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Long Time Behavior for a Wave Equation with Time Delay

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Abstract. In this paper, we consider the wave equation with internal time delay and source terms

$$u_{tt}(x,t) - \triangle u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) + f(x,u) = h(x)$$

in a bounded domain. By virtue of Galerkin method combined with the priori estimates, we prove the existence and uniqueness of global solution under initial-boundary data for the above equation. Moreover, under suitable conditions on the forcing term $f(x,u)$ and μ_1, μ_2, the existence of a compact global attractor is proved. Further, the asymptotic behavior and the decay property of global solution are discussed.

1. Introduction

In this paper, we investigate the following wave equation with time delay term in the feedback

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) + f(x,u) = h(x), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ 0 < t < \tau \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ with a sufficiently smooth boundary $\partial\Omega$. Here f and h are external forcing terms, the source term $f(x,u) \approx |u|^\alpha u + |u|^\beta u$, $\alpha > \beta \geq 0$, μ_1, μ_2 are some constants, $\tau > 0$ represents the time delay, u_0, u_1, f_0 are given functions belonging to some suitable spaces.

In absence of delay ($\mu_2 = 0$), the problem (1.1) becomes

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) + u_t(x,t) + f(x,u) = h(x), & x \in \Omega,\ t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad u(x,t)\big|_{\partial\Omega} = 0l \end{cases} \tag{1.2}$$

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which was considered by several authors. For example, the existence of the global attractor in the energy space $H_0^1(\Omega)\times L^2(\Omega)$ was well discussed and it was a standard result that if $0<\alpha<2/[N-2]^+$, (see [13, 15]). The proof is based on the exponential decay of the solutions for the case $f(x,u)=h(x)=0$ and the compactness of the embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega)$. The critical or super critical cases were also considered, see [3, 9, 12]. When the damping term is replaced by the nonlinear term $\rho(u_t)$, the similar problem was also discussed by Chueshov and Lasiecka [4], Nakao [21].

Introducing the delay term $\mu_2 u_t(x,t-\tau)$ makes the problem different from those considered in the literatures. Time delay arises in many applications depending not only on the present state but also on some past occurrences. It may turn a well-behaved system into a wild one. The presence of delay may be a source of instability. For example, it was shown in [7, 8, 14, 22, 23, 29] that an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used.

In [22], the authors examined a system with the linear damping and a delay inside the domain. More precisely, they considered the following system

$$(1.3)\qquad \begin{cases} u_{tt}(x,t)-\triangle u(x,t)+\mu_1 u_t(x,t)+\mu_2 u_t(x,t-\tau)=0, & x\in\Omega,\ t>0,\\ u(x,t)=0, & x\in\partial\Omega,\ t\geq 0,\\ u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x), & x\in\Omega,\\ u_t(x,t-\tau)=f_0(x,t-\tau), & x\in\Omega,\ 0<t<\tau, \end{cases}$$

and proved the energy of the problem (1.3) is exponentially stable provided $0<\mu_2<\mu_1$. On the contrary, it is also showed that in the case $\mu_2\geq\mu_1$ that there exists a sequence of delays for which the corresponding solution is unstable. The method used in [22] is an observability inequality and a Carleman estimate. The same results were also obtained when both the damping and the delay act on the boundary. We also refer the readers to [29], where the authors obtained the same results as in [22] for the one space dimension by use of the spectral analysis approach.

Recently, in [25], the authors considered abstract semilinear evolution equations with a time-delay and some source term. They show that, if the C_0 semigroup describing the linear part of the model is exponentially stable, then the whole system retains this property when a suitable smallness condition on the time-delay is satisfied.

The case of the time-varying delay in wave equation has also been studied by several authors, see for example, [18, 19, 24, 26, 27] and the references therein.

In the works mentioned above, the authors must used the damping term $\mu_1 u_t(x,t)$ to control the delay term in the priori estimate of the solution and the decay estimate of the energy. By the way, in [6], the authors improve earlier results in the literature by making using of the viscoelastic term to control the time constant delay term.

But, to our best knowledge, there is no results on the existence, energy decay and the global attractors for the whole system which includes the source term $f(u)$ and non-homogeneous term $h(x)$. Recently, we considered the well-posedness for a class of wave equation with past history and a delay [20]. Motivated by there results, in this paper, we will investigate the problem (1.1) under suitable assumptions. Our main difficulty in handing this model is that we have the delay term $\mu_2 u_t(t-\tau)$, which may induce some instabilities. To overcome this difficulty, we introduce the linear damping term to control the delay term as usual, we also need some modified energy functional to study the global attractors and the decay result of the problem (1.1).

