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Restrictions on Seshadri Constants on Surfaces

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Abstract. Starting with the pioneering work of Ein and Lazarsfeld [9] restrictions on values of Seshadri constants on algebraic surfaces have been studied by many authors [2, 5, 10, 12, 18, 20, 22, 24]. In the present note we show how approximation involving continued fractions combined with recent results of Küronya and Lozovanu on Okounkov bodies of line bundles on surfaces [13, 14] lead to effective statements considerably restricting possible values of Seshadri constants. These results in turn provide strong additional evidence to a conjecture governing the Seshadri constants on algebraic surfaces with Picard number 1.

1. Introduction

Let X be a smooth algebraic variety and let L be a nef line bundle on X. For any point $x \in X$ the real number

$$\varepsilon(L; x) = \inf_{C \ni x} \frac{(L \cdot C)}{\operatorname{mult}_x C},$$

where the infimum is taken over all irreducible curves C passing through x, measures in effect the *local positivity of* L *at* x. We say that a curve $C \subset X$ is a *Seshadri curve of* L *at* x if $\varepsilon(L; x) = (L \cdot C)/\operatorname{mult}_x C$.

These numbers, the Seshadri constants, were introduced by Demailly in [7] in connection with his works on the Fujita Conjecture and they have become a subject of considerable interest ever since. The well known Seshadri criterion of ampleness gives a fundamental positivity restriction on the Seshadri constants of ample line bundles.

Theorem 1.1 (Seshadri criterion of ampleness)**.** *Let X be a smooth algebraic variety and let L be a line bundle on X. Then*

$$L \textit{ is ample if and only if } \inf_{x \in X} \varepsilon(L; x) > 0.$$

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It is natural to wonder if there are any other constrains on the values of Seshadri constants of ample line bundles. Whereas examples of Miranda and Viehweg show that the Seshadri constants of ample line bundles can become arbitrarily small, in the groundbreaking paper [9] Ein and Lazarsfeld showed that there is a positive uniform lower bound when restricting to very general points. Oguiso in [19] showed that the Seshadri function

$$\varepsilon(L;\ \cdot\ )\colon X \ni x \to \varepsilon(L;x) \in \mathbb{R} \tag{1.1}$$

is lower semi-continuous in the topology whose closed sets are countable unions of Zariski closed sets. In particular there is an open and dense subset of X in this topology where the Seshadri function attains its maximal value. We denote this maximal value by $\varepsilon(L;1)$. The number 1 here indicates that the Seshadri constant is taken at a very general point of X without specifying this point. In this terminology the aforementioned result of Ein and Lazarsfeld is the following.

Theorem 1.2 (Ein-Lazarsfeld)**.** *Let X be an algebraic surface and let L be an ample line bundle on X. Then*

$$\varepsilon(L;1) \geq 1.$$

This result cannot be improved in general even under the assumption that the self-intersection $d = (L^2)$ of L is very large, see Example 2.5. The main result of this note shows that nevertheless the set of potential values $\varepsilon(L;1)$ can take on is surprisingly limited.

For a non-square integer d and (p,q) a solution to the Pell equation

$$y^2 - dx^2 = 1 \tag{1.2}$$

(i.e., $q^2 - dp^2 = 1$) we define the following set

$$\operatorname{Exc}(d;p,q) = \left\{1, 2, \ldots, \lfloor\sqrt{d}\rfloor\right\} \cup \left\{\frac{a}{b} \text{ such that } 1 \leq \frac{a}{b} < \frac{p}{q}d \text{ and } 2 \leq b < q^2\right\}.$$

Theorem 1.3. *Let X be a smooth projective surface, $x \in X$, L an ample line bundle on X such that $(L^2) = d$ is not a square. Let (p,q) be an arbitrary solution to the Pell equation* (1.2)*. Then either*

$$\varepsilon(L;1) \geq \frac{p}{q}d,$$

or $\varepsilon(L;1) \in \operatorname{Exc}(d;p,q)$.

