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Analysis of a Frictionless Contact Problem with Adhesion for Piezoelectric Materials

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Abstract. This paper is devoted to the study of the mathematical model involving a frictionless contact between an electro-elasto-viscoplastic body and a conductive adhesive foundation. The process is mechanically dynamic and electrically static. The contact is modeled with a normal compliance where the adhesion is taken into account and a regularized electrical conductivity condition. We derive a variational formulation of the problem and prove its unique weak solution. The proof is based on nonlinear evolution equations with monotone operators, differential equations and fixed point arguments.

1. Introduction

Considerable progress has been achieved recently in modeling and mathematical analysis of various processes involved in contact between deformable bodies. Piezoelectricity is the ability of certain crystals, like the quartz (also ceramics ($BaTiO_3$, $KNbO_3$, $LiNbO_3$, ...) and even the human mandible), to produce a voltage when they are subjected to mechanical stress. The piezoelectric effect is characterized by the coupling between the mechanical and the electrical properties of the material: it was observed that the appearance of electric charges on some crystals was due to the action of body forces and surface tractions and, conversely, the action of the electric field generated strain or stress in the body. This kind of materials appears usually in the industry as switches in radiotronics, electroacoustics or measuring equipments (see [11]). Different models have been developed to describe the interaction between the electric and mechanical fields (see, e.g., [2, 9, 11]) and the references therein. Recently, contact problems involving elasto-piezoelectric materials [1,3,14], viscoelastic piezoelectric materials [21] or elasto-viscoplastic piezoelectric materials [10] have been studied. A quasistatic problem with normal compliance for electro-viscoelastic materials in frictional contact with a conductive foundation was investigated in [13]. The

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adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has also received recently increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [5, 6, 15–17, 19] and recently in the monographs [22]. The bonding field satisfies the restrictions $0 \le \beta \le 1$, when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion, when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active.

In this paper we deal with the study of a dynamic problem of frictionless adhesive contact for general electro-elasto-viscoplastic materials of the form

$$
\begin{aligned}
\boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \mathcal{E}^*\boldsymbol{E}(\varphi(t)) \\
&\quad + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)) + \mathcal{E}^*\boldsymbol{E}(\varphi(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{k}(s))\, ds, \qquad (1.1)
\end{aligned}
$$

$$
\dot{\boldsymbol{k}}(t) = \phi(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{E}^*\boldsymbol{E}(\varphi(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{k}(t)), \qquad (1.2)
$$

$$
\boldsymbol{D}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) - \boldsymbol{B}\nabla\varphi(t), \qquad (1.3)
$$

where $\boldsymbol{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the linearized strain tensor and $\boldsymbol{D}$ is the electric displacement field. Here $\mathcal{A}$ is the viscosity operator, allowed to be nonlinear, $\mathcal{F}$ is the elasticity operator and $\mathcal{G}$ is a nonlinear constitutive function describing the viscoplastic behavior of the material and depending on the internal state variable $\boldsymbol{k}$. ϕ is also a nonlinear constitutive function which depends on $\boldsymbol{k}$. There is a variety of choices for the internal state variables, for reference in the field see [7]. Some commonly used internal state variables are the plastic strain and a number of tensor variables that take into account the spatial display of dislocations and the work-hardening of the material. $\boldsymbol{E}$ is the electric field that satisfies $\boldsymbol{E}(\varphi) = -\nabla\varphi$, where φ is the electric potential. Also, $\mathcal{E}$ represents the third order piezoelectric tensor, $\mathcal{E}^*$ is its transposed and $\boldsymbol{B}$ denotes the electric permittivity tensor.

We assume the decomposition of the form $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{EVP}} + \boldsymbol{\sigma}^{\mathrm{E}}$, where $\boldsymbol{\sigma}^{\mathrm{E}} = -\mathcal{E}^*\boldsymbol{E}(\varphi) = \mathcal{E}^*\nabla\varphi$ is the electric part of the stress and $\boldsymbol{\sigma}^{\mathrm{EVP}}$ is the elastic-viscoplastic part of the stress which satisfies

$$
\begin{gathered}
\boldsymbol{\sigma}^{\mathrm{EVP}}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}^{\mathrm{EVP}}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{k}(s))\, ds, \\
\dot{\boldsymbol{k}}(t) = \phi(\boldsymbol{\sigma}^{\mathrm{EVP}}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{k}(t)).
\end{gathered}
$$

A frictionless contact for elastic-viscoplastic materials with internal state variable was studied in [18, 19].

When $\mathcal{G} = 0$ the constitutive law (1.1)–(1.3) reduces to the electro-viscoelastic law given by (1.3) and

$$
\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t).
$$

When $\mathcal{G} = 0$ and $\mathcal{A} = 0$ the constitutive law (1.1)–(1.3) becomes the electro-elastic constitutive law given by (1.3) and

$$\boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t).$$

The aim of this paper is to study the evolution process of a piezoelectric body under actions of volume and surface forces as well as electrical charge. We assume the mechanical process dynamic and the properties of the body are electro-elasto-viscoplastic described by the constitutive equations (1.1)–(1.3). The contact is frictionless, modeled with a normal compliance with adhesion and a regularized electrical conductivity condition. The novelty in this paper consists on the coupling of adhesion and the conductivity of the foundation which give us a new and nonstandard boundary conditions.

The rest of the paper is structured as follows. Section 2 is devoted to the basic results on functional analysis which are fundamental to the study of the problem treated in this paper. In Section 3 we present the mechanical model, we list the assumptions on the data and give the variational formulation of the problem. In Section 4 we prove an existence and uniqueness result based on nonlinear evolution equations with monotone operators, differential equations and fixed point arguments.

2. Notations and preliminaries

In this section we present the notations and some preliminary material which are necessary in the study of our problem. For further details, we refer the reader to [4, 8]. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ ($d = 2, 3$), while "$\cdot$" and $|\cdot|$ will represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$ respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. Everywhere, the indices i, j run between 1 and d, the summation over repeated indices is implied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the independent spatial variable.

We use the standard notations for Lebesgue and Sobolev spaces associated to Ω and Γ and introduce the spaces

$$\begin{aligned}
H &= L^2(\Omega)^d = \left\{\boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega)\right\}, \\
\mathcal{H} &= \left\{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \\
H_1 &= \{\boldsymbol{u} = (u_i) \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\}, \\
\mathcal{H}_1 &= \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div}\boldsymbol{\sigma} \in H\}.
\end{aligned}$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j}). \tag{2.1}$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned}
(\boldsymbol{u},\boldsymbol{v})_H &= \int_\Omega u_i v_i\,dx, \quad \forall\,\boldsymbol{u},\boldsymbol{v}\in H,\\
(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} &= \int_\Omega \sigma_{ij}\tau_{ij}\,dx, \quad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in \mathcal{H},\\
(\boldsymbol{u},\boldsymbol{v})_{H_1} &= (\boldsymbol{u},\boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall\,\boldsymbol{u},\boldsymbol{v}\in H_1,\\
(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} &= (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\mathrm{Div}\,\boldsymbol{\tau})_H, \quad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in \mathcal{H}_1.
\end{aligned}$$

The associated norms on the spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively. For every $\boldsymbol{v}\in H_1$, we also use the notation $\gamma\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the normal and the tangential components of $\boldsymbol{v}$ on Γ given by

$$v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}. \tag{2.2}$$

We recall that when $\boldsymbol{\sigma}$ is a regular function then the normal component and the tangential part of the stress field $\boldsymbol{\sigma}$ on the boundary are defined by

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}, \tag{2.3}$$

and the following Green's formula holds:

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}\cdot\boldsymbol{v}\,da, \quad \forall\,\boldsymbol{v}\in H_1. \tag{2.4}$$

Let $T>0$, for every real Banach space X we use the notations $C(0,T;X)$ and $C^1(0,T;X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively; $C(0,T;X)$ is a real Banach space with the norm

$$|\boldsymbol{f}|_{C(0,T;X)} = \max_{t\in[0,T]} |\boldsymbol{f}(t)|_X\,,$$

while $C^1(0,T;X)$ is a real Banach space with the norm

$$|\boldsymbol{f}|_{C^1(0,T;X)} = \max_{t\in[0,T]} |\boldsymbol{f}(t)|_X + \max_{t\in[0,T]} \left|\dot{\boldsymbol{f}}(t)\right|_X.$$

We use the dot above to indicate the derivative with respect to the time variable and for a real number r, we use r_+ to represent its positive part, that is $r_+ = \max\{0,r\}$.

Finally, for $k\in\mathbb{N}$ and $p\in[1,\infty]$, we use the classical notation for the Lebesgue spaces $L^p(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1\times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot\,,\cdot)_{X_1\times X_2}$.

