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Multiple Solutions for 4-superlinear Klein-Gordon-Maxwell System Without Odd Nonlinearity

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Abstract. In this paper, we study the following Klein-Gordon-Maxwell system

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where the nonlinearity f and the potential V are allowed to be sign-changing. Under some appropriate assumptions on V and f, we obtain the existence of two different solutions of the system via the Ekeland variational principle and the Mountain Pass Theorem.

1. Introduction

In this paper, we consider the following Klein-Gordon-Maxwell system

$$\text{(KGM)} \qquad \begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where ω is positive constant, the potential V and the nonlinearity f are allowed to be sign-changing. System (KGM) is a modified version of the classical Klein-Gordon-Maxwell system, which has a strong physical meaning since it appears in quantum mechanical models and in semiconductor theory. For more details about the physical background, we refer the reader to [5, 6, 11, 15, 16] and the references therein.

This type of system is settled in the whole space $\mathbb{R}^3$, the Sobolev embedding is not compact for the whole space. A natural idea is study this system on the radial space.

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Interesting reader can see the references [1, 2, 5, 6, 9, 11, 13, 15, 16, 21, 24, 28, 29]. Recently, Carrião, Cunha and Miyagaki [10] studied this type of system with positive periodic potential V. They proved the existence of positive ground state solutions for this system when a periodic potential V is introduced. The method combines the minimization of the corresponding Euler-Lagrange functional on the Nehari manifold with the Brézis and Nirenberg technique. Later, Cunha [14] presented some results on the existence of positive and ground state solutions for the nonlinear (KGM). She introduced a general nonlinearity with subcritical and supercritical growth which does not require the usual Ambrosetti-Rabinowitz condition. Another situation for the potential V is considered by He [20] (see also [12, 22]). He used a coercive potential V which is introduced by Rabinowitz [26]. By means of a variant fountain theorem and the symmetric mountain pass theorem, he obtained the existence of infinitely many large energy solutions. Recently, Li and Tang [22] generalized He's result. In addition, when dealing with nonlinearities which can be negative for small values of u, there are some references [4, 7, 8, 25].

In some of the aforementioned references, the potential V is always assumed to be positive or vanish at infinity and the following famous Ambrosetti-Rabinowitz ((AR) for short) condition is usually required.

(AR) There exists $\mu > 4$ such that

$$0 < \mu F(x,u) \le u f(x,u), \quad u \neq 0.$$

It is well-known that the role of (AR) is to ensure the boundedness of the Palais-Smale (PS) sequences of the energy functional, which is very crucial in applying the critical point theory.

Motivated by [23, 27], in this paper, we consider another case of the potential V and the primitive of f are also allowed to be sign-changing, which is quite different from the previous results. Before stating our main results, we give the following assumption on $V(x)$.

(V1) $V \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} V(x) > -\infty$. Moreover, there exists a constant $d_0 > 0$ such that for any $M > 0$,

$$\lim_{|y|\to\infty} \operatorname{meas}\left\{x \in \mathbb{R}^3 : |x-y| \le d_0, V(x) \le M\right\} = 0,$$

where $\operatorname{meas}(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^3$.

We can find a constant $V_0 > 0$ such that $\widetilde{V}(x) := V(x) + V_0 \ge 1$ for all $x \in \mathbb{R}^3$ which is inspired by Zhang and Xu [31] and let $\widetilde{f}(x,u) := f(x,u) + V_0 u$ for all $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$. Now it is easy to verify the following lemma.

Lemma 1.1. *System* (KGM) *is equivalent to the following problem*

$$\text{(KGM}')\qquad \begin{cases} -\Delta u + \widetilde{V}(x)u - (2\omega+\phi)\phi u = \widetilde{f}(x,u), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega+\phi)u^2, & x \in \mathbb{R}^3. \end{cases}$$

In what follows, we let $\mu > 4$ and give some assumptions on $\widetilde{f}$ and its primitive $\widetilde{F}$ as follows:

(S1) $\widetilde{f} \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, and there exist constants $c_1, c_2 > 0$ and $q \in (4,6)$ such that

$$\left|\widetilde{f}(x,u)\right| \le c_1 |u|^3 + c_2 |u|^{q-1}.$$

(S2) $\lim_{|u|\to\infty} \frac{\left|\widetilde{F}(x,u)\right|}{|u|^4} = \infty$ a.e. $x \in \mathbb{R}^3$ and there exist constants $c_3 \ge 0$, $r_0 \ge 0$ and $\tau \in (0,2)$ such that

$$\inf_{x\in\mathbb{R}^3} \widetilde{F}(x,u) \ge c_3 |u|^\tau \ge 0, \quad \forall\,(x,u) \in \mathbb{R}^3 \times \mathbb{R},\ |u| \ge r_0,$$

where and in the sequel, $\widetilde{F}(x,u) = \int_0^u \widetilde{f}(x,s)\,ds$.

