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A Finiteness Result for Inverse Three Spectra Sturm-Liouville Problems

Ying Yang* and Guangsheng Wei

Abstract. The finiteness for an inverse three spectra Sturm-Liouville problem with potential q on the interval $[0,1]$ and boundary parameters h_0, h_1 is studied in this paper. Under condition that two boundary conditions at a fixed internal rational point a of $(0,1)$ are different and known a priori, we show that there exist at most a finite number of triplets $(q; h_0, h_1)$ corresponding to the three spectra of a Sturm-Liouville equation defined on $[0,1]$, $[0,a]$ and $[a,1]$, respectively, with the same boundary conditions at two endpoints 0 and 1.

1. Introduction

The main goal of this paper is to concern the finiteness problem of recovering the potential q on the interval $[0,1]$ of a Sturm-Liouville equation

$$-u'' + q(x)u = \lambda u \tag{1.1}$$

using three spectra $\sigma(L) = \{\lambda_n\}_{n=0}^{\infty}$, $\sigma(L^-) = \{\mu_n^-\}_{n=0}^{\infty}$ and $\sigma(L^+) = \{\mu_n^+\}_{n=0}^{\infty}$ corresponding to three Sturm-Liouville problems L, L^- and L^+, which are generated respectively by (1.1) defined on $[0,1]$, $[0,a]$ and $[a,1]$ and the following Robin boundary conditions

$$u'(0) + h_0 u(0) = 0 = u'(1) + h_1 u(1), \tag{1.2}$$

$$u'(0) + h_0 u(0) = 0 = u'(a) + h_- u(a), \tag{1.3}$$

$$u'(a) + h_+ u(a) = 0 = u'(1) + h_1 u(1). \tag{1.4}$$

Here all the boundary parameters h_0, h_1, h_-, h_+ belong to $\mathbb{R}$, the potential $q \in L^1[0,1]$ is real-valued and $a \in (0,1)$ is fixed.

In the literature there are many results (see [1, 2, 5, 6, 9–11, 13] and the references therein) related to the inverse three spectra problem. This problem was first investigated by Pivovarchik [10] under condition that $a = 1/2$ and $\sigma(L)$ and $\sigma(L^{\mp})$ are the Dirichlet spectra (i.e., all $h_0, h_1, h_\pm = \infty$). Further investigation has been carried out by Gesztesy

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*Corresponding author.

and Simon [5] under the more general situations of $a \in (0,1)$ and Robin spectra. So far, this inverse problem has been studied in various settings, for example, see [1] for distributional potentials, [9] for Jacobi matrices, [2] for Stieltjes strings and [6] for compound systems. We note that Gesztesy and Simon [5] proved uniqueness of the reconstructed q whenever the three spectra do not overlap and suggested a counterexample to uniqueness otherwise. Hryniv and Mykytyuk [6] also discussed the situation of the overlapping of three Dirichlet spectra for the case of singular potentials. However, all the above studies are restricted to the case of $h_- = h_+$.

Our immediate motivation for this paper is a recent research of the second author and X. Wei [13], who considered the case of $h_- \neq h_+$ and established the following extended inverse three spectra theorem.

Theorem 1.1. *Fix $h_+, h_- \in \mathbb{R}$ with $h_+ > h_-$ and let $a = 1/2$. Suppose the following interlacing property holds:*

$$\mu_n^- < \lambda_{2n} < \mu_n^+ < \lambda_{2n+1} < \mu_{n+1}^- \tag{1.5}$$

for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then h_0, h_1 and q a.e. on $[0,1]$ are uniquely determined by the three spectra $\sigma(L)$ and $\sigma(L^\pm)$.

It is worth mentioning that the interlacing property (1.5) of the associated eigenvalues in general does not hold for all $n \in \mathbb{N}_0$, even the three spectra do not overlap for the case of $h_- \neq h_+$. However, we observe that, in appropriate circumstance, (1.5) remains valid when n is sufficiently large. Motivated by this situation, a natural question occurs:

what if we take out condition (1.5)?

Our purpose here is to consider this question in a more general case of Theorem 1.1 that the fixed interior point a is a rational number, namely,

$$a = \frac{m_1}{m_2}, \tag{1.6}$$

where $m_1 < m_2$ and $m_1, m_2 \in \mathbb{N}$ are co-prime.

In this paper, we shall prove that there exist at most a finite number K_0 (say) of triplets $(q; h_0, h_1)$ corresponding to the three spectra of the Sturm-Liouville problems L and $L^\mp$ provided that $h_- \neq h_+$ and the following condition is satisfied

$$\max\{h_-, h_+\} < \mathrm{B}(h_0, h_1; q) \quad \text{or} \quad \min\{h_-, h_+\} > \mathrm{B}(h_0, h_1; q) \tag{1.7}$$

where

$$\mathrm{B}(h_0, h_1; q) = a h_1 + (1-a) h_0 + \frac{a}{2}\int_a^1 q(t)\,dt - \frac{1-a}{2}\int_0^a q(t)\,dt. \tag{1.8}$$

Here K_0 depends only on the norm $\|q\|_{L^1}$ and the boundary parameters h_0, h_1 and $h_\pm$ (see Section 3 below). The finiteness result does not need precondition that three spectra are pairwise disjoint, that is, condition (1.5) can be dropped. In fact, we shall find out that condition (1.7) implies interlacing property (1.5) for sufficiently large n when a is a rational number (see Section 2 for details). This together with Borg's theorem [4] can ensure that at most a finite number of triplets $(q; h_0, h_1)$ correspond to the three spectra of the problems L and $L^\mp$. By the way, we return to consider the uniqueness problem of recovering the potential q.

Similar results may be obtained for the Dirichlet boundary conditions, where $h_0 = \infty$ and/or $h_1 = \infty$ and for the case of $h_+ \neq h_-$. Moreover, the technique used to obtain our result in the paper is based on Borg's two-spectra theorem [4].

The structure of this paper is as follows. In Section 2 we prove results concerning the interlacing property of the associated eigenvalues for sufficiently large $n > N$. Section 3 presents the way to find the N. The finiteness theorem and its proof will be presented in Section 4.

