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On the Kinematic Formula of the Total Mean Curvature Matrix

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Abstract. In an earlier paper [23] the authors introduced a new ellipsoid associated with a submanifold, and established an integral formula for the total mean curvature matrix of hypersurfaces. In the present paper a kinematic formula for the total mean curvature matrix of submanifolds in $\mathbb{R}^n$ is proved.

1. Introduction

The kinematic formulas, based on invariant measures on the sets of random geometric objects, such as closed curves, convex domains, linear spaces and submanifolds in the Euclidean space $\mathbb{R}^n$ or other space forms, are very important and useful in integral geometry.

Let G be a unimodular Lie group with kinematic density dg and H a closed subgroup of G. Assume that there exists invariant Riemannian metric in the homogeneous space G/H. Let M_0 and M_1 be two compact submanifolds of dimensions q, r in G/H, respectively, M_0 fixed and gM_1 the image of M_1 under a motion $g \in G$. Denote by $I(M_0 \cap gM_1)$ an invariant of the intersected submanifold $M_0 \cap gM_1$. Then evaluating the integral of type

$$\int_{\{g\in G: M_0 \cap gM_1 \neq \emptyset\}} I(M_0 \cap gM_1)\, dg \tag{1.1}$$

and expressing by the integral invariants of M_0 and M_1 is called the kinematic formula for $I(M_0 \cap gM_1)$. For different spaces G/H (such as $\mathbb{R}^n$ or other spaces of constant curvature) and various submanifolds M_0, M_1 (such as closed curves, surfaces, connected domains), letting $I(M_0 \cap gM_1)$ be the volume, area, curvature or other invariants leads to the famous Poincaré formula, Blaschke formula, Chern-Federer kinematic formula, C.-S. Chen kinematic formula and so on.

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For instance, in his classical paper [2] Chern proved the kinematic fundamental formula in $\mathbb{R}^n$. Let D_0 and D_1 be two domains with smooth hypersurfaces ∂D_0 and ∂D_1 in $\mathbb{R}^n$, respectively. Denote by G the group of rigid motions of $\mathbb{R}^n$ with the density dg. Then the kinematic fundamental formula is

$$\begin{aligned}&\int_{\{g\in G: D_0\cap gD_1\neq\emptyset\}} \chi(D_0\cap gD_1)\,dg\\ &= O_{n-2}\cdots O_1\Bigg[O_{n-1}\chi(D_0)\operatorname{Vol}(D_1)+O_{n-1}\chi(D_1)\operatorname{Vol}(D_0)\\ &\qquad + \frac{1}{n}\sum_{h=0}^{n-2}\binom{n}{h+1}\widetilde{H}_h(D_0)\widetilde{H}_{n-h-2}(D_1)\Bigg],\end{aligned}$$

where $\chi(\cdot)$ denotes the Euler characteristic, $\operatorname{Vol}(\cdot)$ the volume, O_{n-1} the volume of the unit sphere S^{n-1} in $\mathbb{R}^n$ and $\widetilde{H}_i(\cdot)$ the ith total mean curvature.

In [27], Zhou obtained the kinematic formula for mean curvature powers of hypersurfaces in $\mathbb{R}^n$, which is the generalization of the formulas of the 3-dimensional case in [1]. Let S_i $(i=0,1)$ be two compact smooth hypersurfaces of class C^2 in $\mathbb{R}^n$. Denote by H the mean curvature of $S_0\cap gS_1$. Then for any integral k with $0\leq 2k\leq n-1$,

$$\int_G\left(\int_{S_0\cap gS_1} H^{2k}\,d\sigma\right)dg = \sum_{\substack{i,j,\ell\\ i+j+\ell=k\\ \ell:\text{even}}} c_{ijk\ell n}\widetilde{\kappa}_n^{\ell+2j}(S_0)\widetilde{\kappa}_n^{\ell+2i}(S_1),$$

where $\widetilde{\kappa}_n^r(S_i)$ is the integral of the principal curvatures of S_i and $c_{ijk\ell n}$ depends on i,j,k,ℓ,n. This is a typical work where the moving frame method is effectively used, and an application of kinematic formulas is given. So far this approach to achieve geometric inequalities and estimate the containment problem has been systematically developed. For the recent developments, interested readers can refer to [5,17,22,24,26,27].

