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STRONG CONVERGENCE THEOREMS OF RELAXED HYBRID STEEPEST-DESCENT METHODS FOR VARIATIONAL INEQUALITIES

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Abstract. Assume that F is a nonlinear operator on a real Hilbert space H which is η-strongly monotone and κ-Lipschitzian on a nonempty closed convex subset C of H. Assume also that C is the intersection of the fixed point sets of a finite number of nonexpansive mappings on H. We develop a relaxed hybrid steepest-descent method which generates an iterative sequence $\{x_n\}$ from an arbitrary initial point $x_0 \in H$. The sequence $\{x_n\}$ is shown to converge in norm to the unique solution u^* of the variational inequality

$$\langle F(u^*), v - u^* \rangle \geq 0 \quad \forall v \in C$$

under the conditions which are more general than those in Ref. 19. Applications to constrained generalized pseudoinverse are included.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, and $F : H \to H$ be an operator. Stampacchia (cf. [1]) initially studied the classical variational inequality problem: find $u^* \in C$ such that

$$\mathrm{VI}(F, C)\ \langle F(u^*), v - u^*) \rangle \geq 0 \quad \forall\ v \in C.$$

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Variational inequalities have been extensively studied because they cover many diverse disciplines such as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance, etc as special cases. The reader is referred to [1-5] and the references therein.

It is well known that if F is strong monotone and Lipschitzian on C, then $\mathrm{VI}(F, C)$ has a unique solution. See, e.g., [1]. In the study of the $\mathrm{VI}(F, C)$, one of the most important problems is how to find a solution of $\mathrm{VI}(F, C)$ if there is any. A great deal of effort has gone into the problem of finding a solution of $\mathrm{VI}(F, C)$; see [2] and [6].

It is also known that the $\mathrm{VI}(F, C)$ is equivalent to the fixed-point equation

$$u^* = P_C(u^* - \mu F(u^*))$$

where P_C is the (nearest point) projection from H onto C; i.e., $P_C x = \mathrm{argmin}_{y \in C} \|x - y\|$ for each $x \in H$ and where $\mu > 0$ is an arbitrarily fixed constant. So if F is strongly monotone and Lipschitzian on C and $\mu > 0$ is small enough, then the mapping determined by the right-hand side of this equation is a contraction. Hence the Banach contraction principle guarantees that the Picard iterates converge in norm to the unique solution of the $\mathrm{VI}(F, C)$. Such a method is called the projection method. It has been widely extended to develop various algorithms for finding solutions of various classes of variational inequalities and complementarity problems; see, e.g., [10, 15, 16]. It is remarkable that the fixed-point equation involves the projection P_C which may not be easy to compute due to the complexity of the convex set C.

Recently Yamada ([7], see also [8]) introduced a hybrid steepest-descent method for solving the $\mathrm{VI}(F, C)$ so as to reduce the complexity probably caused by the projection P_C. His idea is stated as follows: Let C be the fixed point set of a nonexpansive mapping $T : H \to H$; that is, $C = \{x \in H : Tx = x\}$. Recall that T is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$, and let $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$ denote the fixed-point set of T. Let F be η-strongly monotone and κ-Lipschitzian on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and a sequence $\{\lambda_n\}$ of real numbers in $(0, 1)$ satisfying the conditions below:

$(\mathbf{L1})$ $\lim_{n\to\infty} \lambda_n = 0$,

$(\mathbf{L2})$ $\displaystyle\sum_{n=1}^{\infty} \lambda_n = \infty$,

$(\mathbf{L3})$ $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$.

Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following algorithm:

$$u_{n+1} := Tu_n - \lambda_{n+1}\mu F(Tu_n), \quad n \ge 0. \tag{1}$$

Then Yamada [7] proved that $\{u_n\}$ converges strongly to the unique solution of the $\mathrm{VI}(F, C)$. An example of the sequence $\{\lambda_n\}$ which satisfies conditions (L1)-(L3), is given by $\lambda_n = 1/n^\sigma$ where $0 < \sigma < 1$. We note that condition (L3) was first used by Lions [9] to establish the following result.

Theorem 1.1. (See [9]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$. For a sequence $\{\alpha_n\}$ in $[0, 1]$ and an arbitrary point $u \in C$, starting with another arbitrary initial $x_0 \in C$ define a sequence $\{x_n\}$ in C recursively by the formula:*

$$(2) \qquad x_{x+1} := \alpha_n u + (1 - \alpha_n) S x_n \quad n \geq 0.$$

If the sequence $\{\alpha_n\}$ of parameters satisfies conditions $(L1)$, $(L2)$ and $(L3)$, then $\{x_n\}$ converges strongly to an element of $\mathrm{Fix}(S)$.

On the other hand, if C is expressed as the intersection of the fixed-point sets of N nonexpansive mappings $T_i : H \to H$ with $N \geq 1$ an integer, Yamada (Ref. 7) proposed another algorithm,

$$(3) \qquad u_{n+1} := T_{[n+1]} u_n - \lambda_{n+1} \mu F(T_{[n+1]} u_n) \quad n \geq 0$$

where $T_{[k]} := T_{k \bmod N}$ for integer $k \geq 1$ with the mod function taking values in the set $\{1, 2, ..., N\}$; that is, if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_{[k]} = T_N$ if $q = 0$ and $T_{[k]} = T_q$ if $1 \leq q < N$ where $\mu \in (0, 2\eta/\kappa^2)$ and where the sequence $\{\lambda_n\}$ of parameters satisfies conditions (L1), (L2) and the following (L4):

$$(\mathbf{L4}) \qquad \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| \text{ is convergent.}$$

Under these conditions, Yamada [7] proved the strong convergence of $\{u_n\}$ to the unique solution of the $\mathrm{VI}(F, C)$. Note that condition (L4) was first used by Bauschke [11]. In the special case of $N = 1$, Wittmann [12] first introduced condition (L4) and applied (L4) to establish the following theorem.