(S3) $\widetilde{\mathcal{F}}(x,u) := \frac{1}{4}u\widetilde{f}(x,u) - \widetilde{F}(x,u) \ge 0$, and there exist $c_4 > 0$ and $\kappa > 1$ such that

$$\left|\widetilde{F}(x,u)\right|^\kappa \le c_4 |u|^{2\kappa}\, \widetilde{\mathcal{F}}(x,u), \quad \forall\,(x,u) \in \mathbb{R}^3 \times \mathbb{R},\ |u| \ge r_0.$$

Now, our main result is as follows:

Theorem 1.2. *Suppose that conditions* (V1)*,* (S1)*,* (S2) *and* (S3) *are satisfied. Then problem* (KGM) *possesses at least two different solutions.*

Remark 1.3. There are some functions not satisfying the condition (AR) for any $\mu > 4$. For example, the superlinear function $f(x,u) = \sin x \ln(1+|u|)u^2$ does not satisfy condition (AR). In our theorems, $\widetilde{F}(x,u)$ is allowed to be sign-changing. Even if $\widetilde{F}(x,u) \ge 0$, the assumptions (S2) and (S3) seem to be weaker than the superlinear conditions obtained in the aforementioned references. It is easy to check that the following nonlinearities $\widetilde{f}$ satisfy (S2) and (S3):

$$\widetilde{f}(x,u) = a(x)(4.5u^{4.5} + 2u^2 \sin u - 4u\cos u)$$

where $a \in (\mathbb{R}^3, \mathbb{R})$ and $0 < \inf_{\mathbb{R}^3} a(x) \le \sup_{\mathbb{R}^3} a(x) < \infty$.

Remark 1.4. To the best of our knowledge, the condition (V1) is first given in [3], but $\inf_{x\in\mathbb{R}^3} V(x) > 0$ is required. From (V1), one can see that the potential $V(x)$ is allowed to be sign-changing. Therefore, the condition (V1) is weaker than (1.2) in [10, 12, 14, 20, 22].

Remark 1.5. It is not difficult to find functions V satisfying the above conditions. For example, let $V(x)$ be a zig-zag function with respect to $|x|$ defined by

$$V(x) = \begin{cases} 2n\,|x| - 2n(n-1) + a_0, & n-1 \le |x| \le (2n-1)/2, \\ -2n\,|x| + 2n^2 + a_0, & (2n-1)/2 \le |x| \le n, \end{cases}$$

where $n \in \mathbb{N}$ and $a_0 \in \mathbb{R}$.

Remark 1.6. Ding and Li [17] studied (KGM) with sign-changing potential V. They got multiple solutions with odd nonlinearity. Here we do not need the nonlinearity to be odd, and also get two solutions for problem (KGM).

Here, we give the sketch of how to look for two distinct critical points of the functional I (where I is defined by (2.3)). First, we consider a minimization of I constrained in a neighborhood of zero via the Ekeland variational principle (see [18, 30]) and we can find a critical point of I which achieves the local minimum of I and the level of this local minimum is negative (see Step 1 in the proof of Theorem 1.2); and then, around "zero" point, by using Mountain Pass Theorem (see [19]) we can also obtain another critical point of I with its positive level (see Step 2 in the proof of Theorem 1.2). Obviously, these two critical points are different because they are in different levels.

2. Preliminaries and variational setting

Hereafter, we use the following notations:

- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space endowed with the standard scalar product and norm
$$(u,v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv)\,dx, \quad \|u\| = (u,u)^{1/2}.$$
- $D^{1,2}(\mathbb{R}^3)$ denotes the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm
$$\|u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2\,dx.$$
- Let $H = \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \widetilde{V}(x)\,|u|^2\right) dx < \infty\right\}$ with the norm
$$\|u\|_H^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + \widetilde{V}(x)\,|u|^2\right) dx.$$
- H^* denotes the dual space of H.
- $L^s(\mathbb{R}^3)$, $1 \le s < +\infty$, denotes a Lebesgue space with the usual norm $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s\,dx\right)^{1/s}$.