It is known that ellipsoid plays an important role in the study of geometric inequalities, Banach space geometry, PDEs and valuation theory. See [4,6–16,18–20,25,28,29] for detailed information. Besides, ellipsoid has an interesting relationship with kinematic formulas. In the paper [23], Zeng, Xu, Zhou and Ma introduced the total mean curvature ellipsoid E_M associated with a submanifold M in $\mathbb{R}^n$ in differential geometry. The positive semi-definite symmetric matrix corresponding to this ellipsoid is called the total mean curvature matrix $\mathcal{H}(M)$. The concept of total mean curvature matrix extends the scalar invariant (the total mean curvature) to a matrix invariant. The authors also established a kinematic formula of the total mean curvature matrix of hypersurfaces in $\mathbb{R}^n$. As a consequence, taking the trace of the kinematic formula gave a scalar integral formula which is proved by Zhou [27] in $\mathbb{R}^n$.

In this paper, we consider the kinematic formula of the total mean curvature matrix of submanifolds in $\mathbb{R}^n$. Let M be a closed submanifold in $\mathbb{R}^n$ with scalar curvature R_M. Let

$\widetilde{H}^{(2)} = \int_M \left|\overrightarrow{H}_M\right|^2 dS_M$ and $\widetilde{R} = \int_M R_M \, dS_M$ be the total square mean curvature and the total scalar curvature of M, respectively. Denote by $\mathrm{Vol}(M)$ the volume of submanifold M. Denote by $O(n)$ the group of rotations in $\mathbb{R}^n$ and by $d\alpha$ the invariant measure of the orthogonal group $O(n)$ normalized so that the total measure is $O_{n-1}\cdots O_0$, where O_i, $i=0,1,\dots,n-1$, is the i-dimensional surface area of the unit sphere in $\mathbb{R}^{i+1}$. Denote by $G(n)$ the group of rigid motions in $\mathbb{R}^n$, and by dg the invariant measure of the group $G(n)$ which is the product measure of the Lebesgue measure of $\mathbb{R}^n$ and the invariant measure of $SO(n)$, where the invariant measure of $SO(n)$ is normalized so that the total measure is $O_{n-1}\cdots O_1$.

We obtain the following kinematic formula and it is of extrinsic type.

Theorem 1.1. *Let M_i $(i=0,1)$ be a pair of closed submanifolds with dimensions p, q in $\mathbb{R}^n$ with volume $\mathrm{Vol}(M_i)$, total scalar curvature $\widetilde{R}_i$, and total square mean curvature $\widetilde{H}_i^{(2)}$. Let $\mathcal{I}$ be the $n\times n$ identity matrix, then*

$$\int_{\alpha\in O(n), g\in G(n)} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg = c(M_0, M_1)\mathcal{I}, \tag{1.2}$$

where the coefficient $c(M_0,M_1)$ depends only on p, q and n with value

$$\begin{aligned} c(M_0,M_1) = {} & C_0'' \left[(p-1)p^2(p+q-n+2)\widetilde{H}_0^{(2)} - 4(n-q)\widetilde{R}_0\right] \mathrm{Vol}(M_1) \\ & + C_2'' \left[(q-1)q^2(p+q-n+2)\widetilde{H}_1^{(2)} - 4(n-p)\widetilde{R}_1\right] \mathrm{Vol}(M_0), \end{aligned}$$

and

$$\begin{aligned} C_0'' &= \frac{O_{p-1}O_1^2\cdots O_{n-1}^2 O_n O_{q-1} O_{p+q-n+1} O_{p+q-n}}{(p+q-n)(p-1)p(p+2)O_{p+q-n-1}O_{p-1}O_p O_q O_{q+1}}, \\ C_2'' &= \frac{O_{q-1}O_1^2\cdots O_{n-1}^2 O_n O_{p-1} O_{p+q-n+1} O_{p+q-n}}{(p+q-n)(q-1)q(q+2)O_{p+q-n-1}O_{q-1}O_q O_p O_{p+1}}. \end{aligned}$$

In contrast to the usual integral formulas in integral geometry which are scalar-type formulas, the integral formula proved in Theorem 1.1 is a matrix-type formula.