The plan of this paper is as follows. In Section 2, we present some notations and assumptions needed for our work, and then give the main results (Theorems 2.1, 2.5 and 2.6). The proof of Theorem 2.1 is given in Section 3. Section 4 contains some abstract results in the theory on infinite dimensional dynamical systems that will be used. The proofs of Theorems 2.5 and 2.6 are given in Sections 5 and 6, respectively.

2. Preliminaries and main results

In this section, we present some assumptions and state the main results. As usual $(\cdot\,,\cdot)$ denotes L^2-inner product and $\|\cdot\|_p$ denotes L^p-norms. It is well known the norms in $H_0^1(\Omega)$ is given by $\|\nabla\cdot\|_2$, and the dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. Throughout this paper, C and C_i are used to denote the generic positive constant. From now on, we shall omit x and t in all functions of x and t if there is no ambiguity.

We define the phase space

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$$

equipped with the norm

$$\|(u,v)\|_{\mathcal{H}}^2 = \|\nabla u\|_2^2 + \|v\|_2^2\,.$$

Let us state precise assumptions on the term $f(x,u)$ and $h(x)$.

(H1) $f(x,u)$ is measurable in $x \in \Omega$ for all $u \in \mathbb{R}$ and continuous in $u \in \mathbb{R}$ for a.e. $x \in \Omega$, satisfying

$$f(x,0) = 0, \quad |f_u(x,u)| \le k(1+|u|^\alpha), \tag{2.1}$$

where $k > 0$ and α satisfies

$$0 < \alpha \le \frac{2}{N-2} \text{ if } N \ge 3, \text{ or } \alpha > 0 \text{ if } N = 1,2. \tag{2.2}$$

(H2) $h \in L^2(\Omega)$.

(H3) The initial data f_0 satisfies the compatibility condition $f_0(\cdot\,,0) = u_1$.

Then we have the following existence result.

Theorem 2.1. *Assume that* (H1)–(H3) *hold, then we have*

(i) *If* $(u_0, u_1, f_0) \in (H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)\times H_0^1(\Omega, H^1(-\tau,0))$, *then the problem* (1.1) *has a regular solution in the class*

$$u \in L^\infty_{\mathrm{loc}}((0,\infty); H^2(\Omega)\cap H_0^1(\Omega)), \quad u_t \in L^\infty_{\mathrm{loc}}((0,\infty); H_0^1(\Omega)),$$
$$u_{tt} \in L^\infty_{\mathrm{loc}}((0,\infty); L^2(\Omega)).$$

(ii) *If* $(u_0, u_1, f_0) \in H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega,(-\tau,0))$, *then the problem* (1.1) *has a weak solution in the class*

$$u \in L^\infty_{\mathrm{loc}}((0,\infty); H_0^1(\Omega)), \quad u_t \in L^\infty_{\mathrm{loc}}((0,\infty); L^2(\Omega)), \quad u_{tt} \in L^\infty_{\mathrm{loc}}((0,\infty); H^{-1}(\Omega))$$

satisfying

$$(u, u_t) \in C([0,T],\mathcal{H}), \quad \forall T > 0. \tag{2.3}$$

Moreover, the regular (or weak) solutions depend on the initial data $(u_0,u_1)\in\mathcal{H}$, $h \in L^2(\Omega)$ *and* $f_0 \in L^2(\Omega,(-\tau,0))$. *In particular, problem* (1.1) *has uniqueness.*

Remark 2.2. By combining (2.1) and the mean value theorem, we can deduce that there exists a constant $k_0 > 0$ such that

$$|f(x,u) - f(x,v)| \le k_0(1 + |u|^\alpha + |v|^\alpha)\,|u-v| \tag{2.4}$$

for a.e. $x\in\Omega$ and $u,v\in\mathbb{R}$. Also it follows from (2.2) that $H_0^1(\Omega)\hookrightarrow L^{2(\alpha+1)}(\Omega)$.

Remark 2.3. The uniqueness of the problem (1.1) defines the evolution operator

$$S(t)\colon \mathcal{H}\to\mathcal{H}, \quad S(t)(u_0,u_1) = (u(t), u_t(t)), \quad t\ge 0,$$

where $(u(t),u_t(t))$ is the weak solution corresponding to initial data (u_0,u_1). It turns out that $S(t)$ satisfies the semigroup properties

$$S(0) = I \quad \text{and} \quad S(t+s) = S(t)S(s), \quad t,s\ge 0.$$

Moreover, the continuous dependence on the initial data in $\mathcal{H}$ and the regular (2.3) imply that $S(t)$ is strongly continuous on $\mathcal{H}$. Then, the long-time dynamic of the problem (1.1) can be studied by the continuous dynamical system $(\mathcal{H}, S(t))$.