We recall the following standard result for the first order evolution equations (see [22]). Let V and H be real Hilbert spaces such that V is dense in H and the injection map is continuous. The space H is identified with its own dual and with a subspace of the dual V' of V. We write

$$V \subset H \subset V'$$

and we say that the inclusions above define a Gelfand triple. We denote by $|\cdot|_V$, $|\cdot|_H$ and $|\cdot|_{V'}$ the norms on the spaces V, H and V' respectively, and we use $(\cdot\,,\cdot)_{V'\times V}$ for the duality pairing between V' and V. Note that if $\boldsymbol{f} \in H$ then

$$(\boldsymbol{f}, \boldsymbol{v})_{V'\times V} = (\boldsymbol{f}, \boldsymbol{v})_H, \quad \forall\, \boldsymbol{v} \in V.$$

Theorem 2.1. *Let V, H be as above, and let $A\colon V \to V'$ be hemicontinuous and monotone operator which satisfies*

$$(A\boldsymbol{u}, \boldsymbol{u})_{V'\times V} \geq \omega\, |\boldsymbol{u}|_V^2 + \lambda, \quad \forall\, \boldsymbol{u} \in V, \tag{2.5}$$

$$|A\boldsymbol{u}|_{V'} \leq c(|\boldsymbol{u}|_V + 1), \quad \forall\, \boldsymbol{u} \in V, \tag{2.6}$$

for some constants $\omega > 0$, $c > 0$ and $\lambda \in \mathbb{R}$. Then, given $\boldsymbol{u}_0 \in H$ and $\boldsymbol{f} \in L^2(0,T;V')$, there exists a unique function $\boldsymbol{u}$ which satisfies

$$\boldsymbol{u} \in L^2(0,T;V) \cap C(0,T;H), \quad \dot{\boldsymbol{u}} \in L^2(0,T;V'),$$
$$\dot{\boldsymbol{u}}(t) + A\boldsymbol{u}(t) = \boldsymbol{f}(t), \quad a.e.\ t \in (0,T),$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0.$$

This theorem will be used in Section 4 to obtain the existence and uniqueness result.

3. Mechanical and variational formulations

The physical setting is as follows. An electro-elasto-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with a smooth boundary Γ. The body is acted upon by body forces of density $\boldsymbol{f}_0$ and has volume electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. We consider a partition of Γ into three disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two parts Γ_a and Γ_b, on the other hand such that $\operatorname{meas}(\Gamma_1) > 0$ and $\operatorname{meas}(\Gamma_a) > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$, so the displacement field vanishes there. Surface traction of density $\boldsymbol{f}_2$ acts on $\Gamma_2 \times (0,T)$ and a body force of density $\boldsymbol{f}_0$ is applied in $\Omega \times (0,T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0,T)$ and a surface electrical charge of density q_b is prescribed on $\Gamma_b \times (0,T)$. The body is in adhesive contact with a conductive obstacle or foundation, over the contact surface $\Gamma_3 \times (0,T)$. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. Then, the classical formulation of the mechanical contact problem of electro-elasto-viscoplastic material with internal state variable is as follows.

Problem 3.1. Find a displacement field $\boldsymbol{u}\colon \Omega\times[0,T]\to\mathbb{R}^d$, a stress field $\boldsymbol{\sigma}\colon\Omega\times[0,T]\to S_d$, an electric potential field $\varphi\colon\Omega\times[0,T]\to\mathbb{R}$, a bonding field $\beta\colon\Gamma_3\times[0,T]\to\mathbb{R}$ and an internal state variable field $\boldsymbol{k}\colon\Omega\times[0,T]\to\mathbb{R}^m$ such that

$$\boldsymbol{\sigma}(t)=\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))+\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\mathcal{E}^*\nabla\varphi(t) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s)-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s))-\mathcal{E}^*\nabla\varphi(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\boldsymbol{k}(s))\,ds \quad \text{in } \Omega\times(0,T), \tag{3.1}$$

$$\dot{\boldsymbol{k}}=\phi(\boldsymbol{\sigma}-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})-\mathcal{E}^*\nabla\varphi,\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{k}) \quad \text{in } \Omega\times(0,T), \tag{3.2}$$

$$\boldsymbol{D}=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{B}\nabla\varphi \quad \text{in } \Omega\times(0,T), \tag{3.3}$$

$$\rho\ddot{\boldsymbol{u}}=\operatorname{Div}\boldsymbol{\sigma}+\boldsymbol{f}_0 \quad \text{in } \Omega\times(0,T), \tag{3.4}$$

$$\operatorname{div}\boldsymbol{D}=q_0 \quad \text{in } \Omega\times(0,T), \tag{3.5}$$

$$\boldsymbol{u}=\boldsymbol{0} \quad \text{on } \Gamma_1\times(0,T), \tag{3.6}$$

$$\boldsymbol{\sigma\nu}=\boldsymbol{f}_2 \quad \text{on } \Gamma_2\times(0,T), \tag{3.7}$$

$$-\sigma_\nu=p_\nu(u_\nu)-\gamma_\nu\beta^2R_\nu(u_\nu) \quad \text{on } \Gamma_3\times(0,T), \tag{3.8}$$

$$-\boldsymbol{\sigma}_\tau=p_\tau(\beta)\boldsymbol{R}_\tau(\boldsymbol{u}_\tau) \quad \text{on } \Gamma_3\times(0,T), \tag{3.9}$$

$$\dot{\beta}=-(\beta(\gamma_\nu(R_\nu(u_\nu))^2+\gamma_\tau\,|\boldsymbol{R}_\tau(\boldsymbol{u}_\tau)|^2)-\epsilon_a)_+ \quad \text{on } \Gamma_3\times(0,T), \tag{3.10}$$

$$\varphi=0 \quad \text{on } \Gamma_a\times(0,T), \tag{3.11}$$

$$\boldsymbol{D}.\boldsymbol{\nu}=q_b \quad \text{on } \Gamma_b\times(0,T), \tag{3.12}$$

$$\boldsymbol{D}.\boldsymbol{\nu}=\psi(u_\nu)\Phi(\varphi-\varphi_0) \quad \text{on } \Gamma_3\times(0,T), \tag{3.13}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0,\quad \dot{\boldsymbol{u}}(0)=\boldsymbol{v}_0,\quad \boldsymbol{k}(0)=\boldsymbol{k}_0 \quad \text{in } \Omega, \tag{3.14}$$

$$\beta(0)=\beta_0 \quad \text{on } \Gamma_3. \tag{3.15}$$

We now describe (3.1)–(3.15) and provide explanation of the equations and the boundary conditions. First, equations (3.1)–(3.3) represent the electro-elasto-viscoplastic constitutive law with internal state variable introduced in the first section. Equations (3.4)–(3.5) represent the equation of motion and equilibrium equation for the electric-displacement field, respectively. Equations (3.6)–(3.7) are the displacement-traction conditions. Condition (3.8) represents the normal compliance condition with adhesion where p_ν is a given positive function and γ_ν is a given adhesion coefficient. The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu\beta^2R_\nu(u_\nu)$, the adhesive traction is tensile and proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L$. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \le s \le 0, \\ 0 & \text{if } s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\boldsymbol{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive from the physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (3.9) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\boldsymbol{R}_\tau$ is the truncation operator given by

$$\boldsymbol{R}_\tau(\boldsymbol{v}) = \begin{cases} \boldsymbol{v} & \text{if } |\boldsymbol{v}| \le L, \\ L\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|} & \text{if } |\boldsymbol{v}| > L. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted.

Equation (3.10) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [5], see also [22] for details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ϵ_a. Notice that in this model once debonding occurs, adhesion cannot be reestablished, since $\dot{\beta} \le 0$. Equations (3.11)–(3.12) represent the electric boundary conditions.

Next, (3.13) is the electrical contact condition on Γ_3. It represents a regularized condition which may be obtained as follows. First, we assume that the foundation is electrically conductive and its potential is maintained at φ_0. When $u_\nu < 0$ there is no contact, there are no free electrical charges on the surface and the normal component of the electric displacement field vanishes. Thus,

$$u_\nu < 0 \;\Rightarrow\; \boldsymbol{D}.\boldsymbol{\nu} = 0. \tag{3.16}$$

When there is contact then u_ν is positive the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with l as the proportionality factor. Thus,

$$u_\nu \ge 0 \;\Rightarrow\; \boldsymbol{D}.\boldsymbol{\nu} = l(\varphi - \varphi_0). \tag{3.17}$$

We combine (3.16) and (3.17) to obtain

$$\boldsymbol{D}.\boldsymbol{\nu} = l\chi_{[0,\infty)}(u_\nu)(\varphi - \varphi_0), \tag{3.18}$$

where $\chi_{[0,\infty)}$ is the characteristic function of the interval $[0,\infty)$, that is

$$\chi_{[0,\infty)}(r) = \begin{cases} 0 & \text{if } r < 0, \\ 1 & \text{if } r \geq 0. \end{cases}$$

Condition (3.18) describes perfect electrical contact and is somewhat similar to the well-known Signorini contact condition. Both conditions may be over-idealizations in many applications. To make it more realistic, we regularize conditions (3.18) and write it as (3.13) in which $l\chi_{[0,\infty)}(u_\nu)$ is replaced with ψ which is a regular function which will be described below, and Φ is the truncation function

$$\Phi(s) = \begin{cases} 0 & \text{if } s < -L, \\ s & \text{if } -L \leq s \leq L, \\ L & \text{if } s > L, \end{cases}$$

where L is a large positive constant. We note that is truncation does not pose any practical limitations on the applicability of the model, since L my be arbitrarily large, higher than any possible peak voltage in the system, and therefore in applications $\Phi(\varphi - \varphi_0) = \varphi - \varphi_0$.

