it has the following property : For any $x,\, u \in K$ and any net $\{x_\alpha\}$ in K, if

$$x_\alpha \longrightarrow x \;\; \text{and} \;\; f(x_\alpha,\, tu + (1-t)x) \in (-\text{int}C)^c \;\; \text{for all } \alpha \text{ and } 0 \le t \le 1,$$

then $f(x,\, u) \in (-\mathrm{int}C)^c$.

Remark 3.1. Let K be a nonempty convex subset of a Hausdorff topological vector space. Then a bifunction $f : K \times K \longrightarrow \mathcal{Z}$ satisfies the (L)-condition if it is vector topologically pseudomonotone [5, Theorem 2.7]. Vector topological pseudomonotonicity will not be used in the sequel; see [5] for the definition. It follows from [5, Corollary 2.6] that if for every fixed $u \in K$ the function $f_u : K \longrightarrow \mathcal{Z}$ defined by $f_u(x) = f(x,\, u)$ for $x \in K$, is C-upper semicontinuous on K, then f is vector topologically pseudomonotone.

Theorem 3.2. *Let K be a nonempty convex subset of a Hausdorff topological vector space, and let $f : K \times K \longrightarrow \mathcal{Z}$ be a bifunction. For every $E \in \mathcal{F}(K)$, let $\Phi_E : co(E) \longrightarrow 2^{co(E)}$ be the mapping defined by*

$$\Phi_E(u) = \{x \in co(E) \,:\, f(x,\, u) \in (-\mathrm{int}C)^c\}\,.$$

Assume that the following conditions are satisfied.

(i) *For every $E \in \mathcal{F}(K)$, Φ_E is a KKM mapping and has closed values.*

(ii) *f satisfies both the weak $(S)_+$-condition and (L)-condition.*

(iii) *(Coercivity) There exist nonempty $\mathcal{L}$-compact subsets A and B of K with B convex such that if $x \in K \cap A^c$ then $f(x,\, u_x) \in (-\mathrm{int}C)$ for some $u_x \in B$.*

Then VEP$(f,\, K)$ has a solution.

For the proof of Theorem 3.2, we first prove :

Theorem 3.3. *Let K be a nonempty $\mathcal{L}$-compact and convex subset of a Hausdorff topological vector space, and let $f : K \times K \longrightarrow \mathcal{Z}$ be a bifunction. If the conditions (i) and (ii) in Theorem 3.2 are satisfied, then* VEP$(f,\, K)$ *has a solution.*

Proof. The condition (i) together with Fan-KKM Theorem assert that for every $E \in \mathcal{F}(K)$,

$$S_E = \{x \in K : f(x,\, u) \in (-\mathrm{int}C)^c \quad \text{for all } u \in \mathrm{co}(E)\} \neq \emptyset\,.$$

For every $E \in \mathcal{F}(K)$, let $\overline{S_E}^{\mathcal{L}}$ denote the $\mathcal{L}$-closure of S_E in K. Since for E, $E' \in \mathcal{F}(K)$,

$$S_{E \cup E'} \subset S_E \cap S_{E'} \subset \overline{S_E}^{\mathcal{L}} \cap \overline{S_{E'}}^{\mathcal{L}}\,,$$

the family $\{\overline{S_E}^{\mathcal{L}} \,:\, E \in \mathcal{F}(K)\}$ has the finite intersection property. The $\mathcal{L}$-compactness of K implies that

$$S = \bigcap_{E \in \mathcal{F}(K)} \overline{S_E}^{\mathcal{L}} \neq \emptyset\,.$$

We claim that any point $\widehat{x} \in S$ is a solution of VEP$(f,\, K)$.

For every $u \in K$, consider the set $U = \{\widehat{x},\, u\} \in \mathcal{F}(K)$. Since $\widehat{x} \in \overline{S_U}^{\mathcal{L}}$, there is a net $\{x_\alpha\}$ in S_U such that $x_\alpha \xrightarrow{\mathcal{L}} \widehat{x}$. By definition,

$$f(x_\alpha,\, tu + (1-t)\widehat{x}) \in (-\text{int}C)^c \quad \text{for all } \alpha \text{ and } 0 \le t \le 1.$$