Remark 1.2. Considering a special case: if the integrand in (1.2) is the trace of $\mathcal{H}(\alpha M_0 \cap gM_1)$, Theorem 1.1 gives the integral formula proved by Chen in 1973 (see [1]). Note that the result of [1] plays an important role in both differential geometry and integral geometry and are widely used.

In addition, let M_0, M_1 be a pair of closed C^2 hypersurface in $\mathbb{R}^n$ $(n\geq 3)$, Zeng, Xu, Zhou and Ma [23] gave the integral formula for the total mean curvature matrix associated with hypersurfaces. Theorem 1.1 is a generalization of the result in [23].

Next, assume that M_0 and M_1 are a pair of closed minimal submanifolds in $\mathbb{R}^n$. Since the mean curvature of the minimal submanifold vanishes, we have the following results.

Corollary 1.3. *Let M_i $(i = 0, 1)$ be a pair of closed minimal submanifolds with dimensions p, q in $\mathbb{R}^n$ with volume $\mathrm{Vol}(M_i)$ and total scalar curvature $\widetilde{R}_i$. Let $\mathcal{I}$ be the $n \times n$ identity matrix, then*

$$\int_{\alpha\in O(n),g\in G(n)} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg = c(M_0, M_1)\mathcal{I},$$

where $c(M_0, M_1) = -4C_0''(n-q)\widetilde{R}_0 \mathrm{Vol}(M_1) - 4C_2''(n-p)\widetilde{R}_1 \mathrm{Vol}(M_0)$.

2. Preliminaries

In this section, we review some basic facts about the mean curvature vector and the total mean curvature ellipsoid of a submanifold in $\mathbb{R}^n$.

2.1. The mean curvature vector of a submanifold

Let M be a p-dimensional submanifold in $\mathbb{R}^n$. Choose an orthonormal frame $\{e_1, \dots, e_n\}$ at $x \in M$ in $\mathbb{R}^n$ so that $\{e_1, \dots, e_p\}$ is a basis of the tangent space $T_x M$ and $\{e_{p+1}, \dots, e_n\}$ is a basis of the normal space $T_x^\perp M$. We take the following convention on the ranges of indices:

$$1 \le i, j \le p, \quad p+1 \le \alpha, \beta \le n.$$

Let $\{\omega_1, \dots, \omega_n\}$ be the dual orthonormal frame of $\{e_1, \dots, e_n\}$. The fundamental equations of M are the following,

$$\begin{aligned} dx &= \sum_i \omega_i e_i, \\ de_i &= \sum_j \omega_{ij} e_j + \sum_\alpha \omega_{i\alpha} e_\alpha, \\ de_\alpha &= \sum_i \omega_{\alpha i} e_i + \sum_\beta \omega_{\alpha\beta} e_\beta, \end{aligned}$$

where $\omega_{i\alpha} = -\omega_{\alpha i}$, $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ and $\omega_{ij} = -\omega_{ji}$.

Restricted to M, we obtain $0 = d\omega_\alpha = \sum_i \omega_{\alpha i} \wedge \omega_i$. By Cartan's lemma, we have

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

The quantities h_{ij}^α are the components of the second fundamental form of M.

The mean curvature vector $\overrightarrow{H}$ is defined by

$$\overrightarrow{H}_M = \frac{1}{p} \sum_\alpha \left(\sum_i h_{ii}^\alpha \right) e_\alpha,$$

and the mean curvature of M is

$$H_M = \frac{1}{p} \left[\sum_\alpha \left(\sum_i h_{ii}^\alpha \right)^2 \right]^{1/2},$$

that is the norm of the mean curvature vector. Then the total scalar curvature of M is given by

$$\widetilde{H}^{(2)} = \int_M \left|\overrightarrow{H}_M\right|^2 dS_M,$$

where dS_M is the volume element of M.