Remark 2.4. There is no restrictions on μ_1 and μ_2 in Theorem 2.1, that is, the problem (1.1) has a unique global solution for arbitrary numbers μ_1 and μ_2. When a suitable smallness condition on the time-delay feedback is satisfied (i.e., $0 < |\mu_2| < \mu_1$), we have the following Theorems 2.5 and 2.6.

Let $\lambda_1 > 0$ be the first eigenvalue of $-\triangle w = \lambda w$ in Ω with $w = 0$ on $\partial\Omega$. Now, we replace (2.2) by the following condition

$$0 < \alpha < \frac{2}{N-2} \text{ if } N \geq 3, \text{ or } \alpha > 0 \text{ if } N = 1, 2. \tag{2.5}$$

We set

$$F(x,u) = \int_0^u f(x,s)\,ds.$$

In addition, we suppose that there exist constants $L_0 > 0$ and $\beta \in [0, \lambda_1)$ such that

$$-L_0 - \beta u^2 \leq f(x,u)u \quad \text{and} \quad -L_0 - \frac{\beta}{2}u^2 \leq F(x,u) \tag{2.6}$$

for a.e. $x \in \Omega$ and $u \in \mathbb{R}$.

Theorem 2.5. *Assume that the hypotheses of Theorem* 2.1*,* $0 < |\mu_2| < \mu_1$*,* (2.5) *and* (2.6) *hold. Then the dynamical system* $(\mathcal{H}, S(t))$ *corresponds to the problem* (1.1) *possesses a compact global attractor* $\mathcal{A}$*. Moreover, it is characterized by the unstable manifold*

$$\mathcal{A} = \mathcal{M}^u(\mathcal{N})$$

of the set of stationary solutions $\mathcal{N} = \{(u,0) \in \mathcal{H} \mid -\triangle u + f(x,u) = h\}$*.*

A typical assumption that implies the conditions (2.6) on f is

$$\liminf_{|s|\to\infty} \frac{f(x,s)}{s} > -\lambda_1.$$

For the decay property of solution $u(t)$ for problem (1.1), we have

Theorem 2.6. *Let* $u(t)$ *be a weak solution in Theorem* 2.1 *with* $h = 0$*. Let all assumptions in* (H1) *be satisfied, and* $0 < |\mu_2| < \mu_1$*. In addition, we replace* (2.6) *with*

$$-\frac{\beta}{2}u^2 \leq F(x,u) \leq f(x,u)u \tag{2.7}$$

for a.e. $x \in \Omega$ *and* $u \in \mathbb{R}$*. Then there exist two positive constants* κ*,* K *such that*

$$E(u(t)) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \int_\Omega F(x,u(t))\,dx \leq Ke^{-\kappa t}, \quad \forall\, t \geq 0. \tag{2.8}$$

3. Well-posedness

In this section we first prove the existence and uniqueness of regular solutions to problem (1.1) by using Faedo-Galerkin method as in [10,17]. Then, we extend the same result to the weak solutions by using the density arguments.

The energy functional corresponding to the system (1.1) is given by

$$E(u(t)) = \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \int_\Omega F(x,u(t))\,dx - \int_\Omega hu(t)\,dx. \tag{3.1}$$

3.1. Approximate problem

First, we assume $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $f_0 \in L^2(\Omega, H^1(-\tau,0))$. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where $\{w_k\}_{k=1}^\infty$ is a basis of $H^2(\Omega)\cap H_0^1(\Omega)$.

We will seek an approximate solution in the form

$$u_m(x,t) = \sum_{j=1}^m g_{jm}(t)w_j(x)$$

satisfying the following approximate equation

$$(u_m'', w_j) + (\nabla u_m, \nabla w_j) + \mu_1(u_m', w_j) + \mu_2(u_m'(t-\tau), w_j) + (f(u_m) - h, w_j) = 0 \tag{3.2}$$

with initial data

$$(u_m(0), u_m'(0)) = (u_{0m}, u_{1m}), \quad u_m'(t) = f_{0m}(t), \quad t \in [-\tau, 0), \tag{3.3}$$

where

$$\begin{aligned} u_{0m} &\to u_0 && \text{in } H^2(\Omega)\cap H_0^1(\Omega),\\ u_{1m} &\to u_1 && \text{in } H_0^1(\Omega),\\ f_{0m}(t) &\to f_0(t) && \text{in } L^2(\Omega, H^1(-\tau,0)). \end{aligned} \tag{3.4}$$

We note that the approximate problem (3.2)–(3.4) can be reduced to an ordinary differential equation system and by standard existence theory for ODEs, the problem admits a local solution $u_m(t)$ in some interval $[0, T_m)$ with $0 < T_m \le T$. The following estimates imply that the local solutions $u^m(t)$ to the interval $[0,T]$ for any given $T > 0$.