The reasons for the regularization (3.13) of (3.18) are mathematical. First, we need to avoid the discontinuity in the free electric charge when contact is established and, therefore, we regularize the function $l\chi_{[0,\infty)}$ in (3.18) with a Lipschitz continuous function ψ. A possible choice is

$$\psi(r) = \begin{cases} 0 & \text{if } r < 0, \\ l\delta r & \text{if } 0 \leq r \leq 1/\delta, \\ l & \text{if } r > \delta, \end{cases} \tag{3.19}$$

where $\delta > 0$ is a small parameter. This choice means that during the process of contact the electrical conductivity increases as the contact among the surface asperities improves, and stabilizes when u_ν reaches the value δ. Secondly, we need the term $\Phi(\varphi - \varphi_0)$ to control the boundedness of $\varphi - \varphi_0$.

Note that when $\psi \equiv 0$ in (3.13) then

$$\boldsymbol{D}.\boldsymbol{\nu} = 0 \quad \text{on } \Gamma_3 \times (0,T), \tag{3.20}$$

which decouples the electrical and mechanical problems on the contact surface. Condition (3.20) models the case when the obstacle is a perfect insulator and was used in [3, 20, 21]. Condition (3.13), instead of (3.20), introduces strong coupling between the mechanical and the electric boundary conditions and leads to a new and nonstandard mathematical model.

In (3.14), $\boldsymbol{u}_0$ is the initial displacement, $\boldsymbol{v}_0$ is the initial velocity and $\boldsymbol{k}_0$ is the initial internal state variable. Finally (3.15) is the initial condition, in which β_0 denotes the initial bonding field. To obtain the variational formulation of Problem 3.1 we introduce for the bonding field the set

$$Z = \left\{\theta\colon [0,T] \to L^2(\Gamma_3) \mid 0 \le \theta(t) \le 1, \forall\, t \in [0,T],\ \text{a.e. on } \Gamma_3\right\}.$$

For the displacement field we use closed subspace of H_1 defined by

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\}.$$

The set of admissible internal state variables is given by

$$Y = \left\{\boldsymbol{\alpha} = (\alpha_i) \mid \alpha_i \in L^2(\Omega), 1 \le i \le m\right\}.$$

Since $\operatorname{meas}(\Gamma_1) > 0$, the following Korn inequality holds:

$$|\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}} \ge c_k\, |\boldsymbol{v}|_{H_1}, \quad \forall\, \boldsymbol{v} \in V,$$

where $c_k > 0$ is a constant depending only on Ω and Γ_1. We consider the inner product $(\cdot\,,\cdot)_V$ on V given by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \tag{3.21}$$

and let $|\cdot|_V$ be the associated norm. From Korn's inequality it follows that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V. Therefore $(V, |\cdot|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem there exists a positive constant c_0 which depends only on Ω, Γ_1 and Γ_3 such that

$$|\boldsymbol{v}|_{L^2(\Gamma_3)^d} \le c_0\, |\boldsymbol{v}|_V, \quad \forall\, \boldsymbol{v} \in V. \tag{3.22}$$

For the electric displacement field we use Hilbert spaces

$$\mathcal{W} = L^2(\Omega)^d, \quad \mathcal{W}_1 = \left\{\boldsymbol{D} \in \mathcal{W} \mid \operatorname{div} \boldsymbol{D} \in L^2(\Omega)\right\},$$

endowed with the inner product

$$(\boldsymbol{D}, \boldsymbol{E})_{\mathcal{W}} = \int_\Omega D_i E_i\, dx, \quad (\boldsymbol{D}, \boldsymbol{E})_{\mathcal{W}_1} = (\boldsymbol{D}, \boldsymbol{E})_{\mathcal{W}} + (\operatorname{div} \boldsymbol{D}, \operatorname{div} \boldsymbol{E})_{L^2(\Omega)},$$

and the associated norms $|\cdot|_{\mathcal{W}}$ and $|\cdot|_{\mathcal{W}_1}$, respectively.

The electric potential field is to be found in

$$W = \left\{ \zeta \in H^1(\Omega) \mid \zeta = 0 \text{ on } \Gamma_a \right\}.$$

Since $\mathrm{meas}(\Gamma_a) > 0$, the Friedrichs-Poincaré inequality holds:

$$|\nabla\zeta|_{\mathcal{W}} \geq c_F \, |\zeta|_{H^1(\Omega)}\,, \quad \forall\, \zeta \in W, \tag{3.23}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W we use the inner product

$$(\varphi, \zeta)_W = (\nabla\varphi, \nabla\zeta)_{\mathcal{W}}, \tag{3.24}$$

and $|\cdot|_W$ the associated norm. It follows from (3.23) that $|\cdot|_{H_1}$ and $|\cdot|_W$ are equivalent norms on W and therefore $(W, |\cdot|_W)$ is a real Hilbert space. By the Sobolev trace theorem, there exists a constant $\widetilde{c}_0$ depending only on Ω, Γ_a and Γ_3 such that

$$|\zeta|_{L^2(\Gamma_3)} \leq \widetilde{c}_0 \, |\zeta|_W\,, \quad \forall\, \zeta \in W. \tag{3.25}$$

Moreover, when $\boldsymbol{D} \in \mathcal{W}_1$ is a regular function, the following Green's type formula holds:

$$(\boldsymbol{D}, \nabla\zeta)_H + (\operatorname{div} \boldsymbol{D}, \zeta)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D}.\boldsymbol{\nu}\zeta \, da, \quad \forall\, \zeta \in H^1(\Omega). \tag{3.26}$$

In the study of the mechanical problem (3.1)–(3.15) we make the following assumptions. The viscosity operator $\mathcal{A}\colon \Omega \times S_d \to S_d$ satisfies

(3.27a) There exist constants $C_{\mathcal{A}}^1, C_{\mathcal{A}}^2 > 0$ such that

$$|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})| \leq C_{\mathcal{A}}^1 \, |\boldsymbol{\varepsilon}| + C_{\mathcal{A}}^2, \quad \forall\, \boldsymbol{\varepsilon} \in S_d, \text{ a.e. } \boldsymbol{x} \in \Omega.$$

(3.27b) There exists a constant $m_{\mathcal{A}} > 0$ such that

$$(\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|^2\,, \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \boldsymbol{x} \in \Omega.$$

(3.27c) $\boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is Lebesgue measurable on Ω for all $\boldsymbol{\varepsilon} \in S_d$.

(3.27d) $\boldsymbol{\varepsilon} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is continuous on S_d, a.e. $\boldsymbol{x} \in \Omega$.

The elasticity operator $\mathcal{F}\colon \Omega \times S_d \to S_d$ satisfies

(3.28a) There exists a constant $L_{\mathcal{F}} > 0$ such that

$$|\mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{F}} \, |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2|\,, \quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_d, \text{ a.e. } \boldsymbol{x} \in \Omega.$$

(3.28b) $\boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{\varepsilon})$ is Lebesgue measurable on Ω for all $\boldsymbol{\varepsilon} \in S_d$.

(3.28c) $\boldsymbol{x} \mapsto \mathcal{F}(\boldsymbol{x}, \boldsymbol{0})$ is in $\mathcal{H}$.

The viscoplasticity operator $\mathcal{G}\colon \Omega \times S_d \times S_d \times \mathbb{R}^m \to S_d$ satisfies

(3.29a) There exists a constant $L_{\mathcal{G}} > 0$ such that
$$|\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\boldsymbol{k}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\boldsymbol{k}_2)| \le L_{\mathcal{G}}(|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2| + |\boldsymbol{k}_1-\boldsymbol{k}_2|),$$
$\forall\, \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in S_d$ and $\boldsymbol{k}_1,\boldsymbol{k}_2 \in \mathbb{R}^m$, a.e. $\boldsymbol{x}\in\Omega$.

(3.29b) For any $\boldsymbol{\sigma},\boldsymbol{\varepsilon}\in S_d$ and $\boldsymbol{k}\in\mathbb{R}^m$, $\boldsymbol{x}\mapsto \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{k})$ is Lebesgue measurable on Ω.