Similarly, let R_M be the scalar curvature of M. The total scalar curvature is denoted by

$$\widetilde{R} = \int_M R_M \, dS_M.$$

2.2. The mean curvature ellipsoid associated with a submanifold

Recently, Zeng, Xu, Zhou and Ma [23] defined a new ellipsoid E_M associated with a submanifold M in $\mathbb{R}^n$ from the point of differential geometry. Let M be a closed submanifold in $\mathbb{R}^n$ and $\overrightarrow{H}_M(y)$ the mean curvature vector at $y \in M$. The new ellipsoid can be defined by the total mean curvature matrix

$$\mathcal{H}(M) = \frac{1}{2}\int_M \overrightarrow{H}_M(y) \otimes \overrightarrow{H}_M(y) \, dS_M(y).$$

Since $\overrightarrow{H}_M(y) \otimes \overrightarrow{H}_M(y) = \overrightarrow{H}_M(y)\overrightarrow{H}^t_M(y)$, it is a positive semi-define symmetric matrix. Besides the trace of the total mean curvature matrix recovers the total square mean curvature. Notice that the total square mean curvature of a submanifold in $\mathbb{R}^n$ is an important global differential geometric invariant.

The total mean curvature ellipsoid E_M associated with M is defined by

$$E_M = \left\{x \in \mathbb{R}^n : x^t \mathcal{H}(M) x \leq 1\right\}.$$

Now we show some properties of the total mean curvature matrix (see [23]).

Proposition 2.1. *Let M be a submanifold in $\mathbb{R}^n$ and $\mathcal{H}(M)$ be the total mean curvature matrix of M. Then*

(1) *for any rotation α in $\mathbb{R}^n$, there is*

$$\mathcal{H}(\alpha M) = \alpha \mathcal{H}(M) \alpha^t;$$

(2) *for a vector $x \in \mathbb{R}^n$, the quadratic form $x^t\mathcal{H}(M)x$ is given by*

$$x^t \mathcal{H}(M) x = \frac{1}{2}\int_M \left|\overrightarrow{H}_M(y) \cdot x\right|^2 dS_M(y);$$

(3) *for the trace of $\mathcal{H}(M)$, there is*

$$\operatorname{Tr}\mathcal{H}(M) = \frac{1}{2}\int_M \left|\overrightarrow{H}_M(y)\right|^2 dS_M(y);$$

(4) *for $\alpha \in O(n)$, there is*

$$\operatorname{Tr}\mathcal{H}(\alpha M) = \operatorname{Tr}\mathcal{H}(M).$$

3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need the following Lemma 3.1 where we mainly use Chen's idea in [1] and the moving frame method.

Lemma 3.1. *Let M_i $(i = 0, 1)$ be a pair of closed submanifolds with dimensions p, q in $\mathbb{R}^n$ with volume $\mathrm{Vol}(M_i)$, total scalar curvature $\widetilde{R}_i$, and total square mean curvature $\widetilde{H}_i^{(2)}$. Denote by $\mathrm{Tr}(\mathcal{H}(M_0 \cap gM_1))$ the trace of total mean curvature matrix of $M_0 \cap gM_1$. Then*

$$\begin{aligned}
&\int_{G(n)} \mathrm{Tr}(\mathcal{H}(M_0 \cap gM_1))\, dg \\
&= C_0 \left[(p-1)p^2(p+q-n+2)\widetilde{H}_0^{(2)} - 4(n-q)\widetilde{R}_0\right] \mathrm{Vol}(M_1) \\
&\quad + C_2 \left[(q-1)q^2(p+q-n+2)\widetilde{H}_1^{(2)} - 4(n-p)\widetilde{R}_1\right] \mathrm{Vol}(M_0),
\end{aligned} \tag{3.1}$$

where

$$C_0 = \frac{1}{2(p+q-n)(p-1)p(p+2)} \frac{O_{p-1}}{O_{p+q-n-1}} \frac{O_n \cdots O_1 O_{q-1} O_{p+q-n+1} O_{p+q-n}}{O_{p-1} O_p O_q O_{q+1}},$$
$$C_2 = \frac{1}{2(p+q-n)(q-1)q(q+2)} \frac{O_{q-1}}{O_{p+q-n-1}} \frac{O_n \cdots O_1 O_{p-1} O_{p+q-n+1} O_{p+q-n}}{O_{q-1} O_q O_p O_{p+1}}.$$