3.2. Priori estimates

The first estimate. Multiplying the approximate equation in (3.1) by g_{jm}', then summing up the result in j, we deduce that

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\|u_m'\|_2^2 + \frac{1}{2}\|\nabla u_m\|_2^2 + \int_\Omega F(x,u_m)\,dx - \int_\Omega hu_m\,dx\right\}\\ &= -\mu_1\|u_m'\|_2^2 - \mu_2\int_\Omega u_m'(t-\tau)u_m'\,dx. \end{aligned} \tag{3.5}$$

For simplicity, we will omit the index m when we study the energy functional, since it can also be used for existing solutions.

Let us define the modified energy

$$\widetilde{E}(u(t)) = E(u(t)) + L_0\,|\Omega| + \frac{1}{\lambda_1 \varrho}\,\|h\|_2^2\,,$$

with any constant $\varrho > 0$.

Since $\lambda_1\,\|u\|_2^2 \leq \|\nabla u\|_2^2$, it follows from the condition (2.6) that

$$\int_\Omega F(x,u)\,dx \geq -\frac{\beta}{2\lambda_1}\,\|\nabla u\|_2^2 - L_0\,|\Omega|\,,$$

and for such $\varrho > 0$,

$$-\int_\Omega hu\,dx \geq -\frac{\varrho}{4}\,\|\nabla u\|_2^2 - \frac{1}{\lambda_1 \varrho}\,\|h\|_2^2\,.$$

Then, we deduce that

$$\widetilde{E}(u(t)) \geq \left(\frac{1}{2} - \frac{\beta}{2\lambda_1} - \frac{\varrho}{4}\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}\,\|u_t(t)\|_2^2\,.$$

Noticing $\beta \in [0,\lambda_1)$, taking $\varrho > 0$ sufficiently small, we can obtain

$$\widetilde{E}(u(t)) \geq \beta_0\left(\|\nabla u\|_2^2 + \|u_t\|_2^2\right) \tag{3.6}$$

for some positive constant $\beta_0 < \min\{\beta, 1/2\}$ (in the case $\beta = 0$, we choose $\varrho = 1$ and $\beta_0 = 1/4$).

For the last term in (3.5), it follows from Hölder's inequality that

$$-\mu_2 \int_\Omega u_m'(t-\tau)u_m'\,dx \leq \frac{1}{2}\,|\mu_2|\,\left\|u_m'(t)\right\|_2^2 + \frac{1}{2}\,|\mu_2| \int_\Omega u_m'^2(t-\tau)\,dx.$$

Hence, (3.5) implies that

$$\frac{d}{dt}\widetilde{E}(u_m(t)) \leq \left(|\mu_1| + \frac{|\mu_2|}{2}\right)\left\|u_m'(t)\right\|_2^2 + \frac{1}{2}\,|\mu_2| \int_\Omega u_m'^2(t-\tau)\,dx. \tag{3.7}$$

Integrating (3.7) on $[0,t]$, we obtain

$$\begin{aligned}\widetilde{E}(u_m(t)) \leq \widetilde{E}(u_m(0)) &+ \left(|\mu_1| + \frac{1}{2}\,|\mu_2|\right)\int_0^t\!\!\int_\Omega u_{ms}^2(s)\,dxds \\ &+ \frac{1}{2}\,|\mu_2| \int_0^t\!\!\int_\Omega u_{ms}^2(s-\tau)\,dxds.\end{aligned} \tag{3.8}$$