Theorem 1.2. (See [12]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$. For a sequence $\{\alpha_n\}$ in $[0, 1]$ and an arbitrary point $u \in C$, starting with another arbitrary initial $x_0 \in C$ define a sequence $\{x_n\}$ in C recursively by the formula (2). If the sequence $\{\alpha_n\}$ of parameters satisfies conditions $(L1)$, $(L2)$ and $(L4)$ with $N = 1$, then $\{x_n\}$ converges strongly to an element of $\mathrm{Fix}(S)$.*

In 2003 Xu and Kim [19] further considered and studied the hybrid steepest-descent algorithms (1) and (3). Their major contribution is that the strong convergence of algorithms (1) and (3) holds with condition (L3) replaced by the condition

$(\mathbf{L3})'$ $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$ or equivalently $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1} = 0$
and with condition (L4) replaced by the condition

$(\mathbf{L4})'$ $\lim_{n\to\infty} \lambda_n/\lambda_{n+N} = 1$ or equivalently $\lim_{n\to\infty}(\lambda_n - \lambda_{n+N})/\lambda_{n+N} = 0.$

It is clear that condition (L3)' is strictly weaker than condition (L3) coupled with conditions (L1) and (L2); moreover, (L3)' includes the important and natural choice $\{1/n\}$ for $\{\lambda_n\}$ while (L3) does not. It is easy to see that if $\lim_{n\to\infty} \lambda_n/\lambda_{n+N}$ exists, then condition (L4) implies condition (L4)'. However in general, conditions (L4) and (L4)' are not comparable: neither of them implies the other (see [13] for details). Furthermore under conditions (L1), (L2), (L3)' and (L4)', they gave the applications of algorithms (1) and (3) to the constrained generalized pseudoinverses.

In this paper we introduce the following relaxed hybrid steepest-descent algorithms (I) and (II) which are mixed iteration processes of (1)-(3) as follows:

Algorithm (I). Let $\{\alpha_n\} \subset [0,1),\ \{\lambda_n\} \subset (0,1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following iterative scheme

$$u_{n+1} := \alpha_n u_n + (1-\alpha_n)[Tu_n - \lambda_{n+1}\mu F(Tu_n)] \quad n \geq 0.$$

Algorithm (II). Let $\{\alpha_n\} \subset [0,1),\ \{\lambda_n\} \subset (0,1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following iterative scheme

$$u_{n+1} := \alpha_n u_n + (1-\alpha_n)[T_{[n+1]}u_n - \lambda_{n+1}\mu F(T_{[n+1]}u_n)] \quad n \geq 0.$$

On one hand, under the assumption that $\{\alpha_n\}$ satisfies conditions (L1), (L4) with $N = 1$ and under the assumption that $\{\lambda_n\}$ satisfies conditions (L1), (L2), (L3)', we prove that the sequence $\{u_n\}$ generated by Algorithm (I) converges in norm to the unique solution u^* of the $\mathrm{VI}(F, C)$. On the other hand, under the assumption that $\{\alpha_n\}$ satisfies conditions (L1), (L4) and under the assumption that $\{\lambda_n\}$ satisfies conditions (L1), (L2), (L4)', we prove that the sequence $\{u_n\}$ generated by Algorithm (II) converges in norm to the unique solution u^* of the $\mathrm{VI}(F, C)$. Furthermore, we apply these two results to consider the constrained generalized pseudoinverses. Note that whenever the sequence $\{\alpha_n\}$ is a constant sequence $\{0\}$, i.e., $\alpha_n = 0,\ \forall n \geq 0$, Algorithms (I) and (II) immediately reduce to algorithms (1) and (3), respectively. This shows that our results improve, extend and unify corresponding algorithms in [19].

2. Preliminaries

The following lemmas will be used for the proofs of the main results of the paper in Section 3.

Lemma 2.1. (See [20, Lemma 2.5, pp. 243.]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the inequality*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n \quad \forall n \geq 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *or equivalently* $\prod_{n=0}^{\infty}(1-\alpha_n) = 0$;

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;

(iii) $\{\gamma_n\} \subset [0,\infty)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2. (See [14]). *Demiclosedness Principle. Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I-T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

The following lemma is an immediate consequence of an inner product.

Lemma 2.3. *In a real Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.4. *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers with $\limsup_{n\to\infty} \alpha_n < \infty$ and $\{\beta_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \beta_n \leq 0$. Then $\limsup_{n\to\infty} \alpha_n\beta_n \leq 0$.*

Proof. We prove the conclusion in two cases.

Case 1. $\sup_{j\geq n} \beta_j \geq 0 \ \forall n \geq 0$. For any fixed $n \geq 0$, observe that

$$\sup_{i\geq n} \alpha_i\beta_i \leq \sup_{i\geq n}(\alpha_i \cdot \sup_{j\geq n} \beta_j) = (\sup_{i\geq n} \alpha_i)(\sup_{j\geq n} \beta_j).$$

Thus taking the limit as $n \to \infty$, we obtain the conclusion.

Case 2. $\beta = \sup_{n\geq m_0} \beta_n < 0$ for some $m_0 \geq 0$. It is easy to see that $\alpha_n\beta_n \leq \alpha_n\beta \leq 0 \ \forall n \geq m_0$ which implies the conclusion. ■

For a nonempty closed convex subset $C \subset H$, we denote by P_C the (nearest point) projection from H onto C. In what follows, we state some well-known properties of the projection operator which will be used in the sequel; see [21].

Lemma 2.5. *Let C be a nonempty closed convex subset of H. For any $x, y \in H$ and $z \in C$, the following statements hold:*

(i) $\langle P_C x - x, z - P_C x\rangle \geq 0$;

(ii) $\|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|P_C x - x + y - P_C y\|^2$.

Remark 2.1. Obviously, Lemma 2.5 (i) provides also a sufficient condition for a vector u to be the projection of the vector x; i.e., $u = P_C x$ if and only if $\langle u - x, z - u\rangle \geq 0\ \forall z \in C$.