Proof. First, we recall a special case of the basic formula (see (47) of [3]):

$$\Phi_g\, dg = \pm \Delta^2 dh_y \Theta_0 \Theta_1, \tag{3.2}$$

where Φ_g is the density element of all 1-frames of $M_0 \cap gM_1$ and Δ is the generalized angle between the tangent spaces of M_0 and M_1 and dh_y is the density element of the group of isotropy at a 1-form y of M_1, and Θ_0, Θ_1 are density elements of all 1-frames of M_0, M_1 respectively. The kinematic density dg is the invariant measure of $G(n)$ and has the decomposition $dg = dx d\gamma$, where dx is the Lebesgue measure of $\mathbb{R}^n$ and $d\gamma$ is the invariant measure of $SO(n)$. The interested readers can refer to [3, Section 3, pp. 106–109] for the precise definitions of the density elements of Θ_0, Θ_1 and dh_y.

Let ∇ be the covariant derivative of $\mathbb{R}^n$ so that the directional derivative of a vector field X along another Y is $\nabla_Y X$. If M is immersed in $\mathbb{R}^n$ and X, Y are tangent to M, then at $x \in M$,

$$T_X Y = \text{normal component of } \nabla_X Y \text{ with respect to } M_x.$$

Denote by T^p the second fundamental form of M_0, and by T^q and T^{p+q-n} the second fundamental form of M_1 and $M_0 \cap gM_1$ respectively. Assume that M_0 is a submanifold of M_1 which is immersed in $\mathbb{R}^n$. For $X, Y \in T_x^p$, the tangent space of M_0 at x, then

$$T_X^q Y = \text{normal component of } T_X^p Y \text{ with respect to } (M_1)_x.$$

From now on, let t be a unit vector tangent to $M_0 \cap gM_1$ and use $\left\| T_t^{p+q-n} t \right\|^2$ as the integrand of both sides of the formula (3.2). The first step of the integration is carried out by fixed a unit tangent vector t in M_0 and a unit tangent vector in M_1 and integrating over the isotropic group. Then by the moving frame method, Cramer's rule and (3.2), Chen [1] obtained:

$$\int_{G(n)} \tau(M_0 \cap gM_1)\, dg = C_0' \tau(M_0) \operatorname{Vol}(M_1) + C_2' \tau(M_1) \operatorname{Vol}(M_0), \tag{3.3}$$

where $\tau(M_0) = \displaystyle\int \left\| T_t t \right\|^2 \Theta_0$, Θ_0 is the density of all 1-frames of M_0, and

$$C_0' = \frac{O_n O_{n-1} \cdots O_1 O_{q-1} O_{p+q-n+1} O_{p+q-n}}{O_{p-1} O_p O_{q+1} O_q},$$
$$C_2' = \frac{O_n O_{n-1} \cdots O_1 O_{p-1} O_{p+q-n+1} O_{p+q-n}}{O_{q-1} O_q O_{p+1} O_p}.$$

In the next step, for any M_0 of dimension p in $\mathbb{R}^n$, it shows that $\tau(M_0)$ can be expressed in terms of some well-known geometric quantities. One can notice that this is a pointwise calculus problem as was done by Weyl in [21]. Based on Weyl's idea (the average formula), then

$$\tau(M_0) = \frac{3p}{p+2} O_p \widetilde{H}_0^{(2)} - \frac{4}{p(p+2)} O_p \widetilde{R}_0. \tag{3.4}$$