2. Preliminaries

In this section, we shall establish the interlacing property among the associated eigenvalues for sufficiently large n. We begin by considering the initial-value problems of (1.1) with initial conditions

$$u(0) = 1, \quad u'(0) = -h_0, \tag{2.1}$$

$$v(1) = 1, \quad v'(1) = -h_1. \tag{2.2}$$

Let $u := u(x, \lambda)$ and $v := v(x, \lambda)$ denote the solutions of (1.1)–(2.1) and (1.1)–(2.2), respectively. Note that the eigenvalues $\{\lambda_n\}_{n=0}^\infty$ of the problem L are precisely the zeros of the transcendental function

$$\omega(\lambda) = u(x, \lambda)v'(x, \lambda) - u'(x, \lambda)v(x, \lambda), \tag{2.3}$$

where $\omega(\lambda)$ is independent of $x \in [0, 1]$. Similarly, if letting

$$\omega^-(\lambda, h_-) = u'(a, \lambda) + h_- u(a, \lambda), \tag{2.4}$$

$$\omega^+(\lambda, h_+) = v'(a, \lambda) + h_+ v(a, \lambda), \tag{2.5}$$

then the eigenvalues $\{\mu_n^\mp\}_{n=0}^\infty$ of two problems $L^\mp$ are the zeros of the functions $\omega^\mp(\lambda, h_\mp)$, respectively. It is known [4] that $\omega(\lambda)$ and $\omega^\mp(\lambda, h_\mp)$ are entire in λ of the order 1/2 and the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n^{\mp}\}_{n=0}^{\infty}$ have the following asymptotics

$$\lambda_n = \lambda_{1,n} + 2A + \alpha_n, \tag{2.6}$$

$$\mu_n^- = \mu_{1,n}^- + 2A^- + \alpha_n^-, \tag{2.7}$$

$$\mu_n^+ = \mu_{1,n}^+ + 2A^+ + \alpha_n^+, \tag{2.8}$$

where three sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\alpha_n^{\mp}\}_{n=0}^{\infty}$ are infinitely small as $n \to \infty$,

$$\lambda_{1,n} = (n\pi)^2, \quad \mu_{1,n}^- = \left(\frac{m_2}{m_1} n\pi\right)^2, \quad \mu_{1,n}^+ = \left(\frac{m_2}{m_2 - m_1} n\pi\right)^2 \tag{2.9}$$

and

$$\begin{aligned} A &= h_1 - h_0 + \frac{1}{2}\int_0^1 q(t)\,dt, \\ A^- &= \frac{m_2}{m_1}\left(h_- - h_0 + \frac{1}{2}\int_0^{m_1/m_2} q(t)\,dt\right), \\ A^+ &= \frac{m_2}{m_2 - m_1}\left(h_1 - h_+ + \frac{1}{2}\int_{m_1/m_2}^1 q(t)\,dt\right). \end{aligned} \tag{2.10}$$

We first consider the interlacing property between $\{\lambda_{1,n}\}_{n=0}^{\infty}$ and $\{\mu_{1,n}\}_{n=0}^{\infty}$ for all $n \in \mathbb{N}_0$, where

$$\{\mu_{1,n}\}_{n=0}^{\infty} := \{\mu_{1,n}^-\}_{n=0}^{\infty} \cup \{\mu_{1,n}^+\}_{n=0}^{\infty}$$

(counting multiplicity) is an increasing sequence.

Lemma 2.1. *Let $a = m_1/m_2$ be an irreducible fraction with $m_1 < m_2$ and $m_1, m_2 \in \mathbb{N}$. Then for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we have*

$$\mu_{1,n} \le \lambda_{1,n} \le \mu_{1,n+1}. \tag{2.11}$$

Proof. We first prove (2.11) holds for $n = 0, 1, \ldots, m_2$. Without loss of generality, we assume $m_1 > m_2 - m_1$. Consider the sequence of the numbers $a_n^- := nm_2/m_1$ for $n = 0, 1, \ldots, m_1 - 1$ and $a_j^+ := jm_2/(m_2 - m_1)$ for $j = 0, 1, \ldots, m_2 - m_1 - 1$. Once m_1 and m_2 are given, then they can be arrayed as

$$a_0 = a_1 < a_2 < a_3 < \cdots < a_{m_2-1} < a_{m_2}, \tag{2.12}$$

where $a_0 = a_1 = a_0^- = a_0^+$, $a_{m_2-1} = a_{m_1-1}^- = m_2(1 - 1/m_1)$ (because $m_1 > m_2 - m_1$) and $a_{m_2} = m_2$. Note that if $n_0 m_2/m_1 = j_0 m_2/(m_2 - m_1)$ for some $n_0 > 0$ and $j_0 > 0$, then

$$\frac{m_1}{m_2} = \frac{n_0}{n_0 + j_0},$$

which contradicts the precondition that m_1 and m_2 are co-prime. This shows that (2.12) remains valid.

Now, we need to show that the interval (a_j, a_{j+1}) contains j for $j = 1, 2, \ldots, m_2 - 1$. Since $m_1 < m_2$ and therefore $m_2/m_1 > 1$ and $m_2/(m_2 - m_1) > 1$, it follows that if a_j and a_{j+1} are adjacent of $\{a_1^-, \ldots, a_{m_1-1}^-\}$ and $\{a_1^+, \ldots, a_{m_2-m_1-1}^+\}$, respectively, then there exists at least an integer k belonging to (a_j, a_{j+1}). On the other hand, considering another case that two endpoints a_j and a_{j+1} are not adjacent of the a_n^-'s and a_j^+'s, for example, $a_j = n_0 m_2/m_1$ and $a_{j+1} = j_0 m_2/(m_2 - m_1)$; if (a_j, a_{j+1}) does not contain any integer, then $a_{j+1} - a_j \leq 1$ and there exists an integer k_1 satisfying

$$k_1 \leq \frac{m_2}{m_1} n_0 < k_1 + 1 \quad \text{and} \quad k_1 < \frac{m_2}{m_2 - m_1} j_0 \leq k_1 + 1. \tag{2.13}$$

This implies $k_1 < n_0 + j_0 < k_1 + 1$, which is impossible. Therefore, we find each interval (a_j, a_{j+1}) for $j = 1, 2, \ldots, m_2 - 1$ contains at least a positive integer. This together with $a_{m_2-1} = m_2(1 - 1/m_1) < m_2$ yields (a_j, a_{j+1}) contains j. Multiplying π to a_j to (2.13), we conclude that (2.11) holds for $n = 0, 1, \ldots, m_2$.

We next prove (2.11) holds for $n \geq m_2$. In this case, there exists $p \in N$ such that $n = pm_2 + m_0$, where $m_0 \in \{0, 1, \ldots, m_2 - 1\}$. This yields

$$\sqrt{\mu_{1,n}} = (pm_2 + a_{m_0})\pi, \quad \sqrt{\lambda_{1,n}} = (pm_2 + m_0)\pi, \tag{2.14}$$

and therefore (2.11) holds for all $n \in \mathbb{N}_0$. The proof is complete. □

Let us mention that, since a is a rational number and hence it has rotative periodicity, it follows from the above proof that for any $p \in \mathbb{N}$,

$$\begin{aligned} \mu_{1,pm_2} = \lambda_{1,pm_2} = \mu_{1,pm_2+1} < \lambda_{1,pm_2+1} < \mu_{1,pm_2+2} < \cdots \\ < \mu_{1,(p+1)m_2-1} < \lambda_{1,(p+1)m_2-1} < \mu_{1,(p+1)m_2}. \end{aligned} \tag{2.15}$$

Here $\mu_{1,pm_2} = \mu_{1,pm_2+1} = \mu_{1,pm_1}^- = \mu_{1,p(m_2-m_1)}^+$. This together with condition (1.7) will help us to identify the interlacing property between $\{\lambda_n\}_{n=0}^\infty$ and $\{\mu_n\}_{n=0}^\infty$ for sufficiently large n. However, if a is an irrational number, then the rotative periodicity (2.15) does not remain true. Moveover, in general we do not know the exact positions in (2.15) of μ_{1,pm_1+j}^- for $j = 1, 2, \ldots, m_1 - 1$ and $\mu_{1,p(m_2-m_1)+j}^+$ for $j = 1, 2, \ldots, m_2 - m_1 - 1$, but when m_1 and m_2 are given concretely. For example, if $m_1 = 3$ and $m_2 = 10$, then a simple calculation shows that $\mu_{1,10p+5} = \mu_{1,3p+1}^-$ and $\mu_{1,10p+8} = \mu_{1,3p+2}^-$, and other $\mu_{1,10p+k}$ are $\mu_{1,7p+j}^+$ for $j = 1, 2, \ldots, 6$.