- For any $\rho > 0$ and for any $z \in \mathbb{R}^3$, $B_\rho(z)$ denotes the ball with radius ρ centered at z.
- C and C_i denote various positive constants, which may vary from line to line.
- S_i denote the Sobolev constant for the embedding.
- $\to$ denotes the strong convergence and $\rightharpoonup$ denotes the weak convergence.

Throughout this section, we make the following assumption instead of (V1):

(V2) $\widetilde{V} \in C(\mathbb{R}^3, \mathbb{R})$ and $\inf_{x\in\mathbb{R}^3} \widetilde{V}(x) > 0$. Moreover, there exists a constant $d_0 > 0$ such that for any $M > 0$,

$$\lim_{|y|\to\infty} \operatorname{meas}\left\{x \in \mathbb{R}^3 : |x-y| \le d_0, V(x) \le M\right\} = 0.$$

Remark 2.1. Under assumption (V2), we know from [3, Lemma 3.1] that the embedding $H \hookrightarrow L^s(\mathbb{R}^3)$ is compact for $s \in [2,6)$.

Following technical results established in [6] (see also [16]), (KGM′) can be reduced to a single equation with a nonlocal term.

Proposition 2.2. *For any fixed $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ which solves equation*

$$-\Delta\phi + u^2\phi = -\omega u^2. \tag{2.1}$$

Moreover, the map $\Phi\colon u \in H^1(\mathbb{R}^3) \mapsto \Phi[u] := \phi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and

(i) $-\omega \le \phi_u \le 0$ *on the set* $\{x \mid u(x) \neq 0\}$;

(ii) $\|\phi_u\|_{D^{1,2}} \le C\|u\|_H^2$ *and* $\int_{\mathbb{R}^3} |\phi_u| u^2\,dx \le C\|u\|_{12/5}^4 \le C\|u\|_H^4$.

Multiplying (2.1) by ϕ_u and integrating by parts we obtain

$$\int_{\mathbb{R}^3} |\nabla\phi_u|^2\,dx = -\int_{\mathbb{R}^3} \omega\phi_u u^2\,dx - \int_{\mathbb{R}^3} \phi_u^2 u^2\,dx. \tag{2.2}$$

Using (2.2), we define a functional I on H by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + \widetilde{V}(x)u^2 - \omega\phi_u u^2\right)dx - \int_{\mathbb{R}^3} \widetilde{F}(x,u)\,dx \tag{2.3}$$

for all $u \in H$. By condition (S1), we have

$$\left|\widetilde{F}(x,u)\right| \le \frac{c_1}{4}|u|^4 + \frac{c_2}{q}|u|^q, \quad \forall\,(x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.4}$$

Consequently, similar to the discussion in [12, 20, 22], under assumptions (V2), Proposition 2.2 and (2.4), it is easy to prove that the functional I is of class $C^1(H,\mathbb{R})$. Moreover,

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} \Big(\nabla u\cdot\nabla v + \widetilde{V}(x)uv - (2\omega+\phi_u)\phi_u uv - \widetilde{f}(x,u)v\Big)\,dx. \tag{2.5}$$

Hence, if $u\in H$ is a critical point of I, then the pair (u,ϕ_u) is a solution of system (KGM$'$).

Lemma 2.3 (Mountain Pass Theorem)**.** [19] *Let E be a real Banach space with its dual space E^*, and suppose that $I\in C^1(E,\mathbb{R})$ satisfies*

$$\max\{I(0), I(e)\} \le \mu < \eta \le \inf_{\|u\|=\rho} I(u)$$

for some $\mu,\eta,\rho>0$ and $e\in E$ with $\|e\|>\rho$. Let $c\ge\eta$ be characterized by

$$c = \inf_{\gamma\in\Gamma}\max_{0\le\tau\le1} I(\gamma(\tau)),$$

where $\Gamma=\{\gamma\in C([0,1],E):\gamma(0)=0,\gamma(1)=e\}$ is the set of continuous paths joining 0 and e, then there exists a sequence $\{u_n\}\subset E$ such that

$$I(u_n)\to c\ge\eta \quad \text{and} \quad (1+\|u_n\|)\left\|I'(u_n)\right\|_{E^*}\to 0 \quad \text{as } n\to\infty.$$

This kind of sequence is usually called a Cerami sequence. Recall that a C^1 functional I satisfies Cerami condition at level c ((C)$_c$ condition for short) if any sequence $\{u_n\}\subset H$ such that $I(u_n)\to c$ and $(1+\|u_n\|)\,\|I'(u_n)\|_{E^*}\to 0$ has a convergent subsequence.