Then (3.3) and (3.4) imply that (see [1])

$$\begin{aligned} O_{p+q-n-1} \int_{G(n)} \Bigg[& \frac{3(p+q-n)}{p+q-n+2} \widetilde{H}^{(2)}(M_0 \cap gM_1) \\ & - \frac{4}{(p+q-n)(p+q-n+2)} \widetilde{R}(M_0 \cap gM_1) \Bigg] dg \\ = C_0' O_{p-1} & \left[\frac{3p}{p+2} \widetilde{H}_0^{(2)} - \frac{4}{p(p+2)} \widetilde{R}_0 \right] \operatorname{Vol}(M_1) \\ + C_2' O_{q-1} & \left[\frac{3q}{q+2} \widetilde{H}_1^{(2)} - \frac{4}{q(q+2)} \widetilde{R}_1 \right] \operatorname{Vol}(M_0). \end{aligned} \tag{3.5}$$

In [3] the kinematic formula for $\mu_2(X) = \frac{1}{m(m-1)} \widetilde{R}(X)$, where $m = \dim X$, one has

$$\begin{aligned} & \frac{1}{(p+q-n)(p+q-n-1)} \int_{G(n)} \widetilde{R}(M_0 \cap gM_1)\, dg \\ = & C_0' \frac{O_{p-1}}{O_{p+q-n-1}} \frac{1}{p(p-1)} \widetilde{R}_0 \operatorname{Vol}(M_1) + C_2' \frac{O_{q-1}}{O_{p+q-n-1}} \frac{1}{q(q-1)} \widetilde{R}_1 \operatorname{Vol}(M_0). \end{aligned} \tag{3.6}$$

Combining (3.5), (3.6) and from Proposition 2.1(3), we have

$$\begin{aligned}
&\int_{G(n)} \operatorname{Tr}(\mathcal{H}(M_0 \cap gM_1))\, dg \\
&= C_0 \left[(p-1)p^2(p+q-n+2)\widetilde{H}_0^{(2)} - 4(n-q)\widetilde{R}_0\right] \operatorname{Vol}(M_1) \\
&+ C_2 \left[(q-1)q^2(p+q-n+2)\widetilde{H}_1^{(2)} - 4(n-p)\widetilde{R}_1\right] \operatorname{Vol}(M_0),
\end{aligned} \tag{3.7}$$

where

$$C_0 = \frac{C_0'}{2(p+q-n)(p-1)p(p+2)} \frac{O_{p-1}}{O_{p+q-n-1}}, \quad C_2 = \frac{C_2'}{2(p+q-n)(q-1)q(q+2)} \frac{O_{q-1}}{O_{p+q-n-1}}. \quad \square$$

Next we turn our attention to prove Theorem 1.1.

Proof of Theorem 1.1. Consider the following quadratic form

$$Q(x) = x^t \left(\int_{O(n)\times G(n)} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg \right) x.$$

For any rotation α_0, the quadratic form continues as follows:

$$\begin{aligned}
Q(\alpha_0^t x) &= (\alpha_0^t x)^t \left(\int_{O(n)\times G(n)} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg \right) (\alpha_0^t x) \\
&= x^t \left(\int_{O(n)\times G(n)} \mathcal{H}(\alpha_0(\alpha M_0 \cap gM_1))\, d\alpha dg \right) x \\
&= x^t \left(\int_{O(n)\times G(n)} \mathcal{H}((\alpha_0\alpha) M_0 \cap (\alpha_0 g)M_1)\, d(\alpha_0\alpha) d(\alpha_0 g) \right) x \\
&= x^t \left(\int_{O(n)\times G(n)} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg \right) x \\
&= Q(x),
\end{aligned}$$

where in the second and third step we use Proposition 2.1(1) and the invariance of the kinematic density, respectively.