We next consider the interlacing property between $\{\lambda_n\}_{n=0}^\infty$ and $\{\mu_n\}_{n=0}^\infty$ for sufficiently large n, where

$$\{\mu_n\}_{n=0}^\infty := \{\mu_n^-\}_{n=0}^\infty \cup \{\mu_n^+\}_{n=0}^\infty \tag{2.16}$$

(counting multiplicity) is an increasing sequence.

Lemma 2.2. *Let a be defined as in Lemma 2.1. Suppose the boundary parameters h_0, h_1, h_- and h_+ defined in (1.2)–(1.4) satisfy (1.7). Then there exists a positive number N such that, for all $n > N$,*

$$\mu_n < \lambda_n < \mu_{n+1}. \tag{2.17}$$

Proof. Without loss of generality, we assume $\max\{h_-, h_+\} < \mathrm{B}(h_0, h_1; q)$ in (1.7). The similar argument can deal with another case. Note that this assumption implies $A - A^- > 0$ and $A^+ - A > 0$. From (2.6)–(2.10) and Lemma 2.1, we have that if $n = pm_2$ for all $p \in \mathbb{N}_0$ then $\mu_n = \mu^-_{pm_1}$, $\mu_{n+1} = \mu^+_{p(m_2-m_1)}$,

$$\begin{aligned} \lambda_n - \mu_n &= \lambda_{1,pm_2} - \mu^-_{1,pm_1} + A - A^- + \alpha_{pm_2} - \alpha^-_{pm_1} \\ &= A - A^- + \alpha_{pm_2} - \alpha^-_{pm_1} \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \lambda_n - \mu_{n+1} &= \lambda_{1,pm_2} - \mu^+_{1,p(m_2-m_1)} + A - A^+ + \alpha_{pm_2} - \alpha^+_{p(m_2-m_1)} \\ &= A - A^+ + \alpha_{pm_2} - \alpha^+_{p(m_2-m_1)}. \end{aligned} \tag{2.19}$$

Recall that $A - A^- > 0$ and $A^+ - A > 0$. Since three sequences $\{\alpha_n\}_{n=0}^\infty$ and $\{\alpha_n^\pm\}_{n=0}^\infty$ are infinitely small as $n \to \infty$, it follows that there exists a positive integer, denoted by N_e, satisfying

$$\left|\alpha_{pm_2} - \alpha^-_{pm_1}\right| < \frac{A - A^-}{2}, \quad \left|\alpha_{pm_2} - \alpha^+_{p(m_2-m_1)}\right| < \frac{A^+ - A}{2} \tag{2.20}$$

for all $n > N_e$ and therefore $\mu_n < \lambda_n < \mu_{n+1}$ when $n = pm_2$ and $n > N_e$.

On the other hand, if $n = pm_2 + j$ for $j = 1, 2, \ldots, m_2 - 1$, then

$$\begin{aligned} \lambda_n - \mu_n &= (\lambda_{1,n} - \mu_{1,n}) + (A + \alpha_n) - (A^\epsilon + \alpha^\epsilon_m) \\ &\geq n\pi\left(\sqrt{\lambda_{1,n}} - \sqrt{\mu_{1,n}}\right) + (A + \alpha_n) - (A^\epsilon + \alpha^\epsilon_m) \\ &= n\pi^2(j - a_j) + \beta_n + \beta^\epsilon_m \\ &\geq n\pi^2 C_0^- - |\beta_n| - |\beta^\epsilon_m|, \end{aligned} \tag{2.21}$$

where $\epsilon = \pm$ when $\mu_n = \mu^\pm_m$, a_j are defined by (2.12), $\beta_n = A + \alpha_n$, $\beta^\epsilon_m = A^\epsilon + \alpha^\epsilon_m$, and $C_0^- = \min\{(j - a_j) : j = 1, 2, \ldots, m_2 - 1\}$. It is easy to see that

$$C_0^- \geq \max\left\{\frac{1}{m_1}, \frac{1}{m_2 - m_1}\right\}. \tag{2.22}$$

Note that two sequences $\{\beta_n\}_{n=0}^\infty$ and $\{\beta^\pm_m\}_{m=0}^\infty$ are bounded. Therefore, there are $M_0(\beta_n)$ and $M_0(\beta^\pm_m)$ satisfying $|\beta_n| \leq M_0(\beta_n)$ and $|\beta^\pm_m| \leq M_0(\beta^\pm_m)$ for all $n, m \in \mathbb{N}_0$. For the positive constant C_0^- in (2.22), there exists the positive number

$$N_i^- = \frac{M_0(\beta_n) + M_0(\beta^\pm_m)}{\pi^2 C_0^-} \tag{2.23}$$

such that $\lambda_n > \mu_n$ for all $n > N_i^-$ and $n = pm_2 + j$ for $j = 1, 2, \ldots, m_2 - 1$. In accordance with the similar argument, one infers that $\lambda_n < \mu_{n+1}$ for all $n > N_i^+$ and $n = pm_2 + j$ for $j = 1, 2, \ldots, m_2 - 1$, where

$$N_i^+ = \frac{M_0(\beta_n) + M_0(\beta_m^{\pm})}{\pi^2 C_0^+} \tag{2.24}$$

with $C_0^+ = \min\{(a_{j+1} - j) : j = 1, 2, \ldots, m_2 - 1\}$ and C_0^+ satisfies (2.22). Thus, if we choose $N_i = \max\{N_i^+, N_i^-\}$, then the strict inequality $\mu_n < \lambda_n < \mu_{n+1}$ holds for each $n > N_i$ with $n = pm_2 + j$ for $j = 1, 2, \ldots, m_2 - 1$.

By means of the discussion above for two cases, we infer that the interlacing property (2.17) holds for all $n > N := \max\{N_e, N_i\}$. This completes the proof. □

By the proof of Lemma 2.2, we see that there are two positive numbers N_e and N_i so that

$$\begin{cases} \mu_{pm_2} < \lambda_{pm_2} < \mu_{pm_2+1} & \text{if } p > N_e/m_2, \\ \mu_{pm_2+j} < \lambda_{pm_2+j} < \mu_{pm_2+j+1} & \text{if } p > N_i/m_2, \end{cases} \tag{2.25}$$

where $j = 1, 2, \ldots, m_2 - 1$. This fact urges us to find out two positive numbers N_e and N_i to ensure (2.25) holds. Note that N_e and N_i are only related to the L^1 norm of q and the boundary parameters h_0, h_1 and $h_{\pm}$ (see Section 4 for details).