3. Relaxed Hybrid Steepest-descent Algorithms

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $F : H \to H$ be an operator such that for some constants $\kappa, \eta > 0,\ F$ is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies respectively the following conditions:

$$\|Fx - Fy\| \leq \kappa\|x - y\| \quad \forall x, y \in C, \tag{4}$$

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2 \quad \forall x, y \in C. \tag{5}$$

Denote by P_C the projection of H onto C, i.e., for each $x \in H,\ P_C x$ is the only element in C satisfying

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

It is known that the projection P_C is characterized by the inequality

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \quad \forall y \in C.$$

Thus it follows that the $\mathrm{VI}(F, C)$ is equivalent to the fixed-point problem

$$u^* = P_C(I - \mu F(u^*)),$$

where $\mu > 0$ is a constant.

In this section, assume that $T : H \to H$ is a nonexpansive mapping with $\mathrm{Fix}(T) = C$. Note that obviously, $\mathrm{Fix}(P_C) = C$. For any given numbers $\lambda \in (0, 1)$ and $\mu \in (0, 2\eta/\kappa^2)$ associating with $T : H \to H$, we define the mapping $T^\lambda : H \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx) \quad \forall x \in H.$$

Lemma 3.1. *See Ref. 7. T^λ is a contraction provided $0 < \lambda < 1$ and $0 < \mu < 2\eta/\kappa^2$.*

Indeed, one can observe that

(6) $$\|T^{\lambda}x - T^{\lambda}y\| \le (1-\lambda\tau)\|x-y\|, \quad \forall x, y \in H,$$

where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1)$.

Algorithm (I). Let $\{\alpha_n\} \subset [0,1)$, $\{\lambda_n\} \subset (0,1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, let the sequence $\{u_n\}$ be generated by the following iterative scheme

(7) $$u_{n+1} := \alpha_n u_n + (1-\alpha_n)[Tu_n - \lambda_{n+1}\mu F(Tu_n)] \quad n \ge 0.$$

Theorem 3.1. *Assume that the control conditions (L1) and (L4) with $N = 1$ hold for $\{\alpha_n\} \subset [0,1)$. Assume also that the control conditions (L1), (L2) and (L3)' hold for $\{\lambda_n\} \subset (0,1)$. Then the sequence $\{u_n\}$ generated by Algorithm (I) converges strongly to the unique solution u^* of the* $\mathrm{VI}(F,C)$.

Proof. We divide the proof into several steps.

Step 1. $\{u_n\}$ is bounded. Indeed we have (note that $T^{\lambda_{n+1}}u^* = u^* - \lambda_{n+1}\mu F(u^*)$)

(8) $$\begin{aligned}\|u_{n+1} - u^*\| &= \|\alpha_n u_n + (1-\alpha_n)T^{\lambda_{n+1}}u_n - u^*\| \\ &\le \alpha_n\|u_n - u^*\| + (1-\alpha_n)\|T^{\lambda_{n+1}}u_n - u^*\| \\ &\le \alpha_n\|u_n - u^*\| + (1-\alpha_n)[\|T^{\lambda_{n+1}}u_n - T^{\lambda_{n+1}}u^*\| + \|T^{\lambda_{n+1}}u^* - u^*\|] \\ &\le \alpha_n\|u_n - u^*\| + (1-\alpha_n)[(1-\lambda_{n+1}\tau)\|u_n - u^*\| + \lambda_{n+1}\mu\|F(u^*)\|].\end{aligned}$$

By induction, it is easy to see that

$$\|u_n - u^*\| \le \hat{M} \quad \forall n \ge 0.$$

where $\hat{M} = \max\{\|u_0 - u^*\|, (\mu/\tau)\|F(u^*)\|\}$. Indeed when $n = 0$, from (8) we obtain

$$\begin{aligned}\|u_1 - u^*\| &\le \alpha_0\|u_0 - u^*\| + (1-\alpha_0)[(1-\lambda_1\tau)\|u_0 - u^*\| + \lambda_1\mu\|F(u^*)\|] \\ &\le \alpha_0\hat{M} + (1-\alpha_0)[(1-\lambda_1\tau)\hat{M} + \lambda_1\tau\hat{M}] \\ &= \hat{M}.\end{aligned}$$

Suppose that $\|u_n - u^*\| \le \hat{M}$ for $n \ge 1$. Then from (8) we get

$$\begin{aligned}\|u_{n+1} - u^*\| &\le \alpha_n\|u_n - u^*\| + (1-\alpha_n)[(1-\lambda_{n+1}\tau)\|u_n - u^*\| + \lambda_{n+1}\mu\|F(u^*)\|] \\ &\le \alpha_n\hat{M} + (1-\alpha_n)[(1-\lambda_{n+1}\tau)\hat{M} + \lambda_{n+1}\tau\hat{M}] \\ &= \hat{M}.\end{aligned}$$

This shows that $\|u_{n+1} - u^*\| \le \hat{M}$ for $n+1$. Therefore we have $\|u_n - u^*\| \le \hat{M}\ \forall n \ge 0$.

Step 2. $\|u_{n+1} - Tu_n\| \to 0, n \to \infty$. Indeed by Step 1 $\{u_n\}$ is bounded and so are both $\{Tu_n\}$ and $\{F(Tu_n)\}$. Hence

$$\begin{aligned}\|u_{n+1} - Tu_n\| &= \|\alpha_n(u_n - Tu_n) + (1-\alpha_n)(T^{\lambda_{n+1}}u_n - Tu_n)\| \\ &\le \alpha_n\|u_n - Tu_n\| + (1-\alpha_n)\|T^{\lambda_{n+1}}u_n - Tu_n\| \\ &= \alpha_n\|u_n - Tu_n\| + (1-\alpha_n)\lambda_{n+1}\mu\|F(Tu_n)\| \\ &\le \alpha_n\|u_n - Tu_n\| + \lambda_{n+1}\mu\|F(Tu_n)\| \to 0.\end{aligned}$$