3. Proof of Theorem 1.2

First, we prove the functional I satisfies the Cerami condition.

Lemma 3.1. *Assume that the conditions* (V2), (S1), (S2) *and* (S3) *hold. Then the Cerami sequence $\{u_n\}$*

$$I(u_n)\to c>0 \quad \text{and} \quad (1+\|u_n\|_H)\left\|I'(u_n)\right\|_{H^*}\to 0 \quad \text{as } n\to\infty \tag{3.1}$$

is bounded in H.

Proof. Arguing by contradiction, we can assume $\|u_n\|_H\to\infty$. Define $v_n := u_n/\|u_n\|_H$. Clearly, $\|v_n\|_H=1$ and $\|v_n\|_s\le S_s\|v_n\|_H = S_s$ for $2\le s<6$. Observe that for n large enough, from (3.1) and (S3) we have

$$\begin{aligned} c+1 &\ge I(u_n) - \frac14\langle I'(u_n), u_n\rangle \\ &= \frac14\|u_n\|_H^2 + \frac14\int_{\mathbb{R}^3}\phi_{u_n}^2 u_n^2\,dx + \int_{\mathbb{R}^3}\left(\frac14\widetilde{f}(x,u_n)u_n - \widetilde{F}(x,u_n)\right)dx \\ &\ge \int_{\mathbb{R}^3}\widetilde{\mathcal{F}}(x,u_n)\,dx. \end{aligned} \tag{3.2}$$

In view of Proposition 2.2, (2.3) and (3.1), we have

$$
\begin{aligned}
\frac{1}{2} &= \frac{I(u_n)}{\|u_n\|_H^2} + \frac{1}{\|u_n\|_H^2}\int_{\mathbb{R}^3} \widetilde{F}(x,u_n)\,dx + \frac{1}{2\,\|u_n\|_H^2}\int_{\mathbb{R}^3} \omega\phi_{u_n} u_n^2\,dx \\
&\le \frac{I(u_n)}{\|u_n\|_H^2} + \frac{1}{\|u_n\|_H^2}\int_{\mathbb{R}^3} \left|\widetilde{F}(x,u_n)\right| dx \\
&\le \limsup_{n\to\infty}\left[\frac{I(u_n)}{\|u_n\|_H^2} + \frac{1}{\|u_n\|_H^2}\int_{\mathbb{R}^3} \left|\widetilde{F}(x,u_n)\right| dx\right] \\
&\le \limsup_{n\to\infty}\int_{\mathbb{R}^3} \frac{\left|\widetilde{F}(x,u_n)\right|}{\|u_n\|_H^2}\,dx.
\end{aligned}
\tag{3.3}
$$

For $0 \le a < b$, let $\Omega_n(a,b) := \{x \in \mathbb{R}^3 : a \le |u_n(x)| < b\}$. Going to a subsequence, if necessary, we may assume that $v_n \rightharpoonup v$ in H. Then by Remark 2.1, we have $v_n \to v$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$, and $v_n \to v$ a.e. on $\mathbb{R}^3$.

We now consider the following two possible cases about v.

Case 1: If $v = 0$, then $v_n \to 0$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$, and $v_n \to 0$ a.e. on $\mathbb{R}^3$. Hence, it follows from (2.4) and $v_n := u_n/\|u_n\|_H^2$ that

$$
\begin{aligned}
\int_{\Omega_n(0,r_0)} \frac{\left|\widetilde{F}(x,u_n)\right|}{\|u_n\|_H^2}\,dx &= \int_{\Omega_n(0,r_0)} \frac{\left|\widetilde{F}(x,u_n)\right|}{|u_n|^2}\,|v_n|^2\,dx \\
&\le \left(\frac{c_1}{4} r_0^2 + \frac{c_2}{q} r_0^{q-2}\right)\int_{\Omega_n(0,r_0)} |v_n|^2\,dx \\
&\le C_4 \int_{\mathbb{R}^3} |v_n|^2\,dx \to 0 \quad \text{as } n\to\infty.
\end{aligned}
\tag{3.4}
$$