Letting $t = 0$, we have $f(x_\alpha,\, \widehat{x}) \in (-\text{int}C)^c$ for all α and $\operatorname{Liminf} f(x_\alpha,\, \widehat{x}) \subset (-\text{int}C)^c$. The $(S)_+$-condition implies that there is subnet $\{x_\lambda\}$ of $\{x_\alpha\}$ such that $x_\lambda \longrightarrow \widehat{x}$. Since

$$f(x_\lambda,\, tu + (1-t)\widehat{x}) \in (-\text{int}C)^c \quad \text{for all } \lambda \text{ and } 0 \le t \le 1,$$

the (L)-condition implies $f(\widehat{x},\, u) \in (-\text{int}C)^c$. ■

Proof of Theorem 3.2. Since co$(E \cup B)$ is $\mathcal{L}$-compact for every $E \in \mathcal{F}(K)$ [1, Lemma 5.14, p. 171], by Theorem 3.3 there is an $x_E \in$ co$(E \cup B)$ such that

$$f(x_E,\, u) \in (-\text{int}C)^c \ \text{ for all } u \in \text{co}(E \cup B).$$

The condition (iii) implies $x_E \in A$. Thus, for every $E \in \mathcal{F}(K)$,

$$S_E = \{x \in A : f(x,\, u) \in (-\text{int}C)^c \ \text{ for all } u \in \text{co}(E \cup B)\} \neq \emptyset\,.$$

Let $\overline{S}_E^{\mathcal{L}}$ be the $\mathcal{L}$-closure of S_E in A. By the $\mathcal{L}$-compactness of A, we have

$$\bigcap_{E \in \mathcal{F}(K)} \overline{S}_E^{\mathcal{L}} \neq \emptyset\,.$$

Now, by a similar argument as above, the condition (ii) will complete the proof. ■

As applications of Theorem 3.2, we consider *vector* 0-*diagonally convex* and C-*quasiconvex-like* bifunctions. Let K be a nonempty convex subset of a topological vector space X. A bifunction $f : K \times K \longrightarrow \mathcal{Z}$ is called vector 0-diagonally convex [5] if for any finite set $\{u_1,\, \ldots,\, u_n\} \subset K$,

$$x = \sum_{j=1}^{n} t_j u_j \ \text{ with all } t_j \ge 0 \text{ and } \sum_{j=1}^{n} t_j = 1 \Longrightarrow \sum_{j=1}^{n} t_j f(x\,,\, u_j) \in (-\text{int}C)^c\,.$$

While f is called C-quasiconvex-like [2] if for any $x,\, u_1,\, u_2 \in K$ and $0 \le t \le 1$

$$f(x,\, tu_1 + (1-t)u_2) \in f(x,\, u_1) - C \quad \text{or} \quad f(x,\, tu_1 + (1-t)u_2) \in f(x,\, u_2) - C\,.$$

For a bifunction $f : K \times K \longrightarrow \mathcal{Z}$ as given above, consider the multivalued mapping $\Phi : K \longrightarrow 2^X$ defined by

$$\Phi(u) = \{x \in K \;:\; f(x,\, u) \in (-\text{int}C)^c\} \quad \text{for } u \in K.$$

From the proofs of [5, Lemmas 3.6 and 3.9], we conclude that if either

(a) f is vector 0-diagonally convex, or

(b) f is C-quasiconvex-like with $f(x, x) \in (-\text{int}C)^c$ for all $x \in K$,

then Φ is a KKM mapping. Moreover,

$$\Phi(u) = f_u^{-1}((-\text{int}C)^c) \quad \text{for every } u \in K,$$

where $f_u : K \longrightarrow \mathcal{Z}$ is the function given in Remark 3.1. Then Φ has closed values if K is closed, and if every f_u is C-upper semicontinuous on K.

Now, from Remark 3.1 and Theorem 3.2, we obtain the following corollaries.

Corollary 3.4. *Let $f : K \times K \longrightarrow \mathcal{Z}$ be a bifunction, where K is a nonempty convex subset of a Hausdorff topological vector space. Then* VEP(f, K) *has a solution if the following conditions are satisfied.*

(i) f is vector 0-diagonally convex.

(ii) For every $u \in K$, the function $x \longmapsto f(x, u)$ is C-upper semicontinuous on co(E) for every $E \in \mathcal{F}(K)$.

(iii) f satisfies both the weak $(S)_+$-condition and (L)-condition.

(iv) (Coercivity) There exist nonempty $\mathcal{L}$-compact subsets A and B of K with B convex such that if $x \in K \cap A^c$ then $f(x, u_x) \in (-\text{int}C)$ for some $u_x \in B$.