The finiteness of possible values of $\varepsilon(L;1)$ strictly below any rational number smaller than $\sqrt{(L^2)}$ follows already from [19, Theorem 1]. However Oguiso result addresses all line bundles L *separately* and the statement is ineffective. The key point of Theorem 1.3 is that possible values of $\varepsilon(L;1)$ depend in a *uniform* way only on (L^2) and their set is effectively described and relatively small. Under additional assumptions on X or L one can thus try to reduce further the set of exceptional values (i.e., those lower than $\frac{p}{q}d$). A typical assumption of this kind is that the Picard number $\rho(X)$ of X equals 1. The Seshadri constants of ample line bundles on surfaces with $\rho(X) = 1$ were considered in a series of papers [20–22] leading to the following conjecture which has motivated our research here.

Conjecture 1.4. *Let X be a smooth projective surface with Picard number* 1 *and let L be the ample generator of the Néron-Severi space with* $(L^2) = d$. *Assume furthermore that d is a non-square. Then*

$$\varepsilon(L;1) \geq \frac{p_0}{q_0}d,$$

where (p_0, q_0) *is the* primitive solution *of the Pell equation* (1.2).

In other words, the Seshadri constants of generators of the ample cone on surfaces with Picard number 1 taken at very general points are expected not to lie in the set $\mathrm{Exc}(d; p_0, q_0)$.

Remark 1.5. It is well known that there are surfaces such that the equality in Conjecture 1.4 holds, so that the lower bound there cannot be improved without additional conditions on X, see [2].

In Theorems 4.1 and 4.2, we verify Conjecture 1.4 for two infinite series of line bundles L, namely line bundles with (L^2) of the form $n^2 - 1$ or $n^2 + n$ for an arbitrary positive integer n.

We conclude our note with a case by case study of line bundles with low self-intersection numbers in Section 5. More specifically, we study line bundles with self-intersection up to 8. We show in particular how the general statement of Theorem 1.3 can be considerably improved when coupled with Proposition 2.2 and when working with a specific number (L^2).

2. General properties of the Seshadri constants

In this section we recall properties of the Seshadri constants needed in the sequel. For a general introduction to this circle of ideas we refer to the book of Lazarsfeld [15] and the survey [3].

We begin with a useful lower bound on the self-intersection of curves moving in a nontrivial family, see [12, Theorem A] and [1]. This bound will be applied to the Seshadri curves of a fixed line bundle.

Proposition 2.1 (Bounding self-intersection of curves in a family)**.** *Let X be a smooth projective surface. Let (C_t, x_t) be a nontrivial family of pointed curves $C_t \subset X$ such that* *for some integer* $m \geq 2$ *there is* $\operatorname{mult}_{x_t} C_t \geq m$. *Then*

$$(C_t^2) \geq m(m-1) + \operatorname{gon}(C_t).$$

The next proposition shows that if $\varepsilon(L;1)$ is relatively small compared to (L^2), then there are geometric reasons, see [23, Theorem]. We say that X is *fibred by Seshadri curves* if there exists a morphism $f\colon X \to B$ to a curve B such that for a very general point $x \in X$, the curve $f^{-1}(f(x))$ computes $\varepsilon(L;x)$.

Proposition 2.2 (Fibration by Seshadri curves)**.** *Let* X *be a smooth projective complex surface and let* L *be an ample line bundle on* X. *If*

$$\varepsilon(L;1) < \sqrt{\frac{3}{4}(L^2)},$$

then X *is fibred by Seshadri curves. In particular,* $\varepsilon(L;1)$ *is an integer.*

We record also for further reference the following property of line bundles whose Seshadri curves are smooth.

Proposition 2.3 (Smooth Seshadri curves)**.** *Let* X *be a smooth projective surface and let* L *be a primitive ample line bundle on* X *(i.e.,* L *is not divisible in the Picard group of* X*). Assume that* $\varepsilon(L;1)$ *is computed by smooth curves. Then*

(1) *either* X *is fibred by Seshadri curves, or*

(2) $(L^2) = 1$.