Noticing the past history values of $u_{mt}(t)$, $t \in [-\tau, 0]$, the last term in (3.8) can be rewritten as follows

$$\begin{aligned} \int_0^t \int_\Omega u_{ms}^2(s-\tau)\,dxds &= \int_\Omega \int_{-\tau}^{t-\tau} u_{m\rho}^2(\rho)\,d\rho dx \\ &= \int_\Omega \int_{-\tau}^{0} u_{m\rho}^2(\rho)\,d\rho dx + \int_\Omega \int_0^{t-\tau} u_{m\rho}^2(\rho)\,d\rho dx \\ &= \int_\Omega \int_{-\tau}^{0} f_{0m}^2(\rho)\,d\rho dx + \int_\Omega \int_0^{t-\tau} u_{m\rho}^2(\rho)\,d\rho dx \\ &\le \int_\Omega \int_{-\tau}^{0} f_{0m}^2(\rho)\,d\rho dx + \int_\Omega \int_0^{t} u_{m\rho}^2(\rho)\,d\rho dx. \end{aligned} \tag{3.9}$$

From (3.6), (3.8) and (3.9), we have

$$\widetilde{E}(u_m(t)) \le \widetilde{E}(u_m(0)) + \frac{1}{2}|\mu_2| \int_\Omega \int_{-\tau}^{0} f_{0m}^2(\rho)\,d\rho dx + 4(|\mu_1| + |\mu_2|) \int_0^t \widetilde{E}(u_m(s))\,ds.$$

From the choice of u_{0m}, u_{1m} and f_{0m}, using Gronwall's lemma, once $T > 0$ be given, $\forall\, t \in [0, T]$, we can find a positive constant C independent of m such that

$$\widetilde{E}(u_m(t)) \le C, \quad \forall\, t \in [0, T],\ \forall\, m \in \mathbb{N}.$$

Noticing (3.6), we conclude that

$$\|\nabla u_m(t)\|_2^2 + \left\|u_m'(t)\right\|_2^2 \le C, \quad \forall\, t \in [0, T],\ \forall\, m \in \mathbb{N}. \tag{3.10}$$

The second estimate. We first estimate $\|u_m''(0)\|_2$. Replacing w_j by $u_m''(t)$ in (3.2) and taking $t = 0$, we obtain

$$\begin{aligned} &\left\|u_m''(0)\right\|_2^2 \\ &= -(\nabla u_m(0), \nabla u_m''(0)) - \mu_1(u_m'(0), u_m''(0)) - \mu_2(u_m'(-\tau), u_m''(0)) \\ &\quad - (f(\cdot\,, u_m(0)), u_m''(0)) + (h, u_m''(0)) \\ &\le \Big(\|\nabla u_m(0)\|_2 + |\mu_1| \left\|u_m'(0)\right\|_2 + |\mu_2| \left\|u_m'(-\tau)\right\|_2^2 + \|f(\cdot\,, u_m(0))\|_2 + \|h\|_2\Big) \left\|u_m''(0)\right\|_2. \end{aligned}$$

From the choice of u_{0m}, u_{1m}, h and (3.10), we deduce

$$\left\|u_m''(0)\right\|_2 \le C. \tag{3.11}$$

Now, getting derivative of (3.2) with respect to t, we get

$$(u_m''', w_j) + (\nabla u_m', \nabla w_j) + \mu_1(u_m'', w_j) + \mu_2(u_m''(t-\tau), w_j) + (f_u(\cdot\,, u_m)u_m', w_j) = 0.$$

Multiplying by g_{jm}'', summing over j from 1 to m, it follows that

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\left\|u_m''(t)\right\|_2^2 + \frac{1}{2}\left\|\nabla u_m'(t)\right\|_2^2\right) &= -\mu_1 \left\|u_m''(t)\right\|_2^2 - \mu_2 \int_\Omega u_m''(t-\tau)u_m''(t)\,dx \\ &\quad - \int_\Omega f_u(\cdot\,, u_m)u_m'(t)u_m''(t)\,dx. \end{aligned} \tag{3.12}$$

As the proof of (3.9), we have

$$\begin{aligned}\int_0^t\int_\Omega u_{mss}^2(s-\tau)\,dxds &= \int_\Omega\int_{-\tau}^{t-\tau} u_{m\rho\rho}^2(\rho)\,d\rho dx\\ &\le \int_\Omega\int_{-\tau}^{0} f_{0m\rho}^2(\rho)\,d\rho dx + \int_\Omega\int_0^t u_{m\rho\rho}^2(\rho)\,d\rho dx.\end{aligned}$$

Since $\frac{\alpha}{2(\alpha+1)} + \frac{1}{2(\alpha+1)} + \frac{1}{2} = 1$, by the generalized Hölder's inequality, assumption (2.1)–(2.2), and the estimate (3.10), we have

$$\begin{aligned}-\int_\Omega f_u(\cdot\,,u_m)u_m'(t)u_m''(t)\,dx &\le k\int_\Omega (1+|u_m(t)|^\alpha)\left|u_m'(t)\right|\left|u_m''(t)\right| dx\\ &\le k\left(|\Omega|^{\alpha/[2(\alpha+1)]} + \|u_m(t)\|_{2(\alpha+1)}^\alpha\right)\left\|u_m'(t)\right\|_{2(\alpha+1)}\left\|u_m''(t)\right\|_2\\ &\le C_1\left(\left\|u_m''(t)\right\|_2^2 + \left\|\nabla u_m'(t)\right\|_2^2\right).\end{aligned}$$