Step 3. $\|u_{n+1} - u_n\| \to 0,\ n \to \infty$. Indeed we have

$$\begin{aligned}\|u_{n+1} - u_n\| &= \|\alpha_n u_n - \alpha_{n-1}u_{n-1} + (1-\alpha_n)T^{\lambda_{n+1}}u_n - (1-\alpha_{n-1})T^{\lambda_n}u_{n-1}\| \\ &\le \|\alpha_n u_n - \alpha_{n-1}u_{n-1}\| + \|(1-\alpha_n)T^{\lambda_{n+1}}u_n - (1-\alpha_{n-1})T^{\lambda_n}u_{n-1}\| \\ &\le \alpha_n\|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}|\cdot\|u_{n-1}\| + (1-\alpha_n)\|T^{\lambda_{n+1}}u_n - T^{\lambda_{n+1}}u_{n-1}\| \\ &\quad + \|(1-\alpha_n)T^{\lambda_{n+1}}u_{n-1} - (1-\alpha_{n-1})T^{\lambda_n}u_{n-1}\| \\ &\le \alpha_n\|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}|\cdot\|u_{n-1}\| + (1-\alpha_n)(1-\lambda_{n+1}\tau)\|u_n - u_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}|\cdot\|Tu_{n-1}\| + |(1-\alpha_n)\lambda_{n+1} - (1-\alpha_{n-1})\lambda_n|\cdot\mu\|F(Tu_{n-1})\| \\ &= (1-(1-\alpha_n)\lambda_{n+1}\tau)\|u_n - u_{n-1}\| \\ &\quad + |(1-\alpha_n)\lambda_{n+1} - (1-\alpha_{n-1})\lambda_n|\cdot\mu\|F(Tu_{n-1})\| \\ &\quad + |\alpha_n - \alpha_{n-1}|\cdot(\|u_{n-1}\| + \|Tu_{n-1}\|).\end{aligned}$$

Putting $M = \sup\{\|u_n\| + \|Tu_n\| + \|F(Tu_n)\| : n \ge 0\}$, we obtain

$$\|u_{n+1} - u_n\| \le (1-(1-\alpha_n)\lambda_{n+1}\tau)\|u_n - u_{n-1}\| + ((1-\alpha_n)\lambda_{n+1}\tau)\beta_n + \gamma_n$$

where $\gamma_n = |\alpha_n - \alpha_{n-1}|\cdot(\|u_{n-1}\| + \|Tu_{n-1}\|) \le M|\alpha_n - \alpha_{n-1}|$ and

$$\beta_n = \frac{\mu M|(1-\alpha_n)\lambda_{n+1} - (1-\alpha_{n-1})\lambda_n|}{(1-\alpha_n)\lambda_{n+1}\tau} = \frac{\mu M}{\tau}\cdot\left|1 - \frac{1-\alpha_{n-1}}{1-\alpha_n}\cdot\frac{\lambda_n}{\lambda_{n+1}}\right| \to 0$$

by using conditions that $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\lambda_n/\lambda_{n+1} = 1$. Note that condition $\sum_{n=0}^{\infty}\lambda_n = \infty$ implies $\sum_{n=0}^{\infty}(1-\alpha_n)\lambda_{n+1} = \infty$. By Lemma 2.1, we deduce that $\|u_{n+1} - u_n\| \to 0$.

Step 4. $\|u_n - Tu_n\| \to 0$. This is an immediate consequence of Steps 2 and 3.

Step 5. $\limsup_{n\to\infty}\langle -F(u^*), Tu_n - u^*\rangle \le 0$. To prove this, we pick a subsequence $\{Tu_{n_i}\}$ of $\{Tu_n\}$ so that

$$\limsup_{n\to\infty}\langle -F(u^*), Tu_n - u^*\rangle = \lim_{i\to\infty}\langle -F(u^*), Tu_{n_i} - u^*\rangle.$$

Without loss of generality, we may further assume that $Tu_{n_i} \to \widetilde{u}$ weakly for some $\widetilde{u} \in H$. By Step 4, we derive $u_{n_i} \to \widetilde{u}$ weakly. But by Lemma 2.2 and Step 4, we have $\widetilde{u} \in \mathrm{Fix}(T) = C$. Since u^* is the unique solution of the $\mathrm{VI}(F,C)$, we obtain

$$\limsup_{n\to\infty} \langle -F(u^*), Tu_n - u^* \rangle = \langle -F(u^*), \widetilde{u} - u^* \rangle \leq 0.$$

Step 6. $u_n \to u^*$ in norm. Indeed by applying Lemma 2.3, we have to get

$$\begin{aligned}
\|u_{n+1} - u^*\|^2 &= \|\alpha_n(u_n - u^*) + (1-\alpha_n)(T^{\lambda_{n+1}}u_n - u^*)\|^2 \\
&\leq \alpha_n\|u_n - u^*\|^2 + (1-\alpha_n)\|T^{\lambda_{n+1}}u_n - u^*\|^2 \\
&= \alpha_n\|u_n - u^*\|^2 + (1-\alpha_n)\|(T^{\lambda_{n+1}}u_n - T^{\lambda_{n+1}}u^*) \\
&\quad +(T^{\lambda_{n+1}}u^* - u^*)\|^2 \\
&\leq \alpha_n\|u_n - u^*\|^2 + (1-\alpha_n)[\|T^{\lambda_{n+1}}u_n - T^{\lambda_{n+1}}u^*\|^2 \\
&\quad +2\langle T^{\lambda_{n+1}}u^* - u^*, T^{\lambda_{n+1}}u_n - u^*\rangle] \\
&\leq \alpha_n\|u_n - u^*\|^2 + (1-\alpha_n)[(1-\lambda_{n+1}\tau)\|u_n - u^*\|^2 \\
&\quad +2\mu\lambda_{n+1}\langle -F(u^*), Tu_n - u^* - \lambda_{n+1}\mu F(Tu_n)\rangle \\
&= (1-(1-\alpha_n)\lambda_{n+1}\tau)\|u_n - u^*\|^2 \\
&\quad +2\mu\lambda_{n+1}\langle -F(u^*), Tu_n - u^* - \lambda_{n+1}\mu F(Tu_n)\rangle \\
&= (1-(1-\alpha_n)\lambda_{n+1}\tau)\|u_n - u^*\|^2 \\
&\quad +(1-\alpha_n)\lambda_{n+1}\tau \cdot \frac{2\mu\langle -F(u^*), Tu_n - u^* - \lambda_{n+1}\mu F(Tu_n)\rangle}{(1-\alpha_n)\tau}.
\end{aligned} \tag{9}$$