(3.29c) $\boldsymbol{x}\mapsto\mathcal{G}(\boldsymbol{x},\mathbf{0},\mathbf{0},\mathbf{0})$ is in $\mathcal{H}$.

The function $\phi\colon \Omega\times S_d\times S_d\times\mathbb{R}^m\to\mathbb{R}^m$ satisfies

(3.30a) There exists a constant $L_\phi > 0$ such that
$$|\phi(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\boldsymbol{k}_1) - \phi(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\boldsymbol{k}_2)| \le L_\phi(|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2| + |\boldsymbol{k}_1-\boldsymbol{k}_2|),$$
$\forall\, \boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in S_d$ and $\boldsymbol{k}_1,\boldsymbol{k}_2 \in \mathbb{R}^m$, a.e. $\boldsymbol{x}\in\Omega$.

(3.30b) For any $\boldsymbol{\sigma},\boldsymbol{\varepsilon}\in S_d$ and $\boldsymbol{k}\in\mathbb{R}^m$, $\boldsymbol{x}\mapsto \phi(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{k})$ is Lebesgue measurable on Ω.

(3.30c) $\boldsymbol{x}\mapsto\phi(\boldsymbol{x},\mathbf{0},\mathbf{0},\mathbf{0})$ belongs to $L^2(\Omega)^m$.

The electric permittivity tensor $\boldsymbol{B}=(b_{ij})\colon \Omega\times\mathbb{R}^d\to\mathbb{R}^d$ satisfies

(3.31a) $$\boldsymbol{B}(\boldsymbol{x},\boldsymbol{E}) = (b_{ij}(\boldsymbol{x})E_j),\quad \forall\,\boldsymbol{E}=(E_j)\in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega.$$

(3.31b) $$b_{ij}=b_{ji}\in L^\infty(\Omega).$$

(3.31c) There exists a constant $m_B>0$ such that
$$\boldsymbol{B}\boldsymbol{E}.\boldsymbol{E}\ge m_B\,|\boldsymbol{E}|^2,\quad \forall\,\boldsymbol{E}=(E_j)\in\mathbb{R}^d, \text{ a.e. } \boldsymbol{x}\in\Omega.$$

The piezoelectric tensor $\mathcal{E}\colon\Omega\times S_d\to\mathbb{R}^d$ satisfies

(3.32a) $$\mathcal{E}(\boldsymbol{x},\boldsymbol{\tau}) = (e_{ijk}(\boldsymbol{x})\tau_{jk}),\quad \forall\,\boldsymbol{\tau}=(\tau_{jk})\in S_d, \text{ a.e. } \boldsymbol{x}\in\Omega.$$

(3.32b) $$e_{ijk}=e_{ikj}\in L^\infty(\Omega).$$

The normal compliance function $p_\nu\colon\Gamma_3\times\mathbb{R}\to\mathbb{R}_+$ satisfies

(3.33a) There exists a constant $L_\nu>0$ such that
$$|p_\nu(\boldsymbol{x},r_1)-p_\nu(\boldsymbol{x},r_2)|\le L_\nu\,|r_1-r_2|\,,\quad \forall\, r_1,r_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Gamma_3.$$

(3.33b) for any $r\in\mathbb{R}$, $\boldsymbol{x}\mapsto p_\nu(\boldsymbol{x},r)$ is Lebesgue measurable on Γ_3.

(3.33c) $p_\nu(\boldsymbol{x},r)=0$ for all $r\le 0$, a.e. $\boldsymbol{x}\in\Gamma_3$.

The tangential contact function $p_\tau\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

(3.34a) There exists a constant $L_\tau > 0$ such that

$$|p_\tau(\boldsymbol{x}, d_1) - p_\tau(\boldsymbol{x}, d_2)| \le L_\tau\, |d_1 - d_2|\,, \quad \forall\, d_1, d_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.$$

(3.34b) There exists $M_\tau > 0$ such that $|p_\tau(\boldsymbol{x}, d)| \le M_\tau$, $\forall\, d \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

(3.34c) for any $d \in \mathbb{R}$, $\boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, d)$ is Lebesgue measurable on Γ_3.

(3.34d) The mapping $\boldsymbol{x} \mapsto p_\tau(\boldsymbol{x}, 0)$ belongs to $L^2(\Gamma_3)$.

An example of a normal compliance function p_ν which satisfies conditions (3.33) is $p_\nu(r) = c_\nu r_+$ where c_ν is a positive constant and $r_+ = \max\{0, r\}$.

The surface electrical conductivity function $\psi\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies

(3.35a) There exists a constant $L_\psi > 0$ such that

$$|\psi(\boldsymbol{x}, u_1) - \psi(\boldsymbol{x}, u_2)| \le L_\psi\, |u_1 - u_2|\,, \quad \forall\, u_1, u_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3.$$

(3.35b) There exists $M_\psi > 0$ such that $|\psi(\boldsymbol{x}, u)| \le M_\psi$, $\forall\, u \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Gamma_3$.

(3.35c) $\boldsymbol{x} \mapsto \psi(\boldsymbol{x}, u)$ is measurable on Γ_3 for all $u \in \mathbb{R}$.

(3.35d) $\psi(\boldsymbol{x}, u) = 0$ for all $u \le 0$.

We also suppose that the mass density satisfies

(3.36) $\rho \in L^\infty(\Omega)$, there exists $\rho^* > 0$ such that $\rho(\boldsymbol{x}) \ge \rho^*$, a.e. $\boldsymbol{x} \in \Omega$.

The body forces and surface tractions have the regularity

$$\boldsymbol{f}_0 \in L^2(0,T;H), \quad \boldsymbol{f}_2 \in L^2(0,T;L^2(\Gamma_2)^d), \tag{3.37}$$

$$q_0 \in W^{1,p}(0,T;L^2(\Omega)), \quad q_b \in W^{1,p}(0,T;L^2(\Gamma_b)). \tag{3.38}$$

The adhesion coefficients satisfy

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \epsilon_a \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{3.39}$$

The initial data satisfy

$$\boldsymbol{u}_0 \in V, \quad \boldsymbol{v}_0 \in H, \quad \boldsymbol{k}_0 \in Y. \tag{3.40}$$

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \le \beta_0 \le 1 \quad \text{a.e. on } \Gamma_3. \tag{3.41}$$

Finally, the given potential satisfies

$$\varphi_0 \in L^2(\Gamma_3). \tag{3.42}$$

We will use a modified inner product on the Hilbert space $H = L^2(\Omega)^d$ given by

$$((\boldsymbol{u}, \boldsymbol{v}))_H = (\rho \boldsymbol{u}, \boldsymbol{v})_H, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in H,$$

that is, it is weighted with ρ, and we let $\|\cdot\|_H$ the associated norm, i.e.,

$$\|\boldsymbol{v}\|_H = (\rho\boldsymbol{v}, \boldsymbol{v})_H^{1/2}, \quad \forall\, \boldsymbol{v} \in H.$$

It follows from assumption (3.36) that $\|\cdot\|_H$ and $|\cdot|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |\cdot|_V)$ into $(H, \|\cdot\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual we obtain the Gelfand triple $V \subset H \subset V'$. We use the notation $(\cdot\,,\cdot)_{V'\times V}$ for the duality between V' and V and recall that

$$(\boldsymbol{u}, \boldsymbol{v})_{V'\times V} = ((\boldsymbol{u}, \boldsymbol{v}))_H, \quad \forall\, \boldsymbol{u} \in H,\ \forall\, \boldsymbol{v} \in V.$$

We define four mappings $j\colon L^2(\Gamma_3) \times V \times V \to \mathbb{R}$, $h\colon V \times W \to W$, $\boldsymbol{f}\colon [0,T] \to V'$ and $q\colon [0,T] \to W$, respectively, by

$$j(\beta, \boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_3} p_\nu(u_\nu)v_\nu\, da + \int_{\Gamma_3} p_\tau(\beta)\boldsymbol{R}_\tau(\boldsymbol{u}_\tau).\boldsymbol{v}_\tau\, da - \int_{\Gamma_3} \gamma_\nu\beta^2 R_\nu(u_\nu)v_\nu\, da, \tag{3.43}$$

$$(h(\boldsymbol{u}, \varphi), \zeta)_W = \int_{\Gamma_3} \psi(u_\nu)\varPhi(\varphi - \varphi_0)\zeta\, da, \tag{3.44}$$

$$(\boldsymbol{f}(t), \boldsymbol{v})_{V'\times V} = \int_\Omega \boldsymbol{f}_0(t).\boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(t).\boldsymbol{v}\, da, \tag{3.45}$$

$$(q(t), \zeta)_W = \int_\Omega q_0(t)\zeta\, dx - \int_{\Gamma_b} q_b(t)\zeta\, da \tag{3.46}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\varphi, \zeta \in W$ and $t \in [0,T]$. Note that

$$\boldsymbol{f} \in L^2(0,T;V'), \quad q \in W^{1,p}(0,T;W). \tag{3.47}$$

Using standard arguments based on Green's formulas (2.4) and (3.26), we can derive the following variational formulation of Problem 3.1 as follows.