Let us concern with the functions

$$\omega^-(\lambda, h_+) = u'(a, \lambda) + h_+ u(a, \lambda), \tag{2.26}$$

$$\omega^+(\lambda, h_-) = v'(a, \lambda) + h_- v(a, \lambda), \tag{2.27}$$

and denote their zeros by $\{\upsilon_n^{\mp}\}_{n=0}^{\infty}$. Then both sets $\{\upsilon_n^-\}_{n=0}^{\infty}$ and $\{\upsilon_n^+\}_{n=0}^{\infty}$ are the spectra of the following two Sturm-Liouville problems

$$\begin{cases} -u'' + qu = \lambda u & \text{on } [0, a], \\ u'(0) + h_0 u(0) = 0, \\ u'(a) + h_+ u(a) = 0, \end{cases} \tag{2.28}$$

and

$$\begin{cases} -u'' + qu = \lambda u & \text{on } [a, 1], \\ u'(a) + h_- u(a) = 0, \\ u'(1) + h_1 u(1) = 0. \end{cases} \tag{2.29}$$

Finally, we consider the interlacing property between $\{\mu_n\}_{n=0}^{\infty}$ and $\{\upsilon_n\}_{n=0}^{\infty}$ for sufficiently large n, where

$$\{\upsilon_n\}_{n=0}^{\infty} := \{\upsilon_n^-\}_{n=0}^{\infty} \cup \{\upsilon_n^+\}_{n=0}^{\infty} \tag{2.30}$$

(counting multiplicity) is an increasing sequence.

Lemma 2.3. *Fix $n \in \mathbb{N}_0$. Consider the corresponding eigenvalue $\mu_n^-(h_0, h_-)$ of the operator L^- as the function of h_0 and h_-. Then $\mu_n^-(h_0, h_-)$ is a continuous function of h_0, h_- for $(h_0, h_-) \in \mathbb{R}^2$ and, it is strictly decreasing in $h_0 \in \mathbb{R}$ for any fixed $h_- \in \mathbb{R}$; and strictly increasing in $h_- \in \mathbb{R}$ for any fixed $h_0 \in \mathbb{R}$.*

Proof. The proof refers the proof of [14, Theorem 4.4.3] and is therefore omitted. □

Lemma 2.4. *Let a be defined as in Lemma 2.1. Suppose the interlacing property (2.17) holds. For the same N existence of which is proved in Lemma 2.2, then one of the following two interlacing properties holds for each $n > N$,*

$$\mu_n < \upsilon_n < \mu_{n+1} \quad \text{for } h_+ > h_-, \tag{2.31}$$

$$\upsilon_n < \mu_n < \upsilon_{n+1} \quad \text{for } h_+ < h_-. \tag{2.32}$$

Proof. Let $\omega_1(\lambda) = \omega^-(\lambda, h_-)\omega^+(\lambda, h_+)$ and $\omega_2(\lambda) = \omega^-(\lambda, h_+)\omega^+(\lambda, h_-)$. Then from (2.3)–(2.5) and (2.26)–(2.27) we get

$$\begin{aligned} \omega(\lambda) &= \frac{1}{h_+ - h_-} \begin{vmatrix} u'(a,\lambda) + h_+ u(a,\lambda) & v'(a,\lambda) + h_+ v(a,\lambda) \\ u'(a,\lambda) + h_- u(a,\lambda) & v'(a,\lambda) + h_- v(a,\lambda) \end{vmatrix} \\ &= \frac{1}{h_+ - h_-}[\omega_2(\lambda) - \omega_1(\lambda)]. \end{aligned} \tag{2.33}$$

It should be noted that, by adding a constant to the potential q if need be, we can assume that three continuous functions $\omega(\lambda)$, $\omega_1(\lambda)$ and $\omega_2(\lambda)$ are real in interval $(0, \infty)$ and their all zeros are positive. Since $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ are the zeros of the functions $\omega(\lambda)$ and $\omega_1(\lambda)$, respectively, we obtain $\mu_n < \lambda_n < \mu_{n+1}$ for all $n > N$, which is proved in Lemma 2.2.

We first show that there exists $\upsilon_m \in \{\upsilon_n\}_{n=0}^{\infty}$ so that $\upsilon_m \in (\mu_n, \mu_{n+1})$ for each $n > N$, where $\{\upsilon_n\}_{n=0}^{\infty}$ are the zeros of function $\omega_2(\lambda)$. Since μ_n and μ_{n+1} are the adjacent zeros of $\omega_1(\lambda)$, it follows from (2.33) that

$$\omega(\mu_n)\omega(\mu_{n+1}) = \frac{1}{(h_+ - h_-)^2}\omega_2(\mu_n)\omega_2(\mu_{n+1}). \tag{2.34}$$

It is known [4] that

$$\omega(\lambda) = (\lambda_0 - \lambda) \prod_{n=1}^{N} \frac{\lambda_n - \lambda}{n^2\pi^2} \prod_{n=N+1}^{\infty} \frac{\lambda_n - \lambda}{n^2\pi^2},$$

and each λ_n is the only simple zero of $\omega(\lambda)$ in the interval (μ_n, μ_{n+1}) for $n > N$. Then we have $\omega(\mu_n)\omega(\mu_{n+1}) < 0$, which together with (2.34) and $h_+ \neq h_-$ implies

$$\omega_2(\mu_n)\omega_2(\mu_{n+1}) < 0.$$

By the intermediate value theorem of continuous functions, we infer that there exists at least $\upsilon_m \in \{\upsilon_n\}_{n=0}^{\infty}$ so that $\upsilon_m \in (\mu_n, \mu_{n+1})$ for each $n \geq N$.

We next prove that for each $n > N$ the interval (μ_n, μ_{n+1}) contains at most one element of the set $\{\upsilon_n\}_{n=0}^{\infty}$ for the case $h_+ > h_-$. If $\mu_n = \mu_k^- =: \mu_k^-(h_0, h_-)$ for some $k \in \mathbb{N}_0$, then from Lemma 2.3 we see that the eigenvalue $\mu_k^-(h_0, h_-)$ is strictly increasing in $h_- \in \mathbb{R}$ for any fixed $h_0 \in \mathbb{R}$. In this sense, the eigenvalue υ_k^- can be regarded as $\mu_k^-(h_0, h_+)$. On the other hand, $\mu_n = \mu_k^+ =: \mu_k^+(h_+, h_1)$ is strictly decreasing in $h_+ \in \mathbb{R}$ for any fixed $h_1 \in \mathbb{R}$ and therefore υ_k^+ can be regarded as $\mu_k^+(h_-, h_1)$. Since $h_+ > h_-$, it follows that $\mu_k^- < \upsilon_k^-$ and $\mu_k^+ < \upsilon_k^+$. This yields that $\mu_n < \upsilon_n$ and shows that for $n > N$ the interval (μ_n, μ_{n+1}) contains at most one element of the set $\{\upsilon_n\}_{n=0}^{\infty}$. This fact also remains true in the case $h_+ < h_-$.