Since $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\lambda_n = 0$, $\limsup_{n\to\infty}\langle -F(u^*), Tu_n - u^*\rangle \leq 0$ and $\{F(Tu_n)\}$ is bounded, by Lemma 2.4 we conclude that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{2\mu\langle -F(u^*), Tu_n - u^* - \lambda_{n+1}\mu F(Tu_n)\rangle}{(1-\alpha_n)\tau} \\
&\leq \limsup_{n\to\infty} \frac{2\mu}{(1-\alpha_n)\tau} \cdot \langle -F(u^*), Tu_n - u^*\rangle \\
&\quad + \limsup_{n\to\infty} \frac{2\mu^2\lambda_{n+1}}{(1-\alpha_n)\tau} \cdot \langle -F(u^*), -F(Tu_n)\rangle \\
&\leq 0 + 0 = 0.
\end{aligned}$$

Therefore from Lemma 2.1 we obtain $\lim_{n\to\infty}\|u_n - u^*\| = 0$. ■

Next consider a more general case where

$$C = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i),$$

with $N \geq 1$ an integer and $T_i : H \to H$ being nonexpansive for each $1 \leq i \leq N$.

We consider another relaxed hybrid steepest-descent algorithm for solving the $\mathrm{VI}(F, C)$.

Algorithm (II). Let $\{\alpha_n\} \subset [0, 1)$, $\{\lambda_n\} \subset (0, 1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, let the sequence $\{u_n\}$ be generated by the following iterative scheme

$$u_{n+1} := \alpha_n u_n + (1 - \alpha_n)[T_{[n+1]}u_n - \lambda_{n+1}\mu F(T_{[n+1]}u_n)] \quad n \geq 0. \tag{10}$$

We now state and prove the main result of this paper.

Theorem 3.2. *Assume that the control conditions (L1) and (L4) hold for* $\{\alpha_n\} \subset [0, 1)$. *Assume also that the control conditions (L1), (L2) and (L4)' hold for* $\{\lambda_n\} \subset (0, 1)$. *Assume in addition that*

$$\begin{aligned} C = \bigcap_{i=1}^N \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1T_2 ... T_N) = \mathrm{Fix}(T_N T_1 ... T_{N-1}) \\ = \cdots = \mathrm{Fix}(T_2T_3 ... T_N T_1). \end{aligned} \tag{11}$$

Then the sequence $\{u_n\}$ *generated by Algorithm (II) converges in norm to the unique solution* u^* *of the* $\mathrm{VI}(F, C)$.

Proof. We again divide the proof into several steps.

Step 1. $\{u_n\}$ is bounded. Indeed we have (note that $T^{\lambda_n}_{[n]}u^* = u^* - \lambda_n\mu F(u^*)$ for all $n \geq 1$)

$$\begin{aligned} \|u_{n+1} - u^*\| &= \|\alpha_n u_n + (1 - \alpha_n)T^{\lambda_{n+1}}_{[n+1]}u_n - u^*\| \\ &\leq \alpha_n\|u_n - u^*\| + (1 - \alpha_n)[\|T^{\lambda_{n+1}}_{[n+1]}u_n - T^{\lambda_{n+1}}_{[n+1]}u^*\| + \|T^{\lambda_{n+1}}_{[n+1]}u^* - u^*\|] \\ &\leq \alpha_n\|u_n - u^*\| + (1 - \alpha_n)[(1 - \lambda_{n+1}\tau)\|u_n - u^*\| + \lambda_{n+1}\mu\|F(u^*)\|]. \end{aligned}$$

As in Step 1 of the proof of Theorem 3.1, we get by induction

$$\|u_n - u^*\| \leq \max\{\|u_0 - u^*\|, (\mu/\tau)\|F(u^*)\|\} \quad \forall n \geq 0.$$

Step 2. $\|u_{n+1} - T_{[n+1]}u_n\| \to 0$. Indeed by Step 1 $\{u_n\}$ is bounded and so are $\{T_{[n+1]}u_n\}$ and $\{F(T_{[n+1]}u_n)\}$. Thus

$$\begin{aligned} \|u_{n+1} - T_{[n+1]}u_n\| &= \|\alpha_n(u_n - T_{[n+1]}u_n) + (1 - \alpha_n)(T^{\lambda_{n+1}}_{[n+1]}u_n - T_{[n+1]}u_n)\| \\ &\leq \alpha_n\|u_n - T_{[n+1]}u_n\| + (1 - \alpha_n)\|T^{\lambda_{n+1}}_{[n+1]}u_n - T_{[n+1]}u_n\| \\ &\leq \alpha_n\|u_n - T_{[n+1]}u_n\| + (1 - \alpha_n)\lambda_{n+1}\mu\|F(T_{[n+1]}u_n)\| \\ &\leq \alpha_n\|u_n - T_{[n+1]}u_n\| + \lambda_{n+1}\mu\|F(T_{[n+1]}u_n)\| \to 0. \end{aligned}$$