Problem 3.2. Find a displacement field $\boldsymbol{u}\colon [0,T] \to V$, a stress field $\boldsymbol{\sigma}\colon [0,T] \to \mathcal{H}$, an electric potential field $\varphi\colon [0,T] \to W$, a bonding field $\beta\colon [0,T] \to L^2(\Gamma_3)$ and an internal state variable field $\boldsymbol{k}\colon [0,T] \to Y$ such that for a.e. $t \in (0,T)$,

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \mathcal{E}^*\nabla\varphi(t) \\ &\quad + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s)) - \mathcal{E}^*\nabla\varphi(s), \boldsymbol{\varepsilon}(\boldsymbol{u}(s)), \boldsymbol{k}(s))\, ds,\end{aligned} \tag{3.48}$$

$$\dot{\boldsymbol{k}}(t) = \phi(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) - \mathcal{E}^*\nabla\varphi(t), \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{k}(t)), \tag{3.49}$$

$$(\ddot{\boldsymbol{u}}(t), \boldsymbol{v})_{V'\times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + j(\beta(t), \boldsymbol{u}(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{V'\times V}, \quad \forall\, \boldsymbol{v} \in V, \tag{3.50}$$

$$(\boldsymbol{B}\nabla\varphi(t), \nabla\zeta)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \nabla\zeta)_H + (h(\boldsymbol{u}(t), \varphi(t)), \zeta)_W = (q(t), \zeta)_W, \quad \forall\, \zeta \in W, \tag{3.51}$$

$$\dot{\beta}(t) = -(\beta(t)(\gamma_\nu(R_\nu(u_\nu(t)))^2 + \gamma_\tau\, |\boldsymbol{R}_\tau(\boldsymbol{u}_\tau(t))|^2) - \epsilon_a)_+, \tag{3.52}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0, \quad \boldsymbol{k}(0) = \boldsymbol{k}_0, \quad \beta(0) = \beta_0. \tag{3.53}$$

To study this problem we make the following smallness assumption

$$M_\psi \le \frac{m_B}{\widetilde{c}_0^2}, \tag{3.54}$$

where M_ψ, m_B and $\widetilde{c}_0$ are given in (3.35), (3.31) and (3.25) respectively. Removing this assumption remains a task for future research, since it is made for mathematical reasons, and does not seem to relate to any inherent physical constraints of the problem.

The existence of the unique solution to Problem 3.2 is proved in the next section. To this end, we consider the following remark which is used in different places of the paper.

Remark 3.3. We note that, in Problems 3.1 and 3.2 we do not need to impose explicitly the restriction $0 \le \beta \le 1$. Indeed, equation (3.52) guarantees that $\beta(\boldsymbol{x},t) \le \beta_0(\boldsymbol{x})$ and, therefore, assumption (3.41) shows that $\beta(\boldsymbol{x},t) \le 1$ for $t \ge 0$, a.e. $\boldsymbol{x} \in \Gamma_3$. On the other hand, if $\beta(\boldsymbol{x},t_0) = 0$ at time t_0, then it follows from (3.52) that $\beta(\boldsymbol{x},t) = 0$ for all $t \ge t_0$ and therefore, $\beta(\boldsymbol{x},t) = 0$ for all $t \ge t_0$, a.e. $\boldsymbol{x} \in \Gamma_3$. We conclude that $0 \le \beta(\boldsymbol{x},t) \le 1$ for all $t \in [0,T]$, a.e. $\boldsymbol{x} \in \Gamma_3$.

4. Existence and uniqueness result

Our main existence and uniqueness result is the following.

Theorem 4.1. *Let the assumptions* (3.27)–(3.42) *and* (3.54) *hold. Then Problem* 3.2 *has a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}, \boldsymbol{k}, \varphi, \beta\}$ *satisfying*

$$\boldsymbol{u} \in H^1(0,T;V) \cap C^1(0,T;H), \quad \ddot{\boldsymbol{u}} \in L^2(0,T;V'), \tag{4.1}$$

$$\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}), \quad \mathrm{Div}\,\boldsymbol{\sigma} \in L^2(0,T;V'), \tag{4.2}$$

$$\boldsymbol{k} \in W^{1,2}(0,T;Y), \tag{4.3}$$

$$\varphi \in W^{1,p}(0,T;W), \tag{4.4}$$

$$\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z. \tag{4.5}$$

We conclude that, under the assumption (3.27)–(3.42) and (3.54), the mechanical problem (3.1)–(3.15) has a unique weak solution with the regularity (4.1)–(4.5). The proof of this theorem will be carried out in several steps.

In the first step we let $\boldsymbol{\eta} = (\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) \in L^2(0,T;V' \times Y)$ be given, and prove that there exists a unique solution $\boldsymbol{u}_\eta$ of the following intermediate problem.

Problem 4.2. Find a displacement field $\boldsymbol{u}_\eta \colon [0,T] \to V$ such that for a.e. $t \in (0,T)$,

$$(\ddot{\boldsymbol{u}}_\eta(t), \boldsymbol{v})_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\boldsymbol{\eta}^1(t), \boldsymbol{v})_{V'\times V} = (\boldsymbol{f}(t), \boldsymbol{v})_{V'\times V}, \quad \forall \boldsymbol{v} \in V, \tag{4.6}$$

$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}_\eta(0) = \boldsymbol{v}_0. \tag{4.7}$$

Concerning Problem 4.2, we have the following result.

Lemma 4.3. *There exists a unique solution to Problem* 4.2 *with the regularity* (4.1).

Proof. We use the abstract existence and uniqueness result given by Theorem 2.1. We define the operator $A\colon V \to V'$ by

$$(A\boldsymbol{u}, \boldsymbol{v})_{V'\times V} = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{4.8}$$

It follows from (4.8) and (3.21) that

$$|A\boldsymbol{u} - A\boldsymbol{v}|_{V'} \le c\,|\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v})|_{\mathcal{H}}\,, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{4.9}$$

Keeping in mind (3.27) and Krasnoselski theorem (see for example [12]) we deduce that $A\colon V \to V'$ is continuous, and so hemicontinuous.

Now, by (3.27b) and (3.21), we find

$$(A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_{V'\times V} \ge m_{\mathcal{A}}\,|\boldsymbol{u} - \boldsymbol{v}|_V^2\,, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V, \tag{4.10}$$

i.e., $A\colon V \to V'$ is monotone. Choosing $\boldsymbol{v} = \boldsymbol{0}_V$ in (4.10) we obtain

$$\begin{aligned}(A\boldsymbol{u}, \boldsymbol{u})_{V'\times V} &\ge m_{\mathcal{A}}\,|\boldsymbol{u}|_V^2 - |A\boldsymbol{0}_V|_{V'}\,|\boldsymbol{u}|_V \\ &\ge \frac{1}{2} m_{\mathcal{A}}\,|\boldsymbol{u}|_V^2 - \frac{1}{2m_{\mathcal{A}}}\,|A\boldsymbol{0}_V|_{V'}^2\,, \quad \forall\, \boldsymbol{u} \in V.\end{aligned}$$

Thus, A satisfies condition (2.5) with $\omega = \frac{m_{\mathcal{A}}}{2}$ and $\lambda = \frac{-1}{2m_{\mathcal{A}}}\,|A\boldsymbol{0}_V|_{V'}^2$.

Next, by (4.8), (3.27a) and (3.21) we deduce that

$$|A\boldsymbol{u}|_{V'} \le c(|\boldsymbol{u}|_V + 1), \quad \forall\, \boldsymbol{u} \in V,$$

where c is a positive constant. This implies that A satisfies condition (2.6). Finally, we recall that by (3.47) and (3.40) we have $\boldsymbol{f} - \boldsymbol{\eta}^1 \in L^2(0,T;V')$ and $\boldsymbol{v}_0 \in H$. It follows now from Theorem 2.1 that there exists a unique function $\boldsymbol{v}_\eta$ which satisfies

$$\boldsymbol{v}_\eta \in L^2(0,T;V) \cap C(0,T;H), \quad \dot{\boldsymbol{v}}_\eta \in L^2(0,T;V'), \tag{4.11}$$

$$\dot{\boldsymbol{v}}_\eta(t) + A\boldsymbol{v}_\eta(t) + \boldsymbol{\eta}^1(t) = \boldsymbol{f}(t), \quad \text{a.e. } t \in (0,T), \tag{4.12}$$

$$\boldsymbol{v}_\eta(0) = \boldsymbol{v}_0. \tag{4.13}$$

Let $\boldsymbol{u}_\eta\colon [0,T] \to V$ be defined by

$$\boldsymbol{u}_\eta(t) = \int_0^t \boldsymbol{v}_\eta(s)\,ds + \boldsymbol{u}_0, \quad \forall\, t \in [0,T]. \tag{4.14}$$

It follows from (4.8) and (4.11)–(4.14) that $\boldsymbol{u}_\eta$ is a solution to Problem 4.2 with the regularity (4.1). This concludes the existence part of Lemma 4.3. The uniqueness of the solution follows from the uniqueness of the solution to problem (4.12)–(4.13), guaranteed by Theorem 2.1. □

In the second step we use the displacement field obtained in Lemma 4.3 and we consider the following variational problem.