From (S3), we know that $\kappa > 1$. Thus, if we set $\kappa' = \kappa/(\kappa-1)$, then $2\kappa' \in (2,6)$. Hence, it follows from (S3), Proposition 2.2 and (3.2) that

$$
\begin{aligned}
\int_{\Omega_n(r_0,\infty)} \frac{\left|\widetilde{F}(x,u_n)\right|}{\|u_n\|_H^2}\,dx &= \int_{\Omega_n(r_0,\infty)} \frac{\left|\widetilde{F}(x,u_n)\right|}{|u_n|^2}\,|v_n|^2\,dx \\
&\le \left[\int_{\Omega_n(r_0,\infty)} \left(\frac{\left|\widetilde{F}(x,u_n)\right|}{|u_n|^2}\right)^{\kappa} dx\right]^{1/\kappa} \left[\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'}\,dx\right]^{1/\kappa'} \\
&\le c_4^{1/\kappa}\left[\int_{\Omega_n(r_0,\infty)} \widetilde{\mathcal{F}}(x,u_n)\,dx\right]^{1/\kappa} \left[\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'}\,dx\right]^{1/\kappa'} \\
&\le c_4^{1/\kappa}(c+1)^{1/\kappa}\left[\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'}\,dx\right]^{1/\kappa'} \\
&\le C_5\left[\int_{\Omega_n(r_0,\infty)} |v_n|^{2\kappa'}\,dx\right]^{1/\kappa'} \to 0 \quad \text{as } n\to\infty.
\end{aligned}
\tag{3.5}
$$

Combining (3.4) with (3.5), we have

$$\int_{\mathbb{R}^3} \frac{\left|\widetilde{F}(x,u_n)\right|}{\|u_n\|_H^2}\,dx = \int_{\Omega_n(0,r_0)} \frac{\left|\widetilde{F}(x,u_n)\right|}{\|u_n\|_H^2}\,dx + \int_{\Omega_n(r_0,\infty)} \frac{\left|\widetilde{F}(x,u_n)\right|}{\|u_n\|_H^2}\,dx \to 0 \quad \text{as } n\to\infty,$$

which contradicts (3.3).

Case 2: If $v \neq 0$, we set $A := \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. Then $\operatorname{meas}(A) > 0$. For a.e. $x \in A$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$. Hence $A \subset \Omega_n(r_0,\infty)$ for $n \in \mathbb{N}$ large enough. It follows from Proposition 2.2, (2.3), (2.4), (3.1) and Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|_H^4} = \lim_{n\to\infty} \frac{I(u_n)}{\|u_n\|_H^4} \\
&= \lim_{n\to\infty} \left[\frac{1}{2\|u_n\|_H^2} - \frac{1}{2\|u_n\|_H^4}\int_{\mathbb{R}^3} \omega\phi_{u_n} u_n^2\,dx - \int_{\mathbb{R}^3} \frac{\widetilde{F}(x,u_n)}{\|u_n\|_H^4}\,dx \right] \\
&\le \left[\frac{1}{2\|u_n\|_H^2} - \frac{1}{2\|u_n\|_H^4}\int_{\mathbb{R}^3} \omega^2 u_n^2\,dx - \int_{\Omega_n(0,r_0)} \frac{\left|\widetilde{F}(x,u_n)\right|}{|u_n|^4} |v_n|^4\,dx \right. \\
&\qquad \left. - \int_{\Omega_n(r_0,\infty)} \frac{\left|\widetilde{F}(x,u_n)\right|}{|u_n|^4} |v_n|^4\,dx \right] \\
&\le \limsup_{n\to\infty} \int_{\Omega_n(0,r_0)} \left(\frac{c_1}{4} + \frac{c_2}{q}|u_n|^{q-4}\right)|v_n|^4\,dx - \liminf_{n\to\infty}\left[\int_{\Omega_n(r_0,\infty)} \frac{\left|\widetilde{F}(x,u_n)\right|}{|u_n|^4}|v_n|^4\,dx\right] \qquad (3.6) \\
&\le \left(\frac{c_1}{4} + \frac{c_2}{q}|r_0|^{q-4}\right)\limsup_{n\to\infty}\int_{\Omega_n(0,r_0)} |v_n|^4\,dx - \liminf_{n\to\infty}\left[\int_{\Omega_n(r_0,\infty)} \frac{\left|\widetilde{F}(x,u_n)\right|}{|u_n|^4}|v_n|^4\,dx\right] \\
&\le C_7 - \liminf_{n\to\infty} \int_{\Omega_n(r_0,\infty)} \frac{\left|\widetilde{F}(x,u_n)\right|}{|u_n|^4}|v_n|^4\,dx \\
&= C_7 - \liminf_{n\to\infty} \int_{\mathbb{R}^3} \frac{|witF(x,u_n)|}{|u_n|^4}[\chi_{\Omega_n(r_0,\infty)}(x)]\,|v_n|^4\,dx \\
&\le C_7 - \int_{\mathbb{R}^3} \liminf_{n\to\infty} \frac{\left|\widetilde{F}(x,u_n)\right|}{|u_n|^4}[\chi_{\Omega_n(r_0,\infty)}(x)]\,|v_n|^4\,dx \to -\infty \quad \text{as } n\to\infty,
\end{aligned}$$

which is a contradiction. Thus $\{u_n\}$ is bounded in H. The proof is completed. □

To complete our proof, we have to cite a result in [27].