Proof. Assume to begin with that

$$\varepsilon(L;1) < \sqrt{(L^2)}.$$

Let C_x be a smooth curve computing $\varepsilon(L;x)$ in a very general point $x \in X$. Then it is

$$(L \cdot C_x) < \sqrt{(L^2)}.$$

Combined with the Hodge Index Theorem, this implies

$$(C_x^2)(L^2) \leq (L \cdot C_x)^2 < (L^2) \tag{2.1}$$

and hence $(C_x^2) < 1$. Since C_x passes through a very general point of X, it must be $(C_x^2) \geq 0$, which gives finally $(C_x^2) = 0$.

Now, there is a standard argument (see [17, Proof of Theorem 2] or [23, Proof of Theorem]) using the countability of components of the Hilbert scheme of curves on X, which implies that there is at least one dimensional algebraic family of curves $\{C_x\}$. Since

$(C_x^2) = 0$, two distinct curves C_x and C_y in this family are disjoint. Thus one can define a map from X to the parameter curve T, whose very general fibers are the curves C_x.

In the remaining case we have

$$\varepsilon(L;1) = \sqrt{(L^2)}.$$

Here the assumption that the Seshadri constant *is actually computed by a curve* is essential to conclude. Indeed, we have then the equality in (2.1). Hence $0 \leq (C_x^2) \leq 1$. If $(C_x^2) = 0$, we conclude as before. If $(C_x^2) = 1$, then we have equality in the Hodge inequality, so it must be that C_x and L are numerically proportional. Since L is primitive, it must be $(L^2) = 1$ and we are done. □

Proposition 2.3 implies immediately the following property of line bundles on surfaces with Picard number 1.

Corollary 2.4. *Let X be a surface with Picard number 1. Assume that there exists an integer k such that the Seshadri constant of the ample generator L at a very general point x of X is computed by a curve $C_x \in |kL|$. Then*

(1) *either* $\operatorname{mult}_x C_x \geq 2$, *or*

(2) $(L^2) = 1$ *and* $\varepsilon(L;1) = 1$.

Proof. This follows from Proposition 2.3 since there are no fibrations on surfaces with Picard number 1. □

The next example shows in particular that the assumption on the Picard number of X in Corollary 2.4 is essential.

Example 2.5. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let $L = sH + V$, where H is the class of the fiber of the projection from X onto the second factor and V is the class of the fiber of the projection onto the first factor. Then $(L^2) = 2s$ and $\varepsilon(L;x) = 1$ for all points $x \in X$. Indeed, the fiber in $|H|$ passing through x is the Seshadri curve of L at x.

3. An application of Okounkov bodies to the Seshadri constants

In this section we prove Theorem 1.3. The proof builds upon ideas of Küronya and Lozovanu from [13]. Relating the Seshadri constants and Okounkov bodies is not new; see, e.g., [8, 11]. The key insight here is to use the *infinitesimal* approach in the form which is a slight generalization of [13, Example 4.4]. Our approach has been also strongly influenced by works of Nakamaye and Cascini [6, 18]. For an introduction to Okounkov bodies we refer to the work of Lazarsfeld and Mustaţă [16].

Proof of Theorem 1.3. If $\varepsilon(L;1) \geq \frac{p}{q}d$, then there is nothing to prove. In the remaining case it must be $\varepsilon(L;x) < \frac{p}{q}d$ for an *arbitrary* point $x \in X$. Let now x be a *very general* point of X, i.e., a point in which the function defined in (1.1) attains its maximal value. Since the value of the Seshadri constant is submaximal [4, Proposition 1.1] guarantees that there exists a curve C computing $\varepsilon(L;x)$. With $a = (L.C)$ and $b = \operatorname{mult}_x C$ we have

$$\varepsilon(L;x) = \frac{a}{b} < \frac{p}{q}d. \tag{3.1}$$

Note that the integers a and b need not be coprime.

If $b = 1$, i.e., $\varepsilon(L;x)$ is computed by a *smooth* curve, then

$$\varepsilon(L;x) \in \left\{1, 2, \ldots, \lfloor\sqrt{d}\rfloor\right\} \subset \operatorname{Exc}(d;p,q) \tag{3.2}$$

and we are done. (Proposition 2.3 provides additional information on X and L in this case.)