Combined with the above discussions, we have $\mu_n < \upsilon_n < \mu_{n+1}$ for each $n \geq N$ when $h_+ > h_-$. Similarly, if $h_+ < h_-$, then we have $\upsilon_n < \mu_n < \upsilon_{n+1}$ for $n \geq N$. This completes the proof. □

3. Finding N

In this section we shall identify $N = \max\{N_i, N_e\}$ in Lemma 2.2 such that the interlacing property (2.17) holds for $n > N$. We first present the estimates of sequences $\{\alpha_n\}_{n=0}^{\infty}$ in (2.6) and $\{\beta_m\}_{m=0}^{\infty}$ in (2.21) to find N_e and N_i such that $|\alpha_n| < \delta$ and $|\beta_m| < M_0$ hold for all $n > N_e$ and $m > N_i$. Here the positive number δ is given a priori. The method used here mainly relies on that of used in [8, 12].

Throughout this section, we always assume that (1.7) holds. Let us consider the Sturm-Liouville problem L which is generated by (1.1)–(1.2). It is well known [4] that its eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ obey the following asymptotic expression

$$\lambda_n = (n\pi)^2 + 2A + c(n) + \gamma_n, \tag{3.1}$$

where $\gamma_n = O(1/n)$ as $n \to \infty$, A is defined by (2.10) and $c(n)$ are the Fourier coefficients for potential q:

$$c(n) = \int_0^1 q(t)\cos(2n\pi t)\,dt. \tag{3.2}$$

We need the following lemma which is a copy from [7, Theorem 4.2.1]. We cite this lemma here without proof.

Lemma 3.1. *Given any positive number $\delta(A)$, there exists a positive $N_f(0,1)$ such that for all $n > N_f(0,1)$,*

$$|c(n)| < \delta(A). \tag{3.3}$$

Remark 3.2. Generally speaking, if the unknown potential q belongs to $L^1[0,1]$, we do not know the exact $N_f(0,1)$ such that (3.3) holds for $n > N_f(0,1)$, although we only know its existence. However, if $q \in W^{1,1}[0,1]$, then it follows from [7] that

$$\left| \int_0^1 q(t)\cos(2n\pi t)\,dt \right| \leq \frac{\|q'\|_{L^1}}{2n\pi}.$$

This yields $N_f(0,1) = \|q'\| / (2\pi\delta(A))$, which only depends on the norm $\|q'\|_{L^1}$.

By a result of Mclaughlin [8], one can prove

Lemma 3.3. *For the problem L, we have*

$$|\omega(\lambda) - \rho\sin(\rho)| \leq B \tag{3.4}$$

and

$$\left|\omega(\lambda) - \rho\sin(\rho) - (h_0 - h_1)\cos(\rho) + \int_0^1 q(t)\cos(\rho(1-t))\cos(\rho t)\,dt\right| \leq \frac{B}{|\rho|} \tag{3.5}$$

for $\lambda \in \mathbb{R}$, where $\lambda = \rho^2$, $\omega(\lambda)$ is defined in (2.3) *and*

$$B = (1 + |h_0|)(1 + |h_1|)\left(1 + \|q\|\, e^{\|q\|}\right) \tag{3.6}$$

with $\|q\| := \|q\|_{L^1[0,1]}$.

In the following, we present the estimates for N_e and N_i in Lemmas 3.4 and 3.5, which will help us to obtain the number of triplets $(q; h_0, h_1)$ corresponding to three spectra. Set

$$\delta(A) = \min\left\{\left|A - A^+\right|, \left|A - A^-\right|, A_0\right\}, \tag{3.7}$$

where $A_0 = \min\{1/m_1, 1/(m_2 - m_1)\}$ and A and $A^\pm$ are defined by (2.10). With the above preliminaries provided, we first need to identify N_e.

Lemma 3.4. *Consider the problem L. Let $\delta(A)$ be given by* (3.7) *and let $N_f(0,1)$ be defined in Lemma* 3.1 *corresponding to the positive constant $\delta(A)/2$. Then, for the eigenvalues asymptotic* (3.1), *there exists $N_e(0,1)$ given by*

$$N_e(0,1) = \max\left\{N_f(0,1), \frac{2C}{\delta(A)}\right\} \tag{3.8}$$

such that for all $n > N_e(0,1)$,

$$|\alpha_n| := |c(n) + \gamma_n| < \delta(A). \tag{3.9}$$

Here

$$C = B(4 + 7B), \tag{3.10}$$

and B is defined in (3.6).

Proof. The proof of this lemma consists of two parts. First, we prove that for $D = 5B$ and $n > 25B/\pi^2$ the eigenvalue λ_n lies in $I_n(D) = \big[(n\pi)^2 - D, (n\pi)^2 + D\big]$. We prove the inequality (3.9) holds in the second part.

We first prove the existence of the eigenvalue λ_n in $I_n(D)$ for $n > 25B/\pi^2$. In order to prove this fact, we only need to show that $\omega(\lambda)$ changes sign in $I_n(D)$. For $\lambda = (n\pi)^2 + D$, in virtue of $1 + t/4 \le \sqrt{1+t} \le 1 + t/2$ for $0 \le t \le 1$, we obtain

$$n\pi\left(1 + \frac{5B}{4n^2\pi^2}\right) \le \sqrt{\lambda} \le n\pi\left(1 + \frac{5B}{2n^2\pi^2}\right). \tag{3.11}$$

Moreover, by using $|\sin(n\pi + t)| > |t|\,(1 - t^2/6)$, we deduce that

$$(-1)^n \sin(\sqrt{\lambda}) > (-1)^n \sin\left(n\pi + \frac{5B}{4n\pi}\right) > \frac{5B}{4n\pi}. \tag{3.12}$$

By (3.4), (3.11) and (3.12), we have $(-1)^n\omega(\lambda) > 0$. Similarly, for $\lambda = (n\pi)^2 - D$, we get

$$n\pi\left(1 - \frac{0.106}{n}\right) < \sqrt{\lambda} \le n\pi\left(1 - \frac{5B}{4n^2\pi^2}\right). \tag{3.13}$$

It follows from (3.4), (3.12) and (3.13) that $(-1)^n\omega(\lambda) < 0$. Therefore, there is at least one eigenvalue λ_n in $I_n(D)$ for $n > 25B/\pi^2$. On the other hand, by means of the asymptotic of $\{\lambda_n\}_{n=0}^{\infty}$ (see (3.1)), we see that there is only one λ_n in $I_n(D)$.