Corollary 3.5. *Let $f : K \times K \longrightarrow \mathcal{Z}$ be a bifunction, where K is a nonempty convex subset of a Hausdorff topological vector space. If f is C-quasiconvex-like with $f(x, x) \in (-\text{int}C)^c$ for all $x \in K$, and satisfies the conditions (ii), (iii) and (iv) in Corollary 3.4, then* VEP(f, K) *has a solution.*

Remark 3.6. Let $f : K \times K \longrightarrow \mathcal{Z}$ be given as above. Since f satisfies the (L)-condition if for every $u \in K$ the function $x \longmapsto f(x, u)$ is C-upper semicontinuous on K, Corollary 3.4 generalizes [6, Theorem 2.1] in the sense that the function h given there is identically zero.

4. Existence Results for Vector Variational Inequalities

In this section, by considering upper semicontinuous operators introduced in [7], we shall use Corollary 3.4 to derive some existence results for vector variational inequalities associated with upper semicontinuous operators satisfying the weak $(S)_+$-condition. To proceed, we need the topology of bounded convergence and the topology of simple convergence on $\mathcal{L}(X, \mathcal{Z})$, where X is a topological vector space. See [16, p. 79-87] for a full discussion on these topologies. To describe these topologies, we denote by $\mathcal{B}_X$ the family of all nonempty bounded subsets of X, and $\mathcal{N}_{\mathcal{Z}}$ the family of 0-neighborhoods in $\mathcal{Z}$.

For a given family $\mathcal{E}$ of nonempty subsets of X, if $E \in \mathcal{E}$ and $V \in \mathcal{N}_{\mathcal{Z}}$, we write

$$[E, V]_{\mathcal{E}} = \{f \in \mathcal{L}(X, \mathcal{Z}) : f(E) \subset V\}.$$

When there is no risk of confusion, we shall simply write $[E, V]_{\mathcal{E}} = [E, V]$.

If $\mathcal{E} = \mathcal{F}(X)$ or $\mathcal{E} = \mathcal{B}_X$, then the family

$$\{[E, V]_{\mathcal{E}} : E \in \mathcal{E} \text{ and } V \in \mathcal{N}_{\mathcal{Z}}\}$$

is the 0-neighborhood base in $\mathcal{L}(X, \mathcal{Z})$ for a unique translation-invariant topology $\mathcal{T}_{\mathcal{E}}$; see [16, p. 79]sch. If $\mathcal{E} = \mathcal{F}(X)$, then $\mathcal{T}_{\mathcal{E}}$ is the topology of simple convergence (or the topology of pointwise convergence). Let $\mathcal{L}_s(X, \mathcal{Z})$ denote the space $\mathcal{L}(X, \mathcal{Z})$ equipped with the topology of simple convergence. If $\mathcal{E} = \mathcal{B}_X$, then $\mathcal{T}_{\mathcal{E}}$ is the topology of bounded convergence. Let $\mathcal{L}_b(X, \mathcal{Z})$ denote the space $\mathcal{L}(X, \mathcal{Z})$ equipped with the topology of bounded convergence. When $\mathcal{Z} = \mathbb{R}$, $\mathcal{L}_s(X, \mathcal{Z})$ coincides with the weak* topology on X^*, and $\mathcal{L}_b(X, \mathcal{Z})$ coincides with the strong topology on X^*.

Note that $\mathcal{L}_s(X, \mathcal{Z})$ and $\mathcal{L}_b(X, \mathcal{Z})$ are Hausdorff topological vector spaces since $\mathcal{Z}$ is Hausdorff [16, pp. 79-80]. Also, note that if X and $\mathcal{Z}$ are normed spaces, the norm

$$\ell \longmapsto \|\ell\| = \sup\{|\ell(x)| : |x| \leq 1\}$$

generates the topology of bounded convergence on $\mathcal{L}(X, \mathcal{Z})$, i.e., $\mathcal{L}_b(X, \mathcal{Z})$ is also a normed space. For a full discussion on the topologies of bounded convergence and simple convergence, see, e.g., [14, 16].

As above, we denote $\mathcal{E}$ by $\mathcal{B}_X$ or $\mathcal{F}(X)$. For any given topological space Y, an operator $T : Y \longrightarrow \mathcal{L}(X, \mathcal{Z})$ is called *$C^*_{\mathcal{E}}$-upper semicontinuous* at $y_0 \in Y$ if for any $(E, v) \in \mathcal{E} \times \text{int}C$, there is a neighborhood U of y_0 such that

$$T(y) \in T(y_0) + [\,E, v - \text{int}C]_{\mathcal{E}} \quad \text{for all } y \in U.$$

While T is called $C^*_{\mathcal{E}}$-upper semicontinuous if it is $C^*_{\mathcal{E}}$-upper semicontinuous at every point of Y. We shall write $C^*_{\mathcal{E}} = C^*_{\mathcal{L}(b)}$ when $\mathcal{E} = \mathcal{B}_X$, and write $C^*_{\mathcal{E}} = C^*_{\mathcal{L}(s)}$ when $\mathcal{E} = \mathcal{F}(X)$. Note that T is $C^*_{\mathcal{L}(s)}$-upper semicontinuous at y_0 if and only if for any $(x, v) \in X \times \text{int}C$ there is a neighborhood U of y_0 such that $T(y) \in T(y_0) + [\,\{x\}, v - \text{int}C\,]$ for all $y \in U$.