Similar as the proof of (3.10), using the initial data (3.3) and (3.4), it follows from (3.12) that

$$F_m(t) \le F_m(0) + |\mu_2|\int_\Omega\int_{-\tau}^0 f_{0m\rho}^2(\rho)\,d\rho dx + C_2\int_0^t F_m(s)\,ds,$$

with $C_2 = C_2(C_1,\mu_1,\mu_2) > 0$, where

$$F_m(t) = \left\|u_m''(t)\right\|_2^2 + \left\|\nabla u_m'(t)\right\|_2^2.$$

Using Gronwall's lemma, once $T > 0$ be given, $\forall\, t \in [0,T]$, there exists a positive constant C independent of m such that

$$\left\|u_m''(t)\right\|_2^2 + \left\|\nabla u_m'(t)\right\|_2^2 \le C. \tag{3.13}$$

The first and second a priori estimates permit us to obtain a subsequence of (u_m) which from now on will be also denoted by (u_m) and a function u satisfying

$$u_m \rightharpoonup u \quad \text{weak star in } L^\infty_{\text{loc}}(0,\infty;H_0^1(\Omega)), \tag{3.14}$$

$$u_m' \rightharpoonup u' \quad \text{weak star in } L^\infty_{\text{loc}}(0,\infty;H_0^1(\Omega)), \tag{3.15}$$

$$u_m'' \rightharpoonup u'' \quad \text{weak star in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)). \tag{3.16}$$

Since $H_0^1 \hookrightarrow L^2(\Omega)$ is compact, thanks to Aubin-Lions theorem, we have that

$$u_m \to u \quad \text{strongly in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)),$$

and consequently, making use of Lion's lemma, we obtain

$$f(x,u_m) \rightharpoonup f(x,u) \quad \text{weakly in } L^\infty_{\text{loc}}(0,\infty;L^2(\Omega)). \tag{3.17}$$

Convergence (3.14)–(3.17) permit us to pass to the limit the approximate problem, as usual, we multiply the approximate equation (3.2) by test function $\theta(t) \in \mathscr{D}(0,T)$, and by integration over $[0,T]$, after passing to the limit, we obtain

$$\int_0^T \{(u_{tt}, w_j) + (\nabla u, \nabla w_j) + \mu_1(u_t, w_j) + \mu_2(u_t(t-\tau), w_j) + (f(u) - h, w_j)\}\,\theta(t)\,dt = 0. \tag{3.18}$$

Since (w_j) is a basis of $H^2(\Omega) \cap H_0^1(\Omega)$, replacing w_j by $v \in \mathscr{D}(\Omega)$, from (3.18), we deduce that

$$u_{tt}(x,t) - \triangle u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) + f(x,u) = h(x) \quad \text{in } \mathscr{D}'(\Omega \times (0,T)).$$

Moreover, since $u'', f(x,u) \in L^2_{\text{loc}}(0,\infty); L^2(\Omega))$ and $u' \in L^2_{\text{loc}}(-\tau,\infty; L^2(\Omega)$, we have $\triangle u \in L^2_{\text{loc}}((0,\infty); L^2(\Omega))$ and hence

$$u_{tt}(x,t) - \triangle u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) + f(x,u) = h(x) \quad \text{in } L^2_{\text{loc}}(\Omega \times (0,T)).$$

The proof of Theorem 2.1(i) is finished.

Weak solution. In order to obtain the existence of weak solutions we use standard arguments of density. Indeed, we have obtained regular solutions under regular initial data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ and $f_0 \in H_0^1(\Omega, H^1(-\tau, 0))$. If we take initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f_0 \in L^2(\Omega \times (-\tau, 0))$, there exists a sequence $(u_0^n, u_1^n) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ and $f_0^n \in H_0^1(\Omega, H^1(-\tau, 0))$ such that

$$u_0^n \to u_0 \text{ in } H_0^1(\Omega), \quad u_1^n \to u_1 \text{ in } L^2(\Omega), \quad f_0^n \to f_0 \text{ in } L^2(\Omega \times (-\tau, 0)).$$

Hence, the uniform estimates on the corresponding regular solutions (u^n, u_t^n) imply the existence of a subsequence (still denote (u^n, u_t^n)) which converges to a weak solution (u, u_t) of the problem (1.1) satisfying (2.3).