So we may assume $b \geq 2$. Then by [13, Proposition 4.2], the generic infinitesimal Newton-Okounkov polygon $\Delta(L;x)$ is contained in the triangle $\triangle_{OAB}$ with vertices at points

$$O = (0,0), \quad A = (a/b, a/b), \quad B = (a/(b-1), 0).$$

Comparing the areas of the two figures, we obtain

$$\frac{a}{b} \cdot \frac{a}{b-1} = 2 \cdot \operatorname{Area}(\triangle_{OAB}) \geq 2 \cdot \operatorname{Area}(\Delta(L;x)) = d. \tag{3.3}$$

From (3.1) and (3.3), we obtain

$$\frac{a^2}{b^2} < \frac{p^2 d}{q^2}d = \frac{q^2-1}{q^2}d \leq \frac{q^2-1}{q^2}\frac{a^2}{b(b-1)},$$

which implies

$$\frac{b-1}{b} < \frac{q^2-1}{q^2}.$$

This inequality can hold if and only if $b < q^2$. This verifies the bound on the multiplicity of the Seshadri curve asserted in Theorem 1.3.

In order to conclude observe that all possible pairs $(a,b) \in \mathbb{N}^2$ with $\frac{a}{b} < \frac{p}{q}d$ lie in the range

$$1 < b < q^2 \quad \text{and} \quad b \leq a < b \cdot \frac{p}{q}d, \tag{3.4}$$

hence they are contained in the set $\operatorname{Exc}(d;p,q)$. Note that the inequality (3.3) restricts the actual set of possible values even further. We will explore this in Section 5. $\square$

4. Towards the conjecture

In this section we prove that Conjecture 1.4 holds for two sequences of integers d such that the primitive solution (p_0, q_0) of the Pell equation (1.2) satisfies $p_0 \in \{1, 2\}$.

Theorem 4.1 (The case $p_0 = 1$)**.** *Let $d = n^2 - 1$ for a positive integer n. Then Conjecture 1.4 holds for all polarized pairs (X, L), with X a smooth projective surface with Picard number one, and L the ample generator of $\operatorname{Pic}(X)$ with $(L^2) = d$, that is*

$$\varepsilon(L; 1) \geq \frac{p_0}{q_0} d = \frac{n^2 - 1}{n}.$$

Proof. For $d = n^2 - 1$ the primitive solution to Pell's equation is $(p_0, q_0) = (1, n)$. Assume to the contrary that for a very general point $x \in X$ there exists a curve $C \in |kL|$ for some $k \geq 1$ computing $\varepsilon(L; x) = a/b$ and $a, b \in \operatorname{Exc}(d; 1, n)$. By Corollary 2.4 we have $b \geq 2$. Thus Proposition 2.1 applies and we have

$$k^2(n^2 - 1) = k^2 d = (C^2) \geq b(b - 1) + 1. \tag{4.1}$$

On the other hand,

$$\frac{a}{b} = \frac{L.C}{b} = \frac{kd}{b} = \frac{k(n^2 - 1)}{b} < \frac{p_0}{q_0} d = \frac{n^2 - 1}{n}$$

implies

$$b \geq kn + 1. \tag{4.2}$$

Now it easy to see that (4.1) and (4.2) cannot be satisfied simultaneously. Indeed, combining both inequalities we get

$$k^2(n^2 - 1) \geq (kn + 1)kn + 1.$$ □

In the previous case we had $p_0 = 1$. Now we pass to the next case, i.e., $p_0 = 2$. Then d is of the form $d = n^2 + n$ for some integer $n \geq 1$ and $q_0 = 2n + 1$.