The following assertions are immediate consequences of the definition.

(a) If T is $C^*_{\mathcal{L}(b)}$-upper semicontinuous, then it is $C^*_{\mathcal{L}(s)}$-upper semicontinuous.

(b) If $T : Y \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ is continuous, then it is $C^*_{\mathcal{L}(b)}$-upper semicontinuous.

(c) If $T : Y \longrightarrow \mathcal{L}_s(X, \mathcal{Z})$ is continuous, then it is $C^*_{\mathcal{L}(s)}$-upper semicontinuous.

A nonempty subset K of a topological vector space will be called *locally bounded* if every $x \in K$ has a neighborhood U such that $U \cap K$ is bounded, i.e., x has a bounded neighborhood in K. Clearly, nonempty bounded subsets of a topological vector space are locally bounded, and nonempty subsets of a locally bounded topological vector space are locally bounded.

Theorem 4.1. *Let $T : K \longrightarrow \mathcal{L}(X, \mathcal{Z})$ be an operator, where K is a nonempty subset of a topological vector space X. For every $u \in K$, let $f_u : K \longrightarrow \mathcal{Z}$ be the function given by*

$$f_u(x) = \langle T(x), u - x\rangle \quad \textit{for } x \in K.$$

If K is locally bounded, and if T is $\mathcal{C}^*_{\mathcal{L}(b)}$-upper semicontinuous, then f_u is C-upper semicontinuous on K for every $u \in K$.

Proof. Let $x_0 \in K$ and $v \in \text{int}C$ be arbitrary. We have to show that there is a neighborhood U of x_0 in K such that $f_u(x) \in f_u(x_0) + v - \text{int}C$ for all $x \in U$. Note that

$$f_u(x) - f_u(x_0) = \langle T(x) - T(x_0), u - x\rangle + \langle T(x_0), x_0 - x\rangle .$$

Let U_0 be a bounded neighborhood of x_0 in K. By assumption, there is a neighborhood U_1 of x_0 in K with $U_1 \subset U_0$ such that

$$\begin{aligned} x \in U_1 &\Longrightarrow T(x) - T(x_0) \in [u - U_0, \frac{v}{2} - \text{int}C] \\ &\Longrightarrow \langle T(x) - T(x_0), u - x\rangle \in \frac{v}{2} - \text{int}C . \end{aligned}$$

By the continuity of the function $x \longmapsto \langle T(x_0), x_0 - x\rangle$, there is a neighborhood U of x_0 in K with $U \subset U_1$ such that

$$x \in U \Longrightarrow \langle T(x_0), x_0 - x\rangle \in \frac{v}{2} - \text{int}C .$$

Now, for $x \in U$ we have $f_u(x) - f_u(x_0) \in v - \text{int}C$. ■

Theorem 4.2. *Let $T : K \longrightarrow \mathcal{L}(X, \mathcal{Z})$ be an operator, where K is a nonempty subset of a topological vector space X. For every $u \in K$, let $f_u : K \longrightarrow \mathcal{Z}$ be the function given in Theorem 4.1. If X is locally bounded, and if T is $\mathcal{C}^*_{\mathcal{L}(s)}$-upper semicontinuous with $T(K)$ bounded in $\mathcal{L}_b(X, \mathcal{Z})$, then f_u is C-upper semicontinuous on K for every $u \in K$.*

Proof. Let $x_0 \in K$ and $v \in \text{int}C$ be arbitrary. Note that

$$f_u(x) - f_u(x_0) = \langle T(x) - T(x_0),\, u - x_0\rangle + \langle T(x),\, x_0 - x\rangle\,.$$

Since every 0-neighborhood contains a balanced 0-neighborhood [15, Theorem 1.14, p.11], there is a balanced and bounded 0-neighborhood $\mathbb{B}$ in X such that

$$\begin{aligned} x \in U_0 = (x_0 + \mathbb{B}) \cap K &\Longrightarrow T(x) - T(x_0) \in [\{u - x_0\},\, \frac{v}{2} - \text{int}C] \\ &\Longrightarrow \langle T(x) - T(x_0),\, u - x_0\rangle \in \frac{v}{2} - \text{int}C\,. \end{aligned}$$