Step 3. $\|u_{n+N} - u_n\| \to 0$. As a matter of fact, observing that $T_{[n+N]} = T_{[n]}$, we have

$$\begin{aligned}
\|u_{n+N} - u_n\| &= \|\alpha_{n+N-1}u_{n+N-1} - \alpha_{n-1}u_{n-1} \\
&\quad +(1-\alpha_{n+N-1})T^{\lambda_{n+N}}_{[n+N]}u_{n+N-1} - (1-\alpha_{n-1})T^{\lambda_n}_{[n]}u_{n-1}\| \\
&\le \alpha_{n+N-1}\|u_{n+N-1} - u_{n-1}\| + |\alpha_{n+N-1} - \alpha_{n-1}| \cdot \|u_{n-1}\| \\
&\quad +(1-\alpha_{n+N-1})\|T^{\lambda_{n+N}}_{[n+N]}u_{n+N-1} - T^{\lambda_{n+N}}_{[n+N]}u_{n-1}\| \\
&\quad +\|(1-\alpha_{n+N-1})T^{\lambda_{n+N}}_{[n+N]}u_{n-1} - (1-\alpha_{n-1})T^{\lambda_n}_{[n]}u_{n-1}\| \\
&\le \alpha_{n+N-1}\|u_{n+N-1} - u_{n-1}\| + |\alpha_{n+N-1} - \alpha_{n-1}| \cdot \|u_{n-1}\| \\
&\quad +(1-\alpha_{n+N-1})(1-\lambda_{n+N}\tau)\|u_{n+N-1} - u_{n-1}\| \\
&\quad +|\alpha_{n+N-1} - \alpha_{n-1}| \cdot \|T_{[n]}u_{n-1}\| \\
&\quad +|(1-\alpha_{n+N-1})\lambda_{n+N} - (1-\alpha_{n-1})\lambda_n| \cdot \mu\|F(T_{[n]}u_{n-1})\| \\
&= (1-(1-\alpha_{n+N-1})\lambda_{n+N}\tau)\|u_{n+N-1} - u_{n-1}\| \\
&\quad +|(1-\alpha_{n+N-1})\lambda_{n+N} - (1-\alpha_{n-1})\lambda_n| \cdot \mu\|F(T_{[n]}u_{n-1})\| \\
&\quad +|\alpha_{n+N-1} - \alpha_{n-1}| \cdot (\|u_{n-1}\| + \|T_{[n]}u_{n-1}\|).
\end{aligned}$$

Putting $M = \sup\{\|u_n\| + \|T_{[n+1]}u_n\| + \|F(T_{[n+1]}u_n)\| : n \ge 0\}$, we obtain

$$\|u_{n+N} - u_n\| \le (1-(1-\alpha_{n+N-1})\lambda_{n+N}\tau)\|u_{n+N-1} - u_{n-1}\| + ((1-\alpha_{n+N-1})\lambda_{n+N}\tau)\beta_n + \gamma_n$$

where $\gamma_n = |\alpha_{n+N-1} - \alpha_{n-1}| \cdot (\|u_{n-1}\| + \|T_{[n]}u_{n-1}\|) \le M|\alpha_{n+N-1} - \alpha_{n-1}|$ and

$$\beta_n = \frac{\mu M|(1-\alpha_{n+N-1})\lambda_{n+N} - (1-\alpha_{n-1})\lambda_n|}{(1-\alpha_{n+N-1})\lambda_{n+N}\tau} = \frac{\mu M}{\tau} \cdot |1 - \frac{1-\alpha_{n-1}}{1-\alpha_{n+N-1}} \cdot \frac{\lambda_n}{\lambda_{n+N}}| \to 0$$

by using conditions that $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \lambda_n/\lambda_{n+N} = 1$. Note that condition $\sum_{n=0}^{\infty} \lambda_n = \infty$ implies $\sum_{n=0}^{\infty}(1-\alpha_{n+N-1})\lambda_{n+N} = \infty$. We deduce that $\|u_{n+N} - u_n\| \to 0$ by Lemma 2.1.

Step 4. $u_n - T_{[n+N]}\ldots T_{[n+1]}u_n \to 0$ in norm. Indeed noting that each T_i is nonexpansive and using Step 2, we get the finite table

$$\begin{aligned}
&u_{n+N} - T_{[n+N]}u_{n+N-1} \to 0, \\
&T_{[n+N]}u_{n+N-1} - T_{[n+N]}T_{[n+N-1]}u_{n+N-2} \to 0, \\
&\qquad\qquad\vdots \\
&T_{[n+N]}\ldots T_{[n+2]}u_{n+1} - T_{[n+N]}\ldots T_{[n+1]}u_n \to 0.
\end{aligned}$$

Adding up this table and using Step 3, we deduce that $u_n - T_{[n+N]} ... T_{[n+1]} u_n \to 0$ in norm.

Step 5. $\limsup_{n\to\infty} \langle -F(u^*), T_{[n+1]}u_n - u^* \rangle \le 0$. To see this, we pick a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\limsup_{n\to\infty} \langle -F(u^*), T_{[n+1]}u_n - u^* \rangle = \lim_{i\to\infty} \langle -F(u^*), T_{[n_i+1]}u_{n_i} - u^* \rangle.$$

Since $\{T_{[n+1]}u_n\}$ is bounded, we may also assume that $T_{[n_i+1]}u_{n_i} \to \widetilde{u}$ weakly for some $\widetilde{u} \in H$. Hence from Step 2 we deduce that $u_{n_i+1} \to \widetilde{u}$ weakly. Since the pool of mappings $\{T_i : 1 \le i \le N\}$ is finite, we may further assume (passing to a further subsequence if necessary) that for some integer $k \in \{1, 2, ..., N\}$, $T_{[n_i+1]} \equiv T_k \ \forall i \ge 1$. Then it follows from Step 4 that $u_{n_i+1} - T_{[k+N]} ... T_{[k+1]} u_{n_i+1} \to 0$. Hence by Lemma 2.2 we deduce that

$$\widetilde{u} \in \mathrm{Fix}(T_{[k+N]} ... T_{[k+1]}).$$

Together with assumption (11) this implies that $\widetilde{u} \in C$. Now since u^* solves the $\mathrm{VI}(F, C)$, we obtain

$$\limsup_{n\to\infty} \langle -F(u^*), T_{[n+1]}u_n - u^* \rangle = \langle -F(u^*), \widetilde{u} - u^* \rangle \le 0.$$