Lemma 3.2. *Assume that $p_1, p_2 > 1$, $r, q \geq 1$ and $\Omega \subseteq \mathbb{R}^N$. Let g be a Carathéodory function on $\Omega \times \mathbb{R}$ and satisfy*

$$|g(x,t)| \le a_1 |t|^{(p_1-1)/r} + a_2 |t|^{(p_2-1)/r}, \quad \forall\,(x,t) \in \Omega\times\mathbb{R},$$

where $a_1, a_2 \geq 0$. If $u_n \to u$ in $L^{p_1}(\Omega) \cap L^{p_2}(\Omega)$, and $u_n \to u$ a.e. $x \in \Omega$, then for any $v \in L^{p_1 q}(\Omega) \cap L^{p_2 q}(\Omega)$,

$$\lim_{n\to\infty} \int_\Omega |g(x,u_n) - g(x,u)|^r |v|^q\,dx \to 0. \tag{3.7}$$

Lemma 3.3. *If the conditions* (V2) *and* (S1) *hold. Then any bounded sequence* $\{u_n\}$ *satisfying* (3.1) *has a convergent subsequence in* H.

Proof. Going to a subsequence, if necessary, we may assume that $u_n \rightharpoonup u$ in H. Then by Remark 2.1, we have $v_n \to v$ in $L^s(\mathbb{R}^3)$, for $2 \le s < 6$. Let us take $r \equiv 1$ in Lemma 3.2 and combine with $u_n \to u$ in $L^s(\mathbb{R}^3)$ for $2 \le s < 6$, one can get

$$\lim_{n\to\infty} \left|\widetilde{f}(x,u_n) - \widetilde{f}(x,u)\right| |u_n - u|\, dx \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

We observe that

$$\left\langle I'(u_n) - I'(u), u_n - u \right\rangle \to 0 \quad \text{as } n \to \infty,$$

and we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[(2\omega + \phi_{u_n})\phi_{u_n} u_n - (2\omega + \phi_u)\phi_u u\right](u_n - u)\, dx \\
&= 2\omega \int (\phi_{u_n} u_n - \phi_u u)(u_n - u) + \int_{\mathbb{R}^3} (\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u)\, dx \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Actually, by Hölder's inequality, Proposition 2.2 and the Sobolev inequality, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) u_n (u_n - u)\, dx\right| &\le \|(\phi_{u_n} - \phi_u)(u_n - u)\|_2 \|u_n\|_2 \\
&\le \|\phi_{u_n} - \phi_u\|_6 \|u_n - u\|_3 \|u_n\|_2 \\
&\le C \|\phi_{u_n} - \phi_u\|_{D^{1,2}} \|u_n - u\|_3 \|u_n\|_2,
\end{aligned}$$

where $C > 0$ is a constant. Because $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $2 \le s < 6$, we have

$$\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) u_n (u_n - u)\, dx \to 0 \quad \text{as } n \to \infty$$

and

$$\int_{\mathbb{R}^3} \phi_u (u_n - u)(u_n - u)\, dx \le \|\phi_u\|_6 \|u_n - u\|_3 \|u_n - u\|_2 \to 0 \quad \text{as } n \to \infty.$$

Thus, we get

$$\begin{aligned}
&\int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)\, dx \\
&= \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) u_n (u_n - u)\, dx + \int_{\mathbb{R}^3} \phi_u (u_n - u)(u_n - u)\, dx \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Observe that the sequence $\left\{\phi_{u_n}^2 u_n\right\}$ is bounded in $L^{3/2}(\mathbb{R}^3)$, since

$$\left\|\phi_{u_n}^2 u_n\right\|_{3/2} \le \|\phi_{u_n}\|_6^2 \|u_n\|_3,$$

so

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(\phi_{u_n}^2-\phi_u^2)(u_n-u)\,dx\right| &\le \left\|\phi_{u_n}^2-\phi_u^2\right\|_{3/2}\|u_n-u\|_3\\ &\le \left(\left\|\phi_{u_n}^2\right\|_{3/2}+\left\|\phi_u^2\right\|_{3/2}\right)\|u_n-u\|_3\\ &\to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Now, using (3.8), we can get