We secondly prove (3.9) holds. Using (3.5) we obtain that for $\lambda_n = \rho_n^2$,

$$\left|\sin(\rho_n) - \frac{A}{\rho_n}\cos(\rho_n) - \frac{\cos(\rho_n)}{2\rho_n}\int_0^1 q(t)\cos(2\rho_n t)\,dt - \frac{\sin(\rho_n)}{2\rho_n}\int_0^1 q(t)\sin(2\rho_n t)\,dt\right| \le \frac{B}{|\lambda_n|},$$

where A is defined by (2.10). Denote $\lambda_n = (n\pi)^2 + C_0$ with $|C_0| < 5B$. We know

$$\sqrt{\lambda_n} = n\pi\left(1 + \frac{C_0}{2(n\pi)^2} + C_1\right) \tag{3.14}$$

with $|C_1| < C_0^2/[4(n\pi)^4]$ and we have

$$\left|\sin(\sqrt{\lambda_n}) - (-1)^n\frac{\left(\lambda_n - (n\pi)^2\right)}{2n\pi}\right| < \frac{B(1+B)}{4n^2\pi}. \tag{3.15}$$

From (3.14), we get $\sqrt{\lambda_n} = (1+C_2)n\pi$, where $|C_2| < 3B/(n\pi)^2$ and $1/\sqrt{\lambda_n} = (1+C_3)/(n\pi)$ with $|C_3| < 2\,|C_2|$. Therefore,

$$\left|\frac{\sin(\rho_n)}{2\rho_n}\int_0^1 q(t)\sin(2\rho_n t)\,dt\right| < \frac{2B\,\|q\|}{(n\pi)^2}. \tag{3.16}$$

By $\cos(\sqrt{\lambda_n}) = \cos(n\pi + C_2 n\pi) = (-1)^n + C_4$ with $|C_4| < |C_2 n\pi|^2/2$, we calculate

$$\left|A\frac{\cos(\rho_n)}{\rho_n}\right| < \frac{(-1)^n A}{n\pi} + C_5, \tag{3.17}$$

where $|C_5| \le |A|\,(|C_3|+|C_4|+|C_3C_4|)/(n\pi) < |A|\,(1+B)/[2(n\pi)^2]$. Let $2\sqrt{\lambda_n} = 2n\pi + C_6$ with $|C_6| < 6B/(n\pi)$. Then

$$\cos(2\sqrt{\lambda_n}t) = \cos(2n\pi t) + C_7, \tag{3.18}$$

where $|C_7| < 9B/(n\pi)$. (3.17) and (3.18) together yield

$$\frac{\cos(\rho_n)}{2\rho_n}\int_0^1 q(t)\cos(2\rho_n t)\,dt = \frac{(-1)^n c(n)}{2n\pi} + C_8, \tag{3.19}$$

where $|C_8| < (1+2B)\,\|q\|\,/(n^2\pi)$. Furthermore, combined with the above discussions, one infers that

$$\begin{aligned}\left|\frac{\lambda_n-(n\pi)^2}{2n\pi} - \frac{A}{n\pi} - \frac{c(n)}{2n\pi}\right| < \frac{B(1+B)}{4n^2\pi} + \frac{2B\,\|q\|}{(n\pi)^2} + \frac{|A|\,(1+B)}{2(n\pi)^2}\\ + \frac{(1+2B)\,\|q\|}{n^2\pi} + \frac{B}{(n\pi)^2},\end{aligned} \tag{3.20}$$

which implies $\left|\lambda_n - (n\pi)^2 - 2A - c(n)\right| < C/n$ by a simple calculation, where C is defined by (3.10). In all, when $n > \max\left\{25B/\pi^2, 2C/\delta(A)\right\}$ we find

$$\left|\lambda_n - (n\pi)^2 - 2A\right| < c(n) + \frac{\delta(A)}{2}. \tag{3.21}$$

By virtue of (3.7), one shows that $\delta(A) \le C_0 \le 1$ and, therefore,

$$\frac{2C}{\delta(A)} \ge 2C = 2B(4+7B) > \frac{25B}{\pi^2}, \tag{3.22}$$

which means (3.21) holds for all $n > 2C/\delta(A)$. Moreover, by Lemma 3.1, we deduce that $|c(n)| < \delta(A)/2$, when $n > N_f(0,1)$. Substituting the above inequality into (3.21), we obtain the representation (3.9) for each $n > N_e(0,1) := \max\{N_f(0,1), 2C/\delta(A)\}$ and the proof of Lemma 3.4 is complete. □

Consider two other Sturm-Liouville problems $L_{[0,a]}$ and $L_{[a,1]}$, which are defined on interval $[0,a]$ and $[a,1]$, respectively, where $a = m_1/m_2$ is an irreducible fraction with $m_1 < m_2$ and $m_1, m_2 \in \mathbb{N}$. Recall that $\{\mu_n^-\}_{n=0}^\infty$ and $\{\mu_n^+\}_{n=0}^\infty$ are their spectra. Thus, by the same argument in the proof of Lemma 3.4, for the given positive $\delta(A)$ defined by (3.7), we easily infer that there exist two positive numbers $N_e(0,a)$ and $N_e(a,1)$ defined as

$$N_e(0,a) = \max\left\{N_f(0,a), \frac{2C_-}{\delta(A)}\right\} \quad \text{and} \quad N_e(a,1) = \max\left\{N_f(a,1), \frac{2C_+}{\delta(A)}\right\} \tag{3.23}$$

such that for all $n > \max\{N_e(0,a), N_e(a,1)\}$ the following inequalities hold:

$$\left|\alpha_n^-\right| < \delta(A) \quad \text{and} \quad \left|\alpha_n^+\right| < \delta(A). \tag{3.24}$$

Here $\alpha_n^{\pm}$ are defined in (2.7)–(2.8) and $N_f(0,a)$ and $N_f(a,1)$ are given by Lemma 3.1 corresponding to the Fourier coefficients for potential q defined on $[0,a]$ and $[a,1]$, respectively. Moreover, C_- and C_+ in (3.23) are given by

$$C_- = B_- + 5B_-^2 + \frac{m_2}{m_1}(3B_- + 2B_-^2), \tag{3.25}$$

where $B_- = (1+|h_0|)(1+|h_-|)\left(1+\|q\|\,e^{\|q\|}\right)$ with $\|q\| := \|q\|_{L[0,a]}$, and

$$C_+ = B_+ + 5B_+^2 + \frac{m_2}{m_2-m_1}(3B_+ + 2B_+^2), \tag{3.26}$$

where $B_+ = (1+|h_+|)(1+|h_1|)\left(1+\|q\|\,e^{\|q\|}\right)$ with $\|q\| := \|q\|_{L[a,1]}$.

Set

$$N_e = \max\left\{N_e(0,1), \frac{m_2}{m_1}N_e(0,a), \frac{m_2}{m_2-m_1}N_e(a,1)\right\}. \tag{3.27}$$

It is easy to see that the N_e defined above only depends on the associated norms of the potential q and the boundary parameters h_0, h_1 and $h_{\pm}$.

We next need to identify N_i.

Lemma 3.5. *Let B be defined as in Lemma* 3.3 *(see* (3.6)*). Then for the eigenvalues asymptotic* (3.1) *there exists $N_i(0,1)$ defined by*

$$N_i(0,1) = \frac{25B}{\pi^2}$$

such that for all $n > N_i(0,1)$,

$$|\beta_n| := |2A + \alpha_n| < 5B, \tag{3.28}$$

where A is defined by (2.10).