Step 6. $u_n \to u^*$ in norm. Indeed, applying Lemma 2.3, we get

$$\begin{aligned}
\|u_{n+1} - u^*\|^2 &= \|\alpha_n(u_n - u^*) + (1-\alpha_n)(T^{\lambda_{n+1}}_{[n+1]}u_n - u^*)\|^2 \\
&\le \alpha_n \|u_n - u^*\|^2 + (1-\alpha_n)\|T^{\lambda_{n+1}}_{[n+1]}u_n - u^*\|^2 \\
&= \alpha_n \|u_n - u^*\|^2 + (1-\alpha_n)\|(T^{\lambda_{n+1}}_{[n+1]}u_n - T^{\lambda_{n+1}}_{[n+1]}u^*) + (T^{\lambda_{n+1}}_{[n+1]}u^* - u^*)\|^2 \\
&\le \alpha_n \|u_n - u^*\|^2 + (1-\alpha_n)[\|T^{\lambda_{n+1}}_{[n+1]}u_n - T^{\lambda_{n+1}}_{[n+1]}u^*\|^2 \\
&\quad + 2\langle T^{\lambda_{n+1}}_{[n+1]}u^* - u^*, T^{\lambda_{n+1}}_{[n+1]}u_n - u^* \rangle] \\
&\le \alpha_n \|u_n - u^*\|^2 + (1-\alpha_n)[(1-\lambda_{n+1}\tau)\|u_n - u^*\|^2 \\
&\quad + 2\mu\lambda_{n+1}\langle -F(u^*), T_{[n+1]}u_n - u^* - \lambda_{n+1}\mu F(T_{[n+1]}u_n)\rangle \\
&= (1 - (1-\alpha_n)\lambda_{n+1}\tau)\|u_n - u^*\|^2 \\
&\quad + 2\mu\lambda_{n+1}\langle -F(u^*), T_{[n+1]}u_n - u^* - \lambda_{n+1}\mu F(T_{[n+1]}u_n)\rangle \\
&= (1 - (1-\alpha_n)\lambda_{n+1}\tau)\|u_n - u^*\|^2 \\
&\quad + (1-\alpha_n)\lambda_{n+1}\tau \cdot \frac{2\mu\langle -F(u^*), T_{[n+1]}u_n - u^* - \lambda_{n+1}\mu F(T_{[n+1]}u_n)\rangle}{(1-\alpha_n)\tau}.
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\lambda_n = 0$, $\limsup_{n\to\infty}\langle -F(u^*), T_{[n+1]}u_n - u^*\rangle \le 0$ and $\{F(T_{[n+1]}u_n)\}$ is bounded, so by Lemma 2.4 we conclude that

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{2\mu\langle -F(u^*), T_{[n+1]}u_n - u^* - \lambda_{n+1}\mu F(T_{[n+1]}u_n)\rangle}{(1-\alpha_n)\tau}\\
&\le \limsup_{n\to\infty}\frac{2\mu}{(1-\alpha_n)\tau}\cdot\langle -F(u^*), T_{[n+1]}u_n - u^*\rangle\\
&\quad + \limsup_{n\to\infty}\frac{2\mu^2\lambda_{n+1}}{(1-\alpha_n)\tau}\cdot\langle -F(u^*), -F(T_{[n+1]}u_n)\rangle\\
&\le 0+0=0.
\end{aligned}$$

Therefore from Lemma 2.1 we obtain $\lim_{n\to\infty}\|u_n - u^*\| = 0$. ∎

Remark 3.1. Recall that a self-mapping of a nonempty closed convex subset K of a Hilbert space H is said to be attracting nonexpansive [11, 17] if T is nonexpansive and if $\|Tx-p\| < \|x-p\|$ for $x,p\in K$ with $x\notin \mathrm{Fix}(T)$ and $p\in\mathrm{Fix}(T)$. Recall also that T is firmly nonexpansive [11, 17] if $\langle x-y, Tx-Ty\rangle \ge \|Tx-Ty\|^2$ for all $x,y\in K$. It is known that assumption (11) in Theorem 3.2 is automatically satisfied if each T_i is attracting nonexpansive. Since a projection is firmly nonexpansive, we have the following consequence of Theorem 3.2.

Corollary 3.1. *Let $\mu\in(0,2\eta/\kappa^2)$. Assume that the control conditions (L1) and (L4) hold for $\{\alpha_n\}\subset[0,1)$. Assume also that the control conditions (L1), (L2) and (L4)' hold for $\{\lambda_n\}\subset(0,1)$. Let $u_0\in H$ and let the sequence $\{u_n\}$ be generated by the iterative algorithm*

$$u_{n+1} := \alpha_n u_n + (1-\alpha_n)[P_{[n+1]}u_n - \lambda_{n+1}\mu F(P_{[n+1]}u_n)] \quad n\ge 0$$

where $P_k = P_{C_k}, 1\le k\le N$. Then $\{u_n\}$ converges strongly to the unique solution u^* of the $\mathrm{VI}(F,C)$, with $C=\bigcap_{k=1}^N C_k$. In particular, the sequence $\{u_n\}$ determined by the algorithm

$$u_{n+1} := (1/(n+1))u_n + (n/(n+1))[P_{[n+1]}u_n - (\mu/(n+1))F(P_{[n+1]}u_n) \quad n\ge 0$$

converges in norm to the unique solution u^* of the $\mathrm{VI}(F,C)$. ∎

4. Applications to Constrained Generalized Pseudoinverse

Let K be a nonempty closed convex subset of a real Hilbert space H. Let A be a bounded linear operator on H. Given an element $b\in H$, consider the minimization problem

$$\min_{x\in K}\|Ax-b\|^2. \tag{12}$$

Let S_b denote the solution set. Then S_b is closed and convex. It is known that S_b is nonempty if and only if $P_{\overline{A(K)}}(b) \in A(K)$. In this case, S_b has a unique element with minimum norm; that is, there exists a unique point $\hat{x} \in S_b$ satisfying

$$\|\hat{x}\|^2 = \min\{\|x\|^2 : x \in S_b\}. \tag{13}$$

Definition 4.1. (See [18]). The K-constrained pseudoinverse of A (symbol $\hat{A}_K$) is defined as

$$D(\hat{A}_K) = \{b \in H : P_{\overline{A(K)}}(b) \in A(K)\}, \quad \hat{A}_K(b) = \hat{x} \quad \text{and} \quad b \in D(\hat{A}_K)$$

where $\hat{x} \in S_b$ is the unique solution to (13).