Theorem 4.2 (The case $p_0 = 2$)**.** *Let $d = n^2 + n$ for a positive integer n. Then Conjecture 1.4 holds for all polarized pairs (X, L), where X is a smooth projective surface with Picard number one, and L is the ample generator of $\operatorname{Pic}(X)$ with $(L^2) = d$, that is*

$$\varepsilon(L; 1) \geq \frac{p_0}{q_0} d = \frac{2n(n + 1)}{2n + 1}.$$

Proof. We assume to the contrary that for a very general (hence any) point on X, there exists a curve $C_x \in |kL|$ for some $k \geq 1$ such that with $a = (L.C_x) = kL^2$ and $b = \operatorname{mult}_x C_x$ there is

$$\frac{a}{b} < \frac{2}{2n + 1}(n^2 + n).$$

Equivalently we have

$$\frac{k(2n+1)}{2} < b. \tag{4.3}$$

The standard argument with the Hilbert scheme of curves revoked in the proof of Proposition 2.3 implies that such curves move in a nontrivial family of dimension at least 1.

We have $b \geq 2$ by Corollary 2.4. Hence, by Proposition 2.1 we have

$$b(b-1)+1 \leq k^2(n^2+n). \tag{4.4}$$

Now the argument splits according to the parity of k.

Case 1: Assume that $k = 2\ell$. Then (4.3) reads $b > 2\ell n + \ell$. This implies

$$b \geq 2\ell n + \ell + 1,$$

and in turn we get

$$b(b-1)+1 \geq (2\ell n + \ell + 1)(2\ell n + \ell). \tag{4.5}$$

On the other hand from (4.4) we get

$$b(b-1)+1 \leq 4\ell^2(n^2+n). \tag{4.6}$$

It is elementary to check that (4.5) and (4.6) together give a contradiction.

Case 2: The case $k = 2\ell + 1$ follows similarly. From (4.3) we get $b > (2\ell+1)n + \frac{2\ell+1}{2}$ so that

$$b \geq (2\ell+1)n + \ell + 1.$$

Hence

$$b(b-1)+1 \geq [(2\ell+1)n + \ell + 1]\,[(2\ell+1)n + \ell] + 1$$

and this contradicts inequality (4.4) in this case as well. We leave the details to the reader. □

Thus the first remaining case is $p_0 = 3$. The first d with $p_0 = 3$ is $d = 7$. We will see in the next section that already in this case our approach leaves over some possibilities which require additional arguments.

5. Line bundles with small self-intersection

In this section we analyze consequences of Theorem 1.3 on the distribution of values of the Seshadri constants in general points of line bundles with fixed low degree both in general and in the $\rho(X) = 1$ cases.

5.1. Line bundles of degree 1

If $(L^2) = 1$, then Theorem 1.2 immediately yields $\varepsilon(L; 1) = 1$. Additionally, in this case it is known that the number of points where $\varepsilon(L; x)$ attains a value strictly less than 1 is finite.

The following example shows that there is little hope to obtain any classification of line bundles with self-intersection 1.

Example 5.1. Let X be a smooth projective surface with Picard number $\rho(X) = 1$ and let L be the ample generator on X with $d = (L^2)$. Let $f\colon Y \to X$ be the blow up of X at $d-1$ very general points, with the exceptional divisor $\mathbb{E}$ (being the union of $d-1$ exceptional curves). Then $M := f^*L - \mathbb{E}$ is an ample line bundle with $(M^2) = 1$.

5.2. Line bundles of degree 2

In this case the primitive solution to the Pell's equation is $p_0 = 2$ and $q_0 = 3$ so that

$$\mathrm{Exc}(2; 2, 3) = \left\{1, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{7}, \frac{9}{8}, \frac{10}{8}\right\}.$$

The extremal value 1 is attained for example by the line bundle L of bidegree $(1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

If $\rho(X) = 1$, then Conjecture 1.4 holds by Theorem 4.2 and we have

$$\varepsilon(L; 1) \geq \frac{4}{3}.$$

The value $4/3$ is actually attained on a principally polarized simple abelian surface, see [20, Proposition 2].