Continuous dependence and uniqueness. First we consider the case of the regular solutions. Let $u(t)$ and $v(t)$ be two regular solutions of the problem (1.1) with respect to the initial data $\{u_0, u_1, h_1, f_{01}\}$ and $\{v_0, v_1, h_2, f_{02}\}$ respectively. Then setting $w = u - v$, we have that w is a regular solution of the problem

$$w_{tt} - \triangle w + \mu_1 w_t + \mu_2 w_t(t-\tau) + f(u) - f(v) = h_1 - h_2, \tag{3.19}$$

with Dirichlet boundary condition and initial data

$$w(0) = u_0 - v_0, \quad w_t(0) = u_1 - v_1. \tag{3.20}$$

Hence, we can multiply (3.19) with w_t in $L^2(\Omega)$ and integrate over Ω. Then we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\{\|w_t\|_2^2 + \|\nabla w\|_2^2\right\} &= -\int_\Omega (f(\cdot\,,u) - f(\cdot\,,v)) w_t\,dx - \mu_1 \|w_t\|_2^2 \\ &\quad - \mu_2 \int_\Omega w_t w_t(t-\tau)\,dx + \int_\Omega (h_1 - h_2) w_t\,dx. \end{aligned} \tag{3.21}$$

To simply notations, let

$$G(t) = \frac{1}{2}\left(\|w_t\|_2^2 + \|\nabla w\|_2^2\right),$$

and the norm of the initial data is bounded by some $R > 0$. Then for any given $T > 0$, we use $C_{R,T}$ to denote some positive constants which depend on R and T.

It remains to estimate each term on the right-hand side of (3.21). It follows from the generalized Hölder's inequality that

$$\begin{aligned}\int_\Omega (f(\cdot\,,u) - f(\cdot\,,v))w_t\,dx &\le k_0 \int_\Omega (1 + |u|^\alpha + |v|^\alpha)\,|ww_t|\,dx \\ &\le k_0\left(|\Omega|^{\alpha/[2(\alpha+1)]} + \|u\|_{2(\alpha+1)}^\alpha + \|v\|_{2(\alpha+1)}^\alpha\right)\|w\|_{2(\alpha+1)}\,\|w_t\|_2 \\ &\le C_{R,T}G(t)\end{aligned}$$

and

$$\int_\Omega (h_1 - h_2)w_t\,dx \le \|h_1 - h_2\|_2^2 + C_{R,T}G(t).$$

Hence (3.21) becomes that

$$\frac{d}{dt}G(t) \le C_{R,T}G(t) + \|h_1 - h_2\|_2^2 + \left(\mu_1 + \frac{|\mu_2|}{2}\right)\|w_t\|_2^2 + \frac{|\mu_2|}{2}\int_\Omega w_t^2(t-\tau)\,dx. \tag{3.22}$$

Moreover,

$$\int_0^t\int_\Omega w_t^2(s-\tau)\,dxds \le \int_\Omega\int_{-\tau}^0 (f_{01}(\rho) - f_{02}(\rho))^2\,d\rho dx + \int_\Omega\int_0^t w_\rho^2(\rho)\,d\rho dx.$$

Integrating (3.22) on $[0,t]$, we obtain

$$G(t) \le A + B\int_0^t G(s)\,ds, \quad t \in [0,T],$$

where

$$A = \|h_1 - h_2\|_2^2 T + G(0) + \|f_{01} - f_{02}\|_{L^2(\Omega\times(-\tau,0))}^2, \quad B = C_{R,T} + 2(|\mu_1| + |\mu_2|).$$

Hence, Gronwall's inequality yields that

$$G(t) \le Ae^{Bt},$$

which implies that

$$\|(w,w_t)\|_{\mathcal{H}}^2 \le C_{R,T}\left(\|(w(0),w_t(0))\|_{\mathcal{H}}^2 + \|h_1 - h_2\|_2^2 + \|f_{01} - f_{02}\|_{L^2(\Omega\times(-\tau,0))}^2\right) \tag{3.23}$$

for all $t \in [0,T]$, which shows that the regular solutions of the problem (1.1) depend continuously on the initial data. In particular, the problem (1.1) has a unique solution.

The same conclusion holds for weak solution by density arguments. This ends the proof of Theorem 2.1.