$$\begin{aligned}\|u_n-u\|_H^2 = \big\langle I'(u_n)-I'(u),u_n-u\big\rangle &\\ -\int_{\mathbb{R}^3}\left[(2\omega+\phi_{u_n})\phi_{u_n}u_n-(2\omega+\phi_u)\phi_u u\right](u_n-u)\,dx&\\ +\int_{\mathbb{R}^3}\left(\widetilde{f}(x,u_n)-\widetilde{f}(x,u)\right)(u_n-u)\,dx&\\ \to 0 \quad \text{as } n\to\infty.&\end{aligned}$$

That is $u_n\to u$ in H and the proof is complete. □

From Lemmas 3.1 and 3.3, we get the functional I satisfies the Cerami condition. Now, we prove the functional I has a mountain pass geometric structure.

Lemma 3.4. *Assume that the conditions* (V2) *and* (S1) *hold. Then there exist* $\rho,\eta>0$ *such that* $\inf\{I(u): u\in H \text{ with } \|u\|_H=\rho\}>\eta$.

Proof. From (2.4) and the Sobolev inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\widetilde{F}(x,u)\,dx\right| &\le \int_{\mathbb{R}^3}\left|\frac{c_1}{4}|u|^4+\frac{c_2}{q}|u|^q\right|dx\\ &= \frac{c_1}{4}\|u\|_4^4+\frac{c_2}{q}\|u\|_q^q\\ &\le S_4\frac{c_1}{4}\|u\|_H^4+S_q\frac{c_2}{q}\|u\|_H^q\end{aligned}\tag{3.9}$$

for any $u\in H$. Combining Proposition 2.2, (2.3) with (3.9), we have

$$\begin{aligned}I(u) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2+\widetilde{V}(x)u^2-\omega\phi_u u^2\right)dx-\int_{\mathbb{R}^3}\widetilde{F}(x,u)\,dx\\ &= \frac{1}{2}\|u\|_H^2-\frac{1}{2}\int_{\mathbb{R}^3}\omega\phi_u u^2\,dx-\int_{\mathbb{R}^3}\widetilde{F}(x,u)\,dx\\ &\ge \frac{1}{2}\|u\|_H^2-\int_{\mathbb{R}^3}\left|\widetilde{F}(x,u)\right|dx\\ &\ge \frac{1}{2}\|u\|_H^2-S_4\frac{c_1}{4}\|u\|_H^4-S_q\frac{c_2}{q}\|u\|_H^q\\ &= \frac{1}{2}\|u\|_H^2-C_1\|u\|_H^4-C_2\|u\|_H^q.\end{aligned}\tag{3.10}$$

Since $q\in(4,6)$, we can easily get that there exists $\eta>0$ such that this lemma holds if we let $\|u\|_H=\rho>0$ small enough. □

Lemma 3.5. *Assume that the conditions* (V2) *and* (S2) *hold. Then there exists* $v \in H$ *with* $\|v\|_H = \rho$ *such that* $I(v) < 0$*, where* ρ *is given in Lemma* 3.4.

Proof. From (2.3), we have

$$\frac{I(tu)}{t^4} = \frac{1}{2t^2}\left(\|u\|_H^2 - \int_{\mathbb{R}^3} \omega\phi_{tu} u^2\,dx\right) - \frac{1}{t^4}\int_{\mathbb{R}^3} \widetilde{F}(x,tu)\,dx.$$

Then, by Proposition 2.2, (S2) and Fatou's lemma we deduce that

$$\begin{aligned}
\lim_{t\to\infty} \frac{I(tu)}{t^4} &= \lim_{t\to\infty}\left[\frac{1}{2t^2}\left(\|u\|_H^2 - \int_{\mathbb{R}^3} \omega\phi_{tu} u^2\,dx\right) - \frac{1}{t^4}\int_{\mathbb{R}^3} \widetilde{F}(x,tu)\,dx\right] \\
&\le \limsup_{t\to\infty}\left[\frac{1}{2t^2}\left(\|u\|_H^2 - \int_{\mathbb{R}^3} \omega^2 u^2\,dx\right) - \frac{1}{t^4}\int_{\mathbb{R}^3} \widetilde{F}(x,tu)\,dx\right] \\
&= -\liminf_{t\to\infty}\int_{\mathbb{R}^3} \frac{\widetilde{F}(x,tu)}{t^4u^4} u^4\,dx \\
&\le -\int_{\mathbb{R}^3} \liminf_{t\to\infty} \frac{\widetilde{F}(x,tu)}{t^4u^4} u^4\,dx \\
&= -\infty \quad \text{as } t\to\infty.
\end{aligned}$$

Thus, the lemma is proved by taking $v = t_0 u$ with $t_0 > 0$ large enough. $\square$

Now, we will complete the proof of Theorem 1.2.