5.3. Line bundles of degree 3

The primitive solution to the Pell's equation is now $p_0 = 1$ and $q_0 = 2$. Hence the exceptional set in this case is

$$\mathrm{Exc}(3; 1, 2) = \left\{1, \frac{4}{3}\right\}.$$

Let $f\colon X \to \mathbb{P}^2$ be the blow up of a point $P \in \mathbb{P}^2$ with the exceptional divisor E and let $H = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Then for the line bundle $L = 2H - E$ we have $(L^2) = 3$ and $\varepsilon(L; 1) = 1$, the Seshadri constant at a point $Q \in X$ being computed by the proper transform of the line passing through P and Q (this applies also to Q infinitesimally near to P).

The exceptional value $4/3$ is excluded by Proposition 2.2. If $\rho(X) = 1$, then Conjecture 1.4 holds by Theorem 4.1, hence $\varepsilon(L; 1) \geq 3/2$.

5.4. Line bundles of degree 5

The primitive solution to Pell's equation is now $p_0 = 4$ and $q_0 = 9$.

The lower bound predicted by Conjecture 1.4 equals in this case 20/9, while Theorem 1.2 leaves us with the set of 2401 possible exceptional pairs (a, b) satisfying

$$2 \leq b \leq 80 \quad \text{and} \quad b + 1 \leq a < \frac{20}{9}b.$$

By making sure that the pairs above satisfy the inequality (3.3), we reduce the number of exceptions to 41, starting out with

$$(4, 2), (6, 3), (8, 4), (10, 5), (11, 5), (13, 6), (15, 7), (17, 8), \ldots$$

and ending with $(151, 68)$. Thus for a line bundle L with $(L^2) = 5$ we have

$$\text{either} \quad \varepsilon(L; 1) \geq \frac{20}{9} \quad \text{or} \quad \varepsilon(L; 1) \in \left\{1, 2, \frac{11}{5}, \frac{13}{6}, \frac{15}{7}, \frac{17}{8}, \ldots\right\} \tag{5.1}$$

where the latter set consists of 28 values (some exceptional pairs give the same value of the Seshadri constant). This list cannot be further reduced with our methods for an arbitrary surface X and an arbitrary line bundle L with $(L^2) = 5$. We show here surfaces with the two least values of $\varepsilon(L; 1)$ in the list (5.1).

Example 5.2 ($N = 5$ and $\varepsilon(L; 1) = 1$)**.** Let $f : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ in a point P. Let as usual $H = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$ and let $L = 3H - 2E$, where E is the exceptional divisor of f. Let $x \in X$ be a generic point. In particular x is not a point on the exceptional divisor, so that x can be viewed also as a point on the projective plane $\mathbb{P}^2$. Let C_x be the line joining x and P. Then its proper transform D_x on X has the class $H - E$. Thus

$$\varepsilon(L; x) \leq \frac{L.D_x}{1} = 1.$$

Since $\varepsilon(L; 1) \geq 1$ by Theorem 1.2, we conclude that $\varepsilon(L; 1) = 1$ in this case.

Example 5.3 ($N = 5$ and $\varepsilon(L; 1) = 2$)**.** Let $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 3 general points P, Q, R with exceptional divisors E, F, G. Let $H = f^*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ and let $L = 2H - E - F - G$. Let x be a generic point on X. In particular x is not contained in the union of the exceptional divisors $E \cup F \cup G$. Let C_x be a curve of type $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through P, Q and x. Then its proper transform D_x on X has the class $H - E - F$. Hence

$$\varepsilon(L; x) \leq \frac{L.D_x}{1} = 2.$$

With a little more care one can show that in this case indeed $\varepsilon(L; 1) = 2$.