4. Nonlinear dynamical systems

In this section, for sake of further references we collect several known results on properties of dynamical systems in mathematical physics. They can be found in, for instance, Babin and Vishik [2], Chueshov and Lasiecka [4], Hale [15] or Temam [28]. Below we follow more closely the book by Chueshov and Lasiecka [5].

A compact set $\mathcal{A} \subset H$ is a global attractor for a dynamical system $(H, S(t))$, if it is fully invariant and uniformly attracting, that is $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and for every bounded subset $B \subset H$,

$$\lim_{t\to+\infty} \operatorname{dist}_H(S(t)B, \mathcal{A}) = 0,$$

where dist_H is the Hausdorff semidistance in H.

A bounded set $\mathcal{B} \subset H$ is an absorbing set for $S(t)$ if for any bounded set $B \subset H$, there exists $t_B = t_B(B) \geq 0$ satisfying

$$S(t)B \subset \mathcal{B}, \quad \forall\, t \geq t_B,$$

which characterizes $S(t)$ as a dissipative semigroup.

A semigroup $S(t)$ is asymptotically smooth in H if for any bounded positive invariant set $B \subset H$, there exists a compact set $K \subset \overline{B}$ such that

$$\lim_{t\to\infty} \operatorname{dist}_H(S(t)B, K) = 0.$$

Then the following is well-known, see for instance, [2, 5, 15].

Theorem 4.1. *A dissipative dynamical system $(H, S(t))$ has a compact global attractor if and only if it is asymptotically smooth.*

We present here a more recent method by Chueshov and Lasiecka [5] to verify the asymptotic smoothness property. See also [16].

Theorem 4.2. *Suppose that for any positively bounded invariant $B \subset H$ and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that*

$$\|S(T)x - S(T)y\|_H \leq \varepsilon + \phi_T(x, y), \quad \forall\, x, y \in B,$$

where $\phi_T \colon B \times B \to \mathbb{R}$ satisfies

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \phi_T(z^n, z^m) = 0 \tag{4.1}$$

for any sequence $(z^n)_{n\in\mathbb{N}}$ in B. Then $S(t)$ is asymptotically smooth in H.

Define the unstable manifold $\mathcal{M}^u(Y)$ emanating from the set $Y \subset H$ such that there exists a full trajectory $\gamma = \{z(t) : t \in \mathbb{R}\}$ with the properties

$$z(0) = z_0 \quad \text{and} \quad \lim_{t \to -\infty} \operatorname{dist}_H(z(t), Y) = 0.$$

Finally, we recall the properties of the gradient systems. A dynamical system $(H, S(t))$ is said to be gradient if there exists a strict Lyapunov function for $(H, S(t))$ on the whole phase space H, that is, (a) a continuous functional $\Phi(z)$ such that the function $t \to \Phi(S(t)z)$ is non-increasing for any $z \in H$, (b) the equation $\Phi(S(t)z) = \Phi(z)$ for all $t > 0$ implies that $S(t)z = z$ for all $t > 0$.

Now, we give the following well-known result on the existence and structure of global attractors for asymptotically compact gradient dynamical system, see for instance, Chueshov and Lasiecka [5], Fatori et al. [11].

Theorem 4.3. *Assume that $(H, S(t))$ is a gradient dynamical system which, moreover, is asymptotically smooth. In addition, assume the Lyapunov function $\Phi(z)$ associated with the system, satisfying*

(i) $\Phi(z)$ *is bounded from above on any bounded subset of H;*

(ii) *the set $\Phi_R = \{z \in H \mid \Phi(z) \le R\}$ is bounded for every R;*

(iii) *the set $\mathcal{N}$ of stationary of $(H, S(t))$ is bounded.*

Then $(H, S(t))$ has a compact global attractor characterized by $\mathcal{A} = \mathcal{M}^u(\mathcal{N})$.

5. Global attractor

In order to prove Theorem 2.5 we will apply the abstract results presented in the previous section. The first step is to prove that the dynamical system $(\mathcal{H}, S(t))$ is gradient under the conditions in Theorem 2.5. The second is to verify the quasi-stable on bounded positively invariant sets.

Inspired by [24], we define the modified energy functional as

$$\begin{aligned} \mathcal{E}(t) := \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{\xi}{2}\int_{t-\tau}^{t}\int_\Omega e^{\sigma(s-t)} u_s^2(s)\,dxds \\ + \int_\Omega F(\cdot\,, u(t))\,dx - \int_\Omega hu(t)\,dx, \end{aligned} \tag{5.1}$$

where ξ and σ are suitable positive constants to be determined later.

5.1