Proof. The proof of this lemma follows Lemma 3.4 and is therefore omitted. □

Furthermore, by Lemma 3.5, we also infer that there exist two numbers $N_i(0,a)$ and $N_i(a,1)$ for the problems $L_{[0,a]}$ and $L_{[a,1]}$, which are defined as

$$N_i(0,a) = \frac{25m_1B_-}{m_2\pi^2} \quad \text{and} \quad N_i(a,1) = \frac{25(m_2-m_1)B_+}{m_2\pi^2}, \tag{3.29}$$

which imply that for all $n > \max\{N_i(0,a), N_i(a,1)\}$ the following inequalities hold:

$$\left|\beta_n^-\right| < \frac{m_2}{m_1}5B_- \quad \text{and} \quad \left|\beta_n^+\right| < \frac{m_2}{m_2-m_1}5B_+, \tag{3.30}$$

where $\beta_n^{\pm} = 2A^{\pm} + \alpha_n^{\pm}$.

Set

$$N_i = \max\left\{N_i(0,1), N_i(0,a), N_i(a,1), \frac{2M_0}{A_0\pi^2}\right\}, \tag{3.31}$$

where $M_0 = \max\{5B, 5m_2B_-/m_1, 5m_2B_+/(m_2-m_1)\}$ and A_0 is defined in (3.7).

Based on the above discussion, we are now in a position to identify N.

Theorem 3.6. *Let N_e and N_i be given by* (3.27) *and* (3.31)*, respectively. Let the assumptions of Lemma* 2.2 *hold. Then $N_e > N_i$, that is,*

$$N = \max\{N_i, N_e\} = N_e. \tag{3.32}$$

Furthermore, we have that the interlacing property (2.17) *remains valid for $n > N_e$.*

Proof. In view of the proof of Lemma 2.2, we only need to verify that $N_e > N_i$. By the same reason as (3.22), we obtain

$$\frac{2C}{\delta(A)} > \frac{25B}{\pi^2}, \quad \frac{2C_-}{\delta(A)} > \frac{25m_1B_-}{m_2\pi^2}, \quad \frac{2C_+}{\delta(A)} > \frac{25(m_2-m_1)B_+}{m_2\pi^2}.$$

This implies

$$N_e > \max\{N_i(0,1), N_i(0,a), N_i(a,1)\}. \tag{3.33}$$

Furthermore, because $\delta(A) \le A_0$, equation (3.10) shows that

$$\frac{2C}{\delta(A)} \ge \frac{2C}{A_0} > \frac{10B}{A_0\pi^2}.$$

Similarly, since $m_2 > m_1$, we have

$$\frac{2C_-m_2}{\delta(A)m_1} > \frac{2C_-}{\delta(A)} > \frac{10m_2B_-}{A_0m_1\pi^2},$$
$$\frac{2C_+m_2}{\delta(A)(m_2-m_1)} > \frac{2C_+}{\delta(A)} > \frac{10m_2B_+}{A_0(m_2-m_1)\pi^2}.$$

It follows from $M_0 = \max\{5B, 5m_2B_-/m_1, 5m_2B_+/(m_2-m_1)\}$ that

$$N_e > \frac{2M_0}{C_0\pi^2}. \tag{3.34}$$

Combined with the above discussions, one infers that $N_e > N_i$ and the proof is complete. □

Conclusion. By the above discussion, if (1.7) holds, then

$$N = N_e = \max\left\{N_e(0,1), \frac{m_2}{m_1}N_e(0,a), \frac{m_2}{m_2-m_1}N_e(a,1)\right\} \tag{3.35}$$

and (2.17) holds provided that $n = pm_2 - j > N$.

4. The finiteness theorem

In this section we state and prove the finiteness result for the inverse Sturm-Liouville problems by three spectra corresponding to the problems $L_{[0,1]}$, $L_{[0,a]}$ and $L_{[a,1]}$, respectively, involved two different interface parameters h_+ and h_-.

The following theorem is our main result of the paper.

Theorem 4.1. *Let $h_+, h_- \in \mathbb{R}$ with $h_+ \neq h_-$ and $a = m_1/m_2 \in (0,1)$ be an irreducible fraction, which all are fixed. Let $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n^-\}_{n=0}^{\infty}$ and $\{\mu_n^+\}_{n=0}^{\infty}$ be three spectra of the Sturm-Liouville problems L, L^- and L^+ defined by (1.1)–(1.2), (1.1)–(1.3) and (1.1)–(1.4), respectively.*

Suppose the inequality (1.7) holds. Let

$$[N_e] = pm_2 - j \tag{4.1}$$

for some $j = 1, 2, \ldots, m_2$, where N_e is defined by (3.35) and $[\,\cdot\,]$ is a rule to round down to the nearest integer. Then there exist at most

$$K_0 := C_{pm_2}^{pm_1} \tag{4.2}$$

triplets $(q; h_0, h_1)$ corresponding to three spectra $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n^-\}_{n=0}^{\infty}$ and $\{\mu_n^+\}_{n=0}^{\infty}$.

Proof. As was well known [4], the specification of the spectra $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n^{\mp}\}_{n=0}^{\infty}$ uniquely determine $\omega(\lambda)$ and $\omega^{\mp}(\lambda, h_{\mp})$, respectively. It follows from (2.33) that

$$\omega^-(\lambda, h_+)\omega^+(\lambda, h_-) = \omega^-(\lambda, h_-)\omega^+(\lambda, h_+) + (h_+ - h_-)\omega(\lambda). \tag{4.3}$$

Denote by $\{\upsilon_n\}_{n=0}^{\infty}$ the increasing sequence of the zeros of $\omega^-(\lambda, h_+)\omega^+(\lambda, h_-)$, which can be obtained from (4.3), since h_+ and h_- are known. Recall that $\{\upsilon_i^-\}_{i=0}^{\infty}$ and $\{\upsilon_j^+\}_{j=0}^{\infty}$ are the zeros of $\omega^-(\lambda, h_+)$ and $\omega^+(\lambda, h_-)$. Therefore,

$$\{\upsilon_n\}_{n=0}^{\infty} = \{\upsilon_i^-\}_{i=0}^{\infty} \cup \{\upsilon_j^+\}_{j=0}^{\infty}. \tag{4.4}$$

Unfortunately, υ_i^- and υ_j^+ cannot be identified immediately from $\{\upsilon_n\}_{n=0}^{\infty}$ for each $i, j \in \mathbb{N}_0$. In other words, we do not know which one of $\{\upsilon_i^-\}_{i=0}^{\infty} \cup \{\upsilon_j^+\}_{j=0}^{\infty}$ is equal to υ_n for every $n \in \mathbb{N}_0$, when $\{\mu_i^-\}_{i=0}^{\infty}$, $\{\mu_j^+\}_{j=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$ are known *a priori*.