Now we recall the K-constrained generalized pseudoinverse of A; see [7, 19].

Let $\theta : H \to R$ be a differentiable convex function such that θ' is a κ-Lipschitzian and η-strongly monotone operator for some $\kappa > 0$ and $\eta > 0$. Under these assumptions, there exists a unique point $\hat{x}_0 \in S_b$ for $b \in D(\hat{A}_K)$ such that

$$\theta(\hat{x}_0) = \min\{\theta(x) : x \in S_b\}. \tag{14}$$

Definition 4.2. (See [19]). The K-constrained generalized pseudoinverse of A associated with θ (symbol $\hat{A}_{K,\theta}$) is defined as

$$D(\hat{A}_{K,\theta}) = D(\hat{A}_K), \quad \hat{A}_{K,\theta}(b) = \hat{x}_0 \quad \text{and} \quad b \in D(\hat{A}_{K,\theta})$$

where $\hat{x}_0 \in S_b$ is the unique solution to (14). Note that if $\theta(x) = \|x\|^2/2$, then the K-constrained generalized pseudoinverse $\hat{A}_{K,\theta}$ of A associated with θ reduces to the K-constrained pseudoinverse $\hat{A}_K$ of A in Definition 4.1.

We now apply the results in Section 3 to construct the K-constrained generalized pseudoinverse $\hat{A}_{K,\theta}$ of A. But first, observe that $\widetilde{x} \in K$ solves the minimization problem (12) if and only if there holds the following optimality condition:

$$\langle A^*(A\widetilde{x} - b), x - \widetilde{x}\rangle \geq 0 \quad x \in K$$

where A^* is the adjoint of A. This is equivalent to for each $\lambda > 0$

$$\langle [\lambda A^*b + (I - \lambda A^*A)\widetilde{x}] - \widetilde{x}, x - \widetilde{x}\rangle \geq 0 \quad x \in K$$

or

$$P_K(\lambda A^*b + (I - \lambda A^*A)\widetilde{x}) = \widetilde{x}. \tag{15}$$

Define a mapping $T : H \to H$ by

$$Tx = P_K(A^*b + (I - \lambda A^*A)x) \quad x \in H. \tag{16}$$

Lemma 4.1. (See [19]). *If* $\lambda \in (0, 2\|A\|^{-2})$ *and if* $b \in D(\hat{A}_K)$, *then* T *is attracting nonexpansive and* $\mathrm{Fix}(T) = S_b$.

Theorem 4.1. *Let* $\mu \in (0, 2\eta/\kappa^2)$. *Assume that the control conditions* (*L1*) *and* (*L4*) *with* $N = 1$ *hold for* $\{\alpha_n\} \subset [0, 1)$. *Assume also that the control conditions* (*L1*), (*L2*) *and* (*L3*)*' hold for* $\{\lambda_n\} \subset (0, 1)$. *Given an initial guess* $u_0 \in H$, *let* $\{u_n\}$ *be the sequence generated by the algorithm*

$$u_{n+1} = \alpha_n u_n + (1 - \alpha_n)[Tu_n - \lambda_{n+1}\mu\theta'(Tu_n)] \quad n \geq 0 \tag{17}$$

where T *is given in (16). Then* $\{u_n\}$ *strongly converges to* $\hat{A}_{K,\theta}(b)$.

Proof. The minimization problem (14) is equivalent to the following variational inequality problem:

$$\langle \theta'(\hat{x}_0), x - \hat{x}_0 \rangle \geq 0 \quad x \in S_b. \tag{18}$$

Since $\mathrm{Fix}(T) = S_b$ and θ' is κ-Lipschitzian and η-strongly monotone, using Theorem 3.1 with $F \coloneqq \theta'$ we conclude that $\{u_n\}$ converges in norm to $\hat{x}_0 = \hat{A}_{K,\theta}(b)$. ∎

Lemma 4.2. (See [11, 17]). *Assume that* N *is a positive integer and assume that* $\{T_i\}_{i=1}^N$ *are* N *attracting nonexpansive mappings on* H *having a common fixed point. Then*

$$\bigcap_{i=1}^N \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1T_2...T_N).$$

Now assume that $\{S_b^1, ..., S_b^N\}$ is a family of N closed convex subsets of K such that $S_b = \bigcap_{i=1}^N S_b^i$. For each $1 \leq i \leq N$, we define $T_i : H \to H$ by

$$T_i x = P_{S_b^i}(A^*b + (I - \lambda A^*A)x) \quad x \in H$$

where $P_{S_b^i}$ is the projection from H onto S_b^i.

Theorem 4.2. *Let* $\mu \in (0, 2\eta/\kappa^2)$. *Assume that the control conditions (L1) and (L4) hold for* $\{\alpha_n\} \subset [0, 1)$. *Assume also that the control conditions (L1), (L2) and (L4)' hold for* $\{\lambda_n\} \subset (0, 1)$. *Let* $u_0 \in H$. *Then the sequence* $\{u_n\}$ *generated by the algorithm*

$$u_{n+1} = \alpha_n u_n + (1 - \alpha_n)T_{[n+1]}^{\lambda_{n+1}} u_n \quad \forall n \geq 0 \tag{19}$$

where $T_{[n+1]}^{\lambda_{n+1}} = T_{[n+1]}u_n - \lambda_{n+1}\mu\theta'(T_{[n+1]}u_n),\ n \geq 0$, *converges in norm to* $\hat{A}_{K,\theta}(b)$.

Proof. In the proof of [19, Theorem 4.2], it is proved that

$$S_b = \mathrm{Fix}(T) = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i). \tag{20}$$

By Lemmas 4.1 and 4.2, we see that assumption (11) in Theorem 3.2 holds. On account of (20), Theorem 3.2 ensures that the sequence $\{u_n\}$ generated by (19) converges strongly to the unique solution $\hat{x}_0 = \hat{A}_{K,\theta}(b)$ of (18). ■

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