Proof of Theorem 1.2. To complete the proof of Theorem 1.2, we need to consider the following two steps.

Step 1. We first show that there exists a function $u_0 \in H$ such that $I'(u_0) = 0$ and $I(u_0) < 0$. Let $r_0 = 1$, for any $|u| \ge 1$, from (S2), we have

$$\widetilde{F}(x,u_n) \ge c_3 |u_n|^\tau > 0. \tag{3.11}$$

By (S1), for a.e. $x \in \mathbb{R}^3$ and $0 \le |u| \le 1$, there exists $M > 0$ such that

$$\left|\frac{\widetilde{f}(x,u)u}{u^2}\right| \le \left|\frac{(c_1|u|^3 + c_2|u|^{q-1})|u|}{|u|^2}\right| \le M,$$

which implies that

$$\widetilde{f}(x,u)u \ge -M|u|^2.$$

Using the equality $\widetilde{F}(x,u) = \int_0^1 \widetilde{f}(x,tu)\,dt$, for a.e. $x \in \mathbb{R}^3$ and $0 \le |u| \le 1$, we obtain

$$\widetilde{F}(x,u) > -\frac{1}{2}M|u|^2. \tag{3.12}$$

In view of (3.11) and (3.12), we have for a.e. $x \in \mathbb{R}^3$ and all $u \in \mathbb{R}$ that

$$\widetilde{F}(x,u) \ge -\frac{1}{2}M|u|^2 + c_3|u|^\tau.$$

Then, we have

$$\widetilde{F}(x, t\psi) \geq -\frac{1}{2}Mt^2 |\psi|^2 + t^\tau c_3 |\psi|^\tau . \tag{3.13}$$

Combining Proposition 2.2, (2.3) with (3.13), we have

$$\begin{aligned} I(tu) &= \frac{t^2}{2} \|u\|_H^2 - \frac{t^2}{2} \int_{\mathbb{R}^3} \omega\phi_{tu} u^2 \, dx - \int_{\mathbb{R}^3} \widetilde{F}(x, tu) \, dx \\ &\leq \frac{t^2}{2} \|u\|_H^2 - \frac{t^2}{2} \int_{\mathbb{R}^3} \omega^2 u^2 \, dx + \frac{t^2 M}{2} \int_{\mathbb{R}^3} |u|^2 \, dx - t^\tau c_3 \int_{\mathbb{R}^3} |u|^\tau \, dx. \end{aligned}$$

Since $\mu > 4$, $\tau \in (0, 2)$, for t small enough, we can get that $I(tu) < 0$. Thus, we obtain

$$c_0 = \inf \left\{ I(u) : u \in \overline{B}_\rho \right\} < 0,$$

where $\rho > 0$ is given by Lemma 3.4, $B_\rho = \{u \in H : \|u\|_H < \rho\}$. By Ekeland's variational principle, there exists a sequence $\{u_n\} \subset B_\rho$ such that

$$c_0 \leq I(u_n) \leq c_0 + \frac{1}{n} \quad \text{and} \quad I(w) \geq I(u_n) - \frac{1}{n} \|w - u_n\|_H$$

for all $w \in B_\rho$. Then, following the idea of [30], we can show that $\{u_n\}$ is a bounded Cerami sequence of I. Therefore, Lemma 3.3 implies that there exists a function $u_0 \in H$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$.

Step 2. We now show that there exists a function $\widetilde{u}_0 \in H$ such that $I'(\widetilde{u}_0) = 0$ and $I(\widetilde{u}_0) = \widetilde{c}_0 > 0$. By Lemmas 3.4, 3.5 and 2.3, there is a sequence $\{u_n\} \in H$ satisfies (3.1). Moreover, Lemmas 3.1 and 3.3 show that this sequence has a convergent subsequence and is bounded in H. So, we complete Step 2.

Therefore, combining the above two steps and Lemma 1.1, the proof of Theorem 1.2 is complete. □

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