In order to identify υ_i^- and υ_j^+ from $\{\upsilon_n\}_{n=0}^{\infty}$ for each $i, j \in \mathbb{N}_0$, we need to show two parts. In the first part we treat the case where $n > N_e$. Without loss of generality, we assume $h_+ > h_-$. Similarly for $h_+ < h_-$. Let

$$\{\mu_n\}_{n=0}^{\infty} = \{\mu_i^-\}_{i=0}^{\infty} \cup \{\mu_j^+\}_{j=0}^{\infty} \tag{4.5}$$

(counting multiplicity). Then by Lemmas 2.2 and 2.4 we see that, for all $n > N_e$, the following strict inequalities

$$\mu_n < \lambda_n < \mu_{n+1} \quad \text{and} \quad \mu_n < \upsilon_n < \mu_{n+1} \tag{4.6}$$

hold, where N_e is defined by (3.35), which together with the interlacing properties $\mu_i^- < \upsilon_i^- < \mu_{i+1}^-$ and $\mu_j^+ < \upsilon_j^+ < \mu_{j+1}^+$ for all $i, j \in \mathbb{N}_0$ implies

$$\upsilon_n = \begin{cases} \upsilon_m^+ & \text{if } \mu_n = \mu_m^+ \text{ and } \mu_{n+1} = \mu_{m+1}^+, \\ \upsilon_m^- & \text{if } \mu_n = \mu_m^- \text{ and } \mu_{n+1} = \mu_{m+1}^-, \\ \upsilon_m^+ & \text{if } \mu_n = \mu_m^+ \text{ and } \mu_{n+1} = \mu_{m'}^-, \\ \upsilon_m^- & \text{if } \mu_n = \mu_m^- \text{ and } \mu_{n+1} = \mu_{m'}^+, \end{cases} \tag{4.7}$$

where m and m' are the indices corresponding to (4.4) and (4.5). The relationship (4.7) helps us to identify υ_i^- and υ_j^+ from $\{\upsilon_n\}_{n=[N_e]}^{\infty}$ provided that $n > N_e$.

Secondly, we treat the case where $n \leq N_e$. In this case, we cannot know *a priori* the number of $\{\upsilon_i^-\}_{i=0}^{\infty}$ belonging to the set $\{\upsilon_n\}_{n=[N_e]}^{pm_2-1}$. Thus, we need to consider

$$\sigma^{\pm} := \{\mu_i^-\}_{i=0}^{pm_1-1} \cup \{\mu_j^+\}_{j=0}^{p(m_2-m_1)-1} \tag{4.8}$$

and

$$\delta^{\pm} := \{\upsilon_i^-\}_{i=0}^{pm_1-1} \cup \{\upsilon_j^+\}_{j=0}^{p(m_2-m_1)-1}, \tag{4.9}$$

and label "$(-, i)$" or "$(+, j)$" to each element of $\{\upsilon_n\}_{n=0}^{pm_2-1}$ as one of $\{\upsilon_i^-\}_{i=0}^{pm_1-1}$ or $\{\upsilon_j^+\}_{j=0}^{p(m_2-m_1)-1}$ so that the interlacing properties

$$\mu_i^- < \upsilon_i^- < \mu_{i+1}^- \quad \text{and} \quad \mu_j^+ < \upsilon_j^+ < \mu_{j+1}^+ \tag{4.10}$$

for $i = 0, 1, \ldots, pm_1 - 1$ and $j = 0, 1, \ldots, p(m_2 - m_1) - 1$ are satisfied, when $\{\mu_i^-\}_{i=0}^{pm_1-1}$ and $\{\mu_j^+\}_{j=0}^{p(m_2-m_1)-1}$ are given *a priori*. Once this labelling is given, by using Borg's two-spectra theorem [4], from (4.6) and (4.10) there exist unique potential q on $[0, a]$ and h_0 corresponding to $\{\upsilon_i^-\}_{i=0}^{\infty}$, $\{\mu_i^-\}_{i=0}^{\infty}$, and unique potential q on $[a, 1]$ and h_1 corresponding to $\{\upsilon_i^+\}_{i=0}^{\infty}$, $\{\mu_j^+\}_{j=0}^{\infty}$.

The number of all the distinguishable permutations of the elements of $\delta^{\pm}$ is $K_0 := C_{pm_2}^{pm_1}$, each of permutations can be regarded as the elements of $\{\upsilon_n\}_{n=0}^{pm_2-1}$. Note that some of the permutations may not satisfy (4.10); however, our goal is to find the most possibility of the triplets $(q; h_0, h_1)$.

Combined with the above discussions, one infers that there exist at most K_0 triplets $(q; h_0, h_1)$ when h_+, h_- and the three spectra $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n^-\}_{n=0}^{\infty}$ and $\{\mu_n^+\}_{n=0}^{\infty}$ are given. This completes the proof. □

Note that we have not considered the condition (4.10) in the above proof, and therefore the number of triplets $(q; h_0, h_1)$ is not more than (4.2) in most situations. The purpose of the following corollary is to extend the above result to a more precise situation that the first element of the set $\sigma^{\pm}$ is fixed and (4.10) is satisfied.

Corollary 4.2. *With the same notation as in Theorem* 4.1*, suppose the inequality* (1.7) *holds. Fix the first element of the set* $\sigma^{\pm}$*, and let* $[N_e]$ *be the same as in Theorem* 4.1*. Then there exist at most*

$$K_1 := C_{pm_2-1}^{pm_1} + C_{pm_2-2}^{pm_1-1} \tag{4.11}$$

triplets $(q; h_0, h_1)$ *corresponding to three spectra* $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n^-\}_{n=0}^{\infty}$ *and* $\{\mu_n^+\}_{n=0}^{\infty}$.

Proof. From the proof of Theorem 4.1, we only need to prove the largest number of the distinguishable permutations of the elements of $\delta^{\pm}$ is K_1 when the first element of $\sigma^{\pm}$ is fixed and (4.10) is satisfied.

If assuming that μ_0^+ is the first element of $\sigma^{\pm}$, then the second one is μ_1^+ or μ_0^-. Obviously, if μ_0^- is its second element, then from (4.10) the largest number of the distinguishable permutations of the elements of $\delta^{\pm}$ is K_1; if μ_1^+ is its second, then their number is $C_{pm_2-1}^{pm_1}$. Moreover, this fact remains valid for the first element of $\sigma^{\pm}$ to be μ_0^-. In all cases, there are at most K_1 possibility of the distinguishable permutations of the elements of $\delta^{\pm}$.

Therefore, there exist at most K_1 triplets $(q; h_0, h_1)$ corresponding to $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n^-\}_{n=0}^{\infty}$ and $\{\mu_n^+\}_{n=0}^{\infty}$. This completes the proof. □

Remark 4.3. In Corollary 4.2, we only consider that the first element of $\sigma^{\pm}$ is fixed. Moreover, we can compute the exact number when a and p are known a priori, which is related to the order of the elements in $\sigma^{\pm}$.

In particular, suppose the interlacing condition (2.17) holds for all $n \in \mathbb{N}_0$, i.e., $[N_e] = 0$. We have the following uniqueness result.

Corollary 4.4. *With the same notation as in Theorem* 4.1*, suppose that one of the interlacing properties* (2.31) *or* (2.32) *holds for all* $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$*. Then* h_0*,* h_1 *and the potential* q *a.e. on* $[0, 1]$ *are uniquely determined by* h_+*,* h_- *and three spectra* $\{\lambda_n\}_{n=0}^{\infty}$*,* $\{\mu_n^-\}_{n=0}^{\infty}$ *and* $\{\mu_n^+\}_{n=0}^{\infty}$*.*

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Ying Yang and Guangsheng Wei
College of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710062, P. R. China
E-mail address: yangyingaall@126.com, weimath@vip.sina.com