Problem 4.4. Find the electric potential field $\varphi_\eta\colon [0,T] \to W$

$$(\boldsymbol{B}\nabla\varphi_\eta(t), \nabla\zeta)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \nabla\zeta)_H + (h(\boldsymbol{u}_\eta(t), \varphi_\eta(t)), \zeta)_W = (q(t), \zeta)_W \tag{4.15}$$

for all $\zeta \in W$ and $t \in [0,T]$.

We have the following result for Problem 4.4.

Lemma 4.5. *Problem* 4.4 *has unique solution* φ_η *which satisfies the regularity* (4.4)*. Moreover, if* φ_i *represent the solution to Problem* QV_{η_i} *for* $\boldsymbol{\eta}_i$*,* $i = 1, 2$*, then there exists* $c > 0$ *such that*

$$|\varphi_1(t) - \varphi_2(t)|_W \le c\,|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|_V\,, \quad \forall\, t \in [0,T]. \tag{4.16}$$

Proof. Let $t \in [0,T]$. We use the Riesz representation theorem to define the operator $A_\eta(t)\colon W \to W$ by

$$(A_\eta(t)\varphi, \zeta)_W = (\boldsymbol{B}\nabla\varphi, \nabla\zeta)_H - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \nabla\zeta)_H + (h(\boldsymbol{u}_\eta(t), \varphi), \zeta)_W \tag{4.17}$$

for all $\varphi, \zeta \in W$. Let $\varphi_1, \varphi_2 \in W$, assumptions (3.31) and (3.44) imply

$$\begin{aligned}&(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W\\ &\ge m_B\,|\varphi_1 - \varphi_2|_W^2 + \int_{\Gamma_3} \psi(u_{\eta\nu}(t))(\varPhi(\varphi_1 - \varphi_0) - \varPhi(\varphi_2 - \varphi_0))(\varphi_1 - \varphi_2)\,da.\end{aligned}$$

The positivity of ψ combined with the monotonicity of the function $\varPhi$ give us

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \ge m_B\,|\varphi_1 - \varphi_2|_W^2\,, \tag{4.18}$$

then $A_\eta(t)$ is a strongly monotone operator on W. On the other hand, using again (3.31), (3.32), (3.35) and (3.44) we have

$$\begin{aligned}&(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \zeta)_W\\ &\le c_B\,|\varphi_1 - \varphi_2|_W\,|\zeta|_W + \int_{\Gamma_3} M_\psi\,|\varphi_1 - \varphi_2|\,|\zeta|\,da, \quad \forall\,\zeta \in W,\end{aligned} \tag{4.19}$$

where c_B is a positive constant which depends on $\boldsymbol{B}$. It follows from (4.19) and (3.25) that

$$(A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2, \zeta)_W \le (c_B + M_\psi\widetilde{c}_0^2)\,|\varphi_1 - \varphi_2|_W\,|\zeta|_W\,,$$

thus,

$$|A_\eta(t)\varphi_1 - A_\eta(t)\varphi_2|_W \le (c_B + M_\psi\widetilde{c}_0^2)\,|\varphi_1 - \varphi_2|_W\,, \tag{4.20}$$

which shows that $A_\eta\colon W \to W$ is Lipschitz continuous. Since A_η is a strongly monotone Lipschitz continuous operator on W, we deduce that there exists a unique element $\varphi_\eta(t) \in W$ such that

$$A_\eta(t)\varphi_\eta(t) = q(t). \tag{4.21}$$

We combine now (4.17) and (4.21) to show that $\varphi_\eta(t) \in W$ is the unique solution to the nonlinear variational equation (4.15).

Next, we show that $\varphi_\eta \in W^{1,p}(0,T;W)$. To this end, let $t_1, t_2 \in [0,T]$ and for the sake of simplicity, we write $\varphi_\eta(t_i) = \varphi_i$, $u_{\eta\nu}(t_i) = u_i$, $q(t_i) = q_i$ for $i = 1,2$. Using (4.15), (3.31), (3.32) and (3.44) we find

$$\begin{aligned} m_B \left|\varphi_1 - \varphi_2\right|_W^2 &\le c_{\mathcal{E}} \left|\boldsymbol{u}_1 - \boldsymbol{u}_2\right|_V \left|\varphi_1 - \varphi_2\right|_W + \left|q_1 - q_2\right|_W \left|\varphi_1 - \varphi_2\right|_W \\ &\quad + \int_{\Gamma_3} \left|\psi(u_1)\Phi(\varphi_1 - \varphi_0) - \psi(u_2)\Phi(\varphi_2 - \varphi_0)\right| \left|\varphi_1 - \varphi_2\right| da, \end{aligned} \tag{4.22}$$

where $c_{\mathcal{E}}$ is a positive constant which depends on the piezoelectric tensor $\mathcal{E}$.

We use the bounds $|\psi(u_i)| \le M_\psi$, $|\Phi(\varphi_1 - \varphi_0)| \le L$, the Lipschitz continuity of the function ψ and Φ, and inequality (3.25) to obtain

$$\begin{aligned} &\int_{\Gamma_3} \left|\psi(u_1)\Phi(\varphi_1 - \varphi_0) - \psi(u_2)\Phi(\varphi_2 - \varphi_0)\right| \left|\varphi_1 - \varphi_2\right| da \\ &\le M_\psi \int_{\Gamma_3} \left|\varphi_1 - \varphi_2\right|^2 da + L_\psi L \int_{\Gamma_3} \left|u_1 - u_2\right| \left|\varphi_1 - \varphi_2\right| da \\ &\le M_\psi \widetilde{c}_0^2 \left|\varphi_1 - \varphi_2\right|_W^2 + L_\psi L c_0 \widetilde{c}_0 \left|u_1 - u_2\right|_V \left|\varphi_1 - \varphi_2\right|_W . \end{aligned}$$

Inserting the last inequality in (4.22) yield

$$m_B \left|\varphi_1 - \varphi_2\right|_W \le (c_{\mathcal{E}} + L_\psi L c_0 \widetilde{c}_0) \left|\boldsymbol{u}_1 - \boldsymbol{u}_2\right|_V + \left|q_1 - q_2\right|_W + M_\psi \widetilde{c}_0^2 \left|\varphi_1 - \varphi_2\right|_W . \tag{4.23}$$

It follows from inequality (4.23) and assumption (3.54) that

$$\left|\varphi_1 - \varphi_2\right|_W \le c(\left|\boldsymbol{u}_1 - \boldsymbol{u}_2\right|_V + \left|q_1 - q_2\right|_W). \tag{4.24}$$

Since $\boldsymbol{u}_\eta \in C^1(0,T;H)$ and $q \in W^{1,p}(0,T;W)$, inequality (4.24) implies that $\varphi_\eta \in W^{1,p}(0,T;W)$.

Let $\varphi_{\eta_i} = \varphi_i$ and $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$ for $i = 1,2$. We use (4.15) and arguments similar to those used in the proof of (4.23) to obtain

$$m_B \left|\varphi_1(t) - \varphi_2(t)\right|_W \le (c_{\mathcal{E}} + L_\psi L c_0 \widetilde{c}_0) \left|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\right|_V + M_\psi \widetilde{c}_0^2 \left|\varphi_1(t) - \varphi_2(t)\right|_W$$

for all $t \in [0,T]$. This inequality combined with assumption (3.54) leads to (4.16). □

In the third step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.3 and we consider the following initial-value problem.

Problem 4.6. Find the adhesion field $\beta_\eta\colon [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\beta}_\eta(t) = -\Big(\beta_\eta(t)\left(\gamma_\nu (R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau\, |\boldsymbol{R}_\tau(\boldsymbol{u}_{\eta\tau}(t))|^2\right) - \epsilon_a\Big)_+, \tag{4.25}$$

$$\beta_\eta(0) = \beta_0. \tag{4.26}$$

We have the following result.

Lemma 4.7. *Problem* 4.6 *has a unique solution* $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap Z$.

Proof. For the sake of simplicity we suppress the dependence of various functions on Γ_3, and note that the equalities and inequalities below are valid a.e. on Γ_3.