It turns out that we can reduce further the number of possibilities by imposing the condition $\rho(X) = 1$ on X. Let L be an ample generator with $(L^2) = 5$. Then for any curve $C \in |kL|$ for some positive integer k, 5 divides $a = (L \cdot C)$. This leaves only the pairs

$$(10,5),(15,7),(35,16),(55,25),(75,34).$$

In summary, if X is a smooth projective surface with Picard number one, L an ample line bundle on X with $(L^2) = 5$, then

$$\text{either} \quad \varepsilon(L;1) \geq \frac{20}{9} \quad \text{or} \quad \varepsilon(L;1) \in \left\{2, \frac{11}{5}, \frac{15}{7}, \frac{35}{16}, \frac{75}{34}\right\}.$$

Remark 5.4. It is generally expected that for a generic surface X of degree $d \geq 5$ in $\mathbb{P}^3$ there is $\varepsilon(\mathcal{O}_X(1);1) = \sqrt{d}$. The best available lower bounds can be read of [22, Theorem 2.1]. For a generic quintic surface this lower bound is 15/7.

5.5. Line bundles of degree 6

The primitive solution to Pell's equation is now $p_0 = 2$ and $q_0 = 5$. A computer count shows that the set $\mathrm{Exc}(6;2,5)$ has 252 elements

$$\mathrm{Exc}(6) = \left\{1, \frac{25}{24}, \frac{24}{23}, \frac{23}{22}, \ldots, \frac{31}{13}, \frac{43}{18}, \frac{55}{23}\right\}.$$

A considerable number of these values can be discarded using Proposition 2.2. The modified set $\mathrm{Exc}'(6;2,5)$ still contains 51 elements

$$\mathrm{Exc}'(6) = \left\{\frac{17}{8}, \frac{49}{23}, \frac{32}{15}, \ldots, \frac{31}{13}, \frac{43}{18}, \frac{55}{23}\right\}.$$

However, if $\rho(X) = 1$, then Theorem 4.2 implies that Conjecture 1.4 holds, so that all these 51 possible values are also discarded under this assumption. This underlines the power of Theorem 4.2. It is also worth to remark here that simple application of Proposition 2.1 (i.e., without taking into account that all involved numbers are integers and subject to certain divisibilities) would leave the following set of 6 exceptional values

$$\mathrm{Exc}''(6;2,5) = \left\{\frac{9}{4}, \frac{7}{3}, \frac{26}{11}, \frac{19}{8}, \frac{31}{13}, \frac{43}{18}\right\}.$$

This shows that arithmetic flavor arguments as in the proof of Theorem 4.2, even though simple, are in fact inevitable.

5.6. Line bundles of degree 7

The primitive solution to Pell's equation is now $p_0 = 3$ and $q_0 = 8$. We consider now only surfaces with $\rho(X) = 1$.

Theorem 5.5. *Assume that X is a surface with $\rho(X) = 1$ and let L be an ample generator with $(L^2) = 7$. Then*

$$\textit{either} \quad \varepsilon(L;1) \geq \frac{21}{8} \quad \textit{or} \quad \varepsilon(L;1) = \frac{28}{11}.$$

Proof. Taking into account that $\rho(X) = 1$ and the divisibility condition $7 \mid a$ the list of possible exceptional values of $\varepsilon(L;1)$ is reduced to

$$(7,3), (28,11), (49,19).$$

We show how to exclude the pairs $(7,3)$ and $(49,19)$. Since the Seshadri curve C is singular in both cases, these curves form a 2-dimensional family. Then by Proposition 2.1 with either $(C^2) = 7$ and $q = 3$, or $(C^2) = 343$ and $q = 19$ we obtain $\operatorname{gon}(C) = 1$. Hence X is covered by rational curves. As these curves intersect, X is actually a rational surface. But the assumption $\rho(X) = 1$ forces X to be $\mathbb{P}^2$. This contradicts the assumption that the ample generator of the Picard group has degree 7. □

5.7. Line bundles of degree 8

The primitive solution to Pell's equation is now $p_0 = 1$ and $q_0 = 3$. The set $\operatorname{Exc}(8;1,3)$ contains 37 elements ranging from 1 to 21/8. Applying Proposition 2.2 the list reduces to

$$\operatorname{Exc}'(8;1,3) = \left\{ \frac{5}{2}, \frac{18}{7}, \frac{13}{5}, \frac{21}{8} \right\}.$$

Assuming additionally that $\rho(X) = 1$, the list gets empty in accordance to Theorem 4.1.

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