Note that $[\mathbb{B},\, \frac{v}{2} - \text{int}C]$ is a 0-neighborhood in $\mathcal{L}_b(X,\, \mathcal{Z})$. There is a positive number $\lambda \leq 1$ such that

$$\lambda T(K) \subset [\mathbb{B},\, \frac{v}{2} - \text{int}C]\,.$$

Consequently, $\ell(\lambda\mathbb{B}) \subset \frac{v}{2} - \text{int}C$ for all $\ell \in T(K)$. Note that $U = (x_0 + \lambda\mathbb{B}) \cap K$ is a neighborhood of x_0 with $U \subset U_0$. Now, if $x \in U$, then $T(x)$ maps $\lambda\mathbb{B}$ into $\frac{v}{2} - \text{int}C$ and $\langle T(x),\, x_0 - x\rangle \in \frac{v}{2} - \text{int}C$. The proof is complete. ■

Theorem 4.3. *Let K be a nonempty $\mathcal{L}$-compact and convex subset of a Hausdorff topological vector space X, and let $T : K \longrightarrow \mathcal{L}(X,\, \mathcal{Z})$ be an operator satisfying the weak $(S)_+$-condition. If K is locally bounded and T is $\mathcal{C}^*_{\mathcal{L}(b)}$-upper semicontinuous, then VVI$(T,\, K)$ has a solution.*

Proof. Let $f : K \times K \longrightarrow \mathcal{Z}$ be the function defined by

$$f(x,\, u) = \langle T(x),\, u - x\rangle \quad \text{for } x,\, u \in K.$$

For every $u \in K$, let $f_u : K \longrightarrow \mathcal{Z}$ be the function given in Theorem 4.1. Note that f is vector 0-diagonally convex and satisfies the weak $(S)_+$-condition. By Theorem 4.1, every f_u is C-upper semicontinuous on K. Thus, f satisfies the (L)-condition; see Remark 3.1. Now, the theorem follows from Corollary 3.4. ■

By the same reasoning as above, the following theorem follows from Corollary 3.4 and Theorem 4.2.

Theorem 4.4. *Let K be a nonempty $\mathcal{L}$-compact and convex subset of a Hausdorff topological vector space X, and let $T : K \longrightarrow \mathcal{L}(X,\, \mathcal{Z})$ be an operator satisfying the weak $(S)_+$-condition. If X is locally bounded and T is $\mathcal{C}^*_{\mathcal{L}(s)}$-upper semicontinuous with $T(K)$ bounded in $\mathcal{L}_b(X,\, \mathcal{Z})$, then* VVI$(T,\, K)$ *has a solution.*

The following corollary is an immediate consequence of Theorem 4.3.

Corollary 4.5. *Let K be a nonempty $\mathcal{L}$-compact and convex subset of a Hausdorff topological vector space X, and let $T : K \longrightarrow \mathcal{L}(X, \mathcal{Z})$ be an operator satisfying the weak $(S)_+$-condition. If K is locally bounded and $T : K \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ is continuous, then* $\mathrm{VVI}(T, K)$ *has a solution.*

Note that if X is a normed space and $\mathcal{Z}$ is a Banach space, then a subset of X is bounded if and only if it is $\mathcal{L}$-bounded [7, Proposition 2.2]. Since $\mathcal{L}$-compactness implies $\mathcal{L}$-boundedness, from Corollary 4.5 we obtain :

Corollary 4.6. *Let K be a nonempty $\mathcal{L}$-compact and convex subset of a normed space X, and assume that $\mathcal{Z}$ is a Banach space. If $T : K \longrightarrow \mathcal{L}_b(X, \mathcal{Z})$ is a continuous operator satisfying the weak $(S)_+$-condition, then* $\mathrm{VVI}(T, K)$ *has a solution.*

The following corollary is a consequence of Theorem 4.4 and is regarded as a vector version of [12, Theorem 2.1].

Corollary 4.7. *Let K be a nonempty $\mathcal{L}$-compact and convex subset of a Hausdorff and locally bounded topological vector space X, and let $T : K \longrightarrow \mathcal{L}(X, \mathcal{Z})$ be an operator. If T satisfies the weak $(S)_+$-condition, and $T : K \longrightarrow \mathcal{L}_s(X, \mathcal{Z})$ is continuous with $T(K)$ bounded in $\mathcal{L}_b(X, \mathcal{Z})$, then VVI$(T, K)$ has a solution.*

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