Consider the mapping $F_\eta\colon [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F_\eta(t,\beta) = -\Big(\beta\left(\gamma_\nu (R_\nu(u_{\eta\nu}(t)))^2 + \gamma_\tau\, |\boldsymbol{R}_\tau(\boldsymbol{u}_{\eta\tau}(t))|^2\right) - \epsilon_a\Big)_+$$

for all $t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R_ν and $\boldsymbol{R}_\tau$ that F_η is Lipschitz continuous with respect to the second variable, uniformly in time. Moreover, for all $\beta \in L^2(\Gamma_3)$, the mapping $t \to F_\eta(t,\beta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Thus using a version of Cauchy-Lipschitz theorem we deduce that there exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ solution to Problem 4.6. Also, the arguments used in Remark 3.3 show that $0 \le \beta_\eta(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set Z, we find that $\beta_\eta \in Z$, which concludes the proof of Lemma 4.7. □

Now, define $\boldsymbol{k}_\eta \in W^{1,2}(0,T;Y)$ by

$$\boldsymbol{k}_\eta(t) = \boldsymbol{k}_0 + \int_0^t \boldsymbol{\eta}^2(s)\,ds. \tag{4.27}$$

In the fourth step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.3 and $\boldsymbol{k}_\eta$ defined in (4.27) to consider the following Cauchy problem for the stress field.

Problem 4.8. Find a stress field $\boldsymbol{\sigma}_\eta\colon [0,T] \to \mathcal{H}$ such that

$$\boldsymbol{\sigma}_\eta(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)), \boldsymbol{k}_\eta(s))\,ds, \quad \forall\, t \in [0,T]. \tag{4.28}$$

In the study of Problem 4.8 we have the following result.

Lemma 4.9. *There is a unique solution to Problem* 4.8 *and it satisfies* $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;\mathcal{H})$. *Moreover, if* $\boldsymbol{\sigma}_i$ *and* $\boldsymbol{u}_i$ *represent the solutions to Problem* SV_{η_i}, PV_{η_i} *respectively, and* $\boldsymbol{k}_i$ *is defined in* (4.27) *for* $\boldsymbol{\eta}_i \in L^2(0,T;V' \times Y)$, $i = 1,2$, *then there exists* $c > 0$ *such that*

$$\begin{aligned} &|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)|^2_{\mathcal{H}} \\ &\le c\left(|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2_V + \int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|^2_V\,ds + \int_0^t |\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)|^2_Y\,ds\right) \end{aligned} \tag{4.29}$$

for all $t \in [0,T]$.

Proof. We introduce the operator $\Lambda_\eta \colon L^2(0,T;\mathcal{H}) \to L^2(0,T;\mathcal{H})$ defined by

$$\Lambda_\eta \boldsymbol{\sigma}(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)), \boldsymbol{k}_\eta(s))\, ds \tag{4.30}$$

for all $\boldsymbol{\sigma} \in L^2(0,T;\mathcal{H})$ and $t \in [0,T]$. For $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in L^2(0,T;\mathcal{H})$ we use (4.30) and (3.29) to obtain, for all $t \in [0,T]$,

$$|\Lambda_\eta \boldsymbol{\sigma}_1(t) - \Lambda_\eta \boldsymbol{\sigma}_2(t)|_{\mathcal{H}} \le L_{\mathcal{G}} \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|_{\mathcal{H}}\, ds.$$

It follows from this inequality that for p large enough, the power Λ_η^p is a contraction on the Banach space $L^2(0,T;\mathcal{H})$, and therefore there exists a unique $\boldsymbol{\sigma}_\eta \in L^2(0,T;\mathcal{H})$ such that $\Lambda_\eta \boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_\eta$. Moreover, $\boldsymbol{\sigma}_\eta$ is the unique solution to Problem 4.8. Using (4.28), the regularity of $\boldsymbol{u}_\eta$, the regularity of $\boldsymbol{k}_\eta$ and the properties of the operators $\mathcal{F}$ and $\mathcal{G}$, it follows that $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;\mathcal{H})$.

Consider now $(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \in L^2(0,T;V' \times Y)$, and for $i = 1,2$, denote $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$ and $\boldsymbol{k}_{\eta_i} = \boldsymbol{k}_i$. We have

$$\boldsymbol{\sigma}_i(t) = \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_i(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_i(s)), \boldsymbol{k}_i(s))\, ds, \quad \forall\, t \in [0,T],$$

and using the properties (3.28) and (3.29) of $\mathcal{F}$ and $\mathcal{G}$, we find

$$\begin{aligned} |\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)|_{\mathcal{H}}^2 \le c \Bigg(&|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|_V^2 + \int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|_V^2\, ds \\ &+ \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|_{\mathcal{H}}^2\, ds + \int_0^t |\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)|_Y^2\, ds \Bigg) \end{aligned} \tag{4.31}$$

for all $t \in [0,T]$. Using now Gronwall argument to deduce the estimate (4.29). □

Finally as a consequence of these results and using the properties of $\mathcal{F}$, $\mathcal{E}$, $\mathcal{G}$, ϕ and j for $t \in [0,T]$, we consider the element

$$\Lambda \boldsymbol{\eta}(t) = (\Lambda^1 \boldsymbol{\eta}(t), \Lambda^2 \boldsymbol{\eta}(t)) \in V' \times Y, \tag{4.32}$$

defined by

$$\begin{aligned} (\Lambda^1 \boldsymbol{\eta}(t), \boldsymbol{v})_{V' \times V} = (&\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{E}^* \nabla \varphi_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ &+ \left(\int_0^t \mathcal{G}(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)), \boldsymbol{k}_\eta(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) \right)_{\mathcal{H}} \\ &+ j(\beta_\eta(t), \boldsymbol{u}_\eta(t), \boldsymbol{v}), \quad \forall\, \boldsymbol{v} \in V \end{aligned} \tag{4.33}$$

and

$$\Lambda^2 \boldsymbol{\eta}(t) = \phi(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)), \boldsymbol{k}_\eta(t)). \tag{4.34}$$

Here, for every $\boldsymbol{\eta} \in L^2(0,T;V' \times Y)$, $\boldsymbol{u}_\eta$, φ_η, β_η and $\boldsymbol{\sigma}_\eta$ represent the displacement field, the electric potential field, the adhesion field and the stress field obtained in Lemmas 4.3, 4.5, 4.7, 4.9 respectively, and $\boldsymbol{k}_\eta$ is the internal state variable given by (4.27). We have the following result.

Lemma 4.10. *The operator Λ has a unique fixed point $\boldsymbol{\eta}^* \in L^2(0,T;V' \times Y)$.*

Proof. Let $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in L^2(0,T;V' \times Y)$. Write for $i = 1, 2$,

$$\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i, \quad \dot{\boldsymbol{u}}_{\eta_i} = \boldsymbol{v}_{\eta_i} = \boldsymbol{v}_i, \quad \varphi_{\eta_i} = \varphi_i, \quad \beta_{\eta_i} = \beta_i, \quad \boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i, \quad \boldsymbol{k}_{\eta_i} = \boldsymbol{k}_i.$$

Using (3.21), (3.28), (3.29), (3.32), (3.33), (3.34) and the definition of R_ν, $\boldsymbol{R}_\tau$ we have

$$\begin{aligned} &\left|\Lambda^1\boldsymbol{\eta}_1(t) - \Lambda^1\boldsymbol{\eta}_2(t)\right|^2_{V'} \\ &\le c\bigg(|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2_V + |\varphi_1(t) - \varphi_2(t)|^2_W + \int_0^t |\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)|^2_{\mathcal{H}}\, ds \\ &\quad + \int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|^2_V\, ds + \int_0^t |\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)|^2_Y\, ds + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Gamma_3)}\bigg), \end{aligned} \tag{4.35}$$

and, keeping in mind (4.29) and (4.16), we find

$$\begin{aligned} \left|\Lambda^1\boldsymbol{\eta}_1(t) - \Lambda^1\boldsymbol{\eta}_2(t)\right|^2_{V'} &\le c\bigg(|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2_V + \int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|^2_V\, ds \\ &\quad + \int_0^t |\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)|^2_Y\, ds + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Gamma_3)}\bigg). \end{aligned} \tag{4.36}$$

By similar arguments, from (4.34), (4.29) and (3.30) it follows that

$$\begin{aligned} &\left|\Lambda^2\boldsymbol{\eta}_1(t) - \Lambda^2\boldsymbol{\eta}_2(t)\right|^2_Y \\ &\le c\left(|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)|^2_{\mathcal{H}} + |\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2_V + |\boldsymbol{k}_1(t) - \boldsymbol{k}_2(t)|^2_Y\right) \\ &\le c\bigg(|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2_V + |\boldsymbol{k}_1(t) - \boldsymbol{k}_2(t)|^2_Y + \int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|^2_V\, ds \\ &\quad + \int_0^t |\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)|^2_Y\, ds\bigg). \end{aligned} \tag{4.37}$$