The above property of rotation invariance of $Q(x)$ shows that

$$Q(x) = x^t \left(\int_{O(n)\times G(n)} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg \right) x = c(M_0, M_1)\, |x|^2 \tag{3.8}$$

for some constant $c(M_0, M_1) > 0$. The equation (3.8) implies that

$$\int_{O(n)\times G(n)} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg = c(M_0, M_1)\mathcal{I}. \tag{3.9}$$

From now on, we come to the position to compute $c(M_0, M_1)$. Note that the coefficient $c(M_0, M_1)$ depends only on p, q and n and the traces of both sides of (3.9) are equal, so

$$\begin{aligned} nc(M_0, M_1) &= \operatorname{Tr}\left(\int_{O(n)\times G(n)} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg\right) \\ &= \sum_{i=1}^{n} e_i^t \left(\int_{O(n)\times G(n)} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg\right) e_i \\ &= \sum_{i=1}^{n} \int_{O(n)\times G(n)} \left(e_i^t \mathcal{H}(\alpha M_0 \cap gM_1) e_i\right) d\alpha dg \\ &= \int_{O(n)\times G(n)} \operatorname{Tr} \mathcal{H}(\alpha M_0 \cap gM_1)\, d\alpha dg, \end{aligned} \tag{3.10}$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$.

Furthermore, by Proposition 2.1(4) and the invariance of dg, we have

$$\begin{aligned} &\int_{O(n)\times G(n)} \operatorname{Tr}\left(\mathcal{H}(\alpha M_0 \cap gM_1)\right) d\alpha dg \\ &= \int_{O(n)} \left(\int_{G(n)} \operatorname{Tr}\left(\mathcal{H}(\alpha M_0 \cap \alpha^{-1} gM_1)\right) dg\right) d\alpha \\ &= \int_{O(n)} \left(\int_{G(n)} \operatorname{Tr}\left(\mathcal{H}(M_0 \cap (\alpha^{-1} g)M_1)\right) d(\alpha^{-1} g)\right) d\alpha \\ &= \int_{O(n)} \left(\int_{G(n)} \operatorname{Tr}\left(\mathcal{H}(M_0 \cap gM_1)\right) dg\right) d\alpha \\ &= O_0 O_1 \cdots O_{n-1} \int_{G(n)} \operatorname{Tr}\left(\mathcal{H}(M_0 \cap gM_1)\right) dg. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), the coefficient in (3.9) is given by

$$c(M_0, M_1) = \frac{1}{n} O_0 O_1 \cdots O_{n-1} \int_{G(n)} \operatorname{Tr}\left(\mathcal{H}(M_0 \cap gM_1)\right) dg.$$

From Lemma 3.1, we have

$$\begin{aligned} &c(M_0, M_1) \\ &= \frac{O_0 O_1 \cdots O_{n-1}}{n} \Big\{ C_0 \left[(p-1)p^2(p+q-n+2)\widetilde{H}_0^{(2)} - 4(n-q)\widetilde{R}_0\right] \operatorname{Vol}(M_1) \\ &\qquad + C_2 \left[(q-1)q^2(p+q-n+2)\widetilde{H}_1^{(2)} - 4(n-p)\widetilde{R}_1\right] \operatorname{Vol}(M_0) \Big\}. \end{aligned} \tag{3.12}$$

Let

$$C_0'' = \frac{C_0 O_0 O_1 \cdots O_{n-1}}{n} = \frac{O_{p-1} O_1^2 \cdots O_{n-1}^2 O_n O_{q-1} O_{p+q-n+1} O_{p+q-n}}{(p+q-n)(p-1)p(p+2) O_{p+q-n-1} O_{p-1} O_p O_q O_{q+1}}$$

and

$$C_2'' = \frac{C_n O_0 O_1 \cdots O_{n-1}}{n} = \frac{O_{q-1} O_1^2 \cdots O_{n-1}^2 O_n O_{p-1} O_{p+q-n+1} O_{p+q-n}}{(p+q-n)(q-1)q(q+2) O_{p+q-n-1} O_{q-1} O_q O_p O_{p+1}}.$$

Therefore,

$$c(M_0, M_1) = C_0'' \left[(p-1)p^2(p+q-n+2)\widetilde{H}_0^{(2)} - 4(n-q)\widetilde{R}_0\right] \mathrm{Vol}(M_1) \\ + C_2'' \left[(q-1)q^2(p+q-n+2)\widetilde{H}_1^{(2)} - 4(n-p)\widetilde{R}_1\right] \mathrm{Vol}(M_0).$$

We complete the proof of Theorem 1.1. □

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