Consequently,

$$\begin{aligned} &|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)|^2_{V'\times Y} \\ &\le c\Bigg(|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|^2_V + |\boldsymbol{k}_1(t) - \boldsymbol{k}_2(t)|^2_Y + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Gamma_3)} \\ &\quad + \int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|^2_V\, ds + \int_0^t |\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)|^2_Y\, ds\Bigg), \quad \forall\, t \in [0,T]. \end{aligned} \tag{4.38}$$

Moreover, from (4.6) we obtain

$$(\dot{\boldsymbol{v}}_1 - \dot{\boldsymbol{v}}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_{V'\times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} + (\boldsymbol{\eta}_1^1 - \boldsymbol{\eta}_2^1, \boldsymbol{v}_1 - \boldsymbol{v}_2)_{V'\times V} = 0$$

a.e. $t \in (0,T)$. We integer this equality with respect to time, and use the initial conditions $\boldsymbol{v}_1(0) = \boldsymbol{v}_2(0) = \boldsymbol{v}_0$, condition (3.21) and (3.27) to find

$$m_{\mathcal{A}} \int_0^t |\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)|_V^2 \, ds \le - \int_0^t \big(\boldsymbol{\eta}_1^1(s) - \boldsymbol{\eta}_2^1(s), \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\big)_{V'\times V} \, ds$$

for all $t \in [0,T]$. Then, using the inequality $2ab \le a^2/\gamma + \gamma b^2$ we obtain

$$\int_0^t |\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)|_V^2 \, ds \le c \int_0^t \big|\boldsymbol{\eta}_1^1(s) - \boldsymbol{\eta}_2^1(s)\big|_{V'}^2 \, ds, \quad \forall\, t \in [0,T]. \tag{4.39}$$

On the other hand, from the Cauchy problem (4.25)–(4.26) we can write

$$\beta_i(t) = \beta_0 - \int_0^t \Big(\beta_i(s)\Big(\gamma_\nu \left(R_\nu(u_{\eta\nu}(s))\right)^2 + \gamma_\tau \left|\boldsymbol{R}_\tau(\boldsymbol{u}_{\eta\tau}(s))\right|^2\Big) - \epsilon_a\Big)_+ \, ds.$$

So

$$\begin{aligned} |\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} &\le c \int_0^t \Big|\beta_1(s)\left(R_\nu(u_{1\nu}(s))\right)^2 - \beta_2(s)\left(R_\nu(u_{2\nu}(s))\right)^2\Big|_{L^2(\Gamma_3)} ds \\ &\quad + c \int_0^t \Big|\beta_1(s)\left|\boldsymbol{R}_\tau(\boldsymbol{u}_{1\tau}(s))\right|^2 - \beta_2(s)\left|\boldsymbol{R}_\tau(\boldsymbol{u}_{2\tau}(s))\right|^2\Big|_{L^2(\Gamma_3)} ds. \end{aligned}$$

Note that $\beta_1 = \beta_1 - \beta_2 + \beta_2$ and using the definitions of R_ν and $\boldsymbol{R}_\tau$ give us

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \le c \int_0^t |\beta_1(s) - \beta_2(s)|_{L^2(\Gamma_3)} \, ds + \int_0^t |\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)|_{L^2(\Gamma_3)^d} \, ds.$$

We apply Gronwall's inequality and use the relation (3.22) to conclude that

$$|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)}^2 \le c \int_0^t |\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|_V^2 \, ds. \tag{4.40}$$

Furthermore, from (4.27) we have

$$|\boldsymbol{k}_1(t) - \boldsymbol{k}_2(t)|_Y^2 \le c \int_0^t \big|\boldsymbol{\eta}_1^2(s) - \boldsymbol{\eta}_2^2(s)\big|_Y^2 \, ds. \tag{4.41}$$

Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ have the same initial value we get

$$|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|_V^2 \le \int_0^t |\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)|_V^2 \, ds.$$

From this inequality, (4.38) and (4.40) we obtain

$$\begin{aligned} &|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)|_{V'\times Y}^2 \\ &\le c\left(\int_0^t |\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)|_V^2 \, ds + |\boldsymbol{k}_1(t) - \boldsymbol{k}_2(t)|_Y^2 + \int_0^t |\boldsymbol{k}_1(s) - \boldsymbol{k}_2(s)|_Y^2 \, ds\right), \quad \forall\, t \in [0,T]. \end{aligned}$$

It follows now from (4.39) and (4.41) that

$$|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)|^2_{V'\times Y}\le c\int_0^t|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|^2_{V'\times Y}\,ds.$$

Reiterating the previous inequality n times, we find that

$$|\Lambda^n\boldsymbol{\eta}_1-\Lambda^n\boldsymbol{\eta}_2|^2_{L^2(0,T;V'\times Y)}\le\frac{(cT)^n}{n!}\int_0^t|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|^2_{V'\times Y}\,ds.$$

This inequality shows that for a sufficiently large n the operator Λ^n is a contraction on the Banach space $L^2(0,T;V'\times Y)$, and so Λ has a unique fixed point. □

Now, we have all the ingredients to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\boldsymbol{\eta}_*=(\boldsymbol{\eta}^1,\boldsymbol{\eta}^2)\in L^2(0,T;V'\times Y)$ be the fixed point of Λ defined by (4.32)–(4.34) and denote

$$\boldsymbol{u}=\boldsymbol{u}_{\eta_*},\quad \boldsymbol{k}=\boldsymbol{k}_{\eta_*},\quad \varphi=\varphi_{\eta_*},\quad \beta=\beta_{\eta_*},\tag{4.42}$$

$$\boldsymbol{\sigma}=\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}})+\mathcal{E}^*\nabla\varphi+\boldsymbol{\sigma}_{\eta_*}.\tag{4.43}$$

We prove that $(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{k},\varphi,\beta)$ satisfies (3.48)–(3.53) and (4.1)–(4.5). Indeed, we write (4.28) for $\boldsymbol{\eta}=\boldsymbol{\eta}_*$ and use (4.42)–(4.43) we obtain (3.48). We use (4.6) for $\boldsymbol{\eta}=\boldsymbol{\eta}_*$ and the first equality in (4.42) to find

$$(\ddot{\boldsymbol{u}},\boldsymbol{v})_{V'\times V}+(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}+(\boldsymbol{\eta}^1(t),\boldsymbol{v})_{V'\times V}=(\boldsymbol{f}(t),\boldsymbol{v})_{V'\times V}\tag{4.44}$$

for all $\boldsymbol{v}\in V$, a.e. $t\in(0,T)$. The equalities $\Lambda^1(\boldsymbol{\eta}_*)=\boldsymbol{\eta}^1$ and $\Lambda^2(\boldsymbol{\eta}_*)=\boldsymbol{\eta}^2$ combined with (4.33), (4.34), (4.42) and (4.43) show that

$$\begin{aligned}(\boldsymbol{\eta}^1(t),\boldsymbol{v})_{V'\times V}&=(\mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}+(\mathcal{E}^*\nabla\varphi,\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}\\&\quad+\left(\int_0^t\mathcal{G}(\boldsymbol{\sigma}(s)-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(s))-\mathcal{E}^*\nabla\varphi(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)),\boldsymbol{k}(s))\,ds,\boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}}\\&\quad+j(\beta(t),\boldsymbol{u}(t),\boldsymbol{v}),\quad\forall\,\boldsymbol{v}\in V\end{aligned}\tag{4.45}$$

and

$$\boldsymbol{\eta}^2(t)=\phi\left(\boldsymbol{\sigma}(t)-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t))-\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)),\boldsymbol{k}_\eta(t)\right).\tag{4.46}$$

From (4.46) and (4.27) we see that (3.49) is satisfied. We substitute (4.45) in (4.44) and use (3.48) to see that (3.50) is satisfied. We write now (4.15) and (4.25) for $\boldsymbol{\eta}=\boldsymbol{\eta}_*$ and use (4.42) to find (3.51) and (3.52).

Next, (3.53), the regularities (4.1), (4.3), (4.4) and (4.5) follow from Lemmas 4.3, 4.5, 4.7 and the relation (4.27). The regularity $\boldsymbol{\sigma}\in L^2(0,T;\mathcal{H})$ follows from Lemma 4.9, assumptions (3.27), (3.32) and (4.43). Finally, (3.50) implies that

$$\rho\ddot{\boldsymbol{u}}=\operatorname{Div}\boldsymbol{\sigma}+\boldsymbol{f}_0\quad\text{in }V',\text{ a.e. }t\in(0,T),$$

and from (3.36) and (3.37) we find that $\operatorname{Div} \boldsymbol{\sigma} \in L^2(0,T;V')$. We deduce that the regularity (4.2) holds. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ combined with the unique solvability of Problems 4.2, 4.4, 4.6 and 4.8 guaranteed by Lemmas 4.3, 4.5, 4.7 and 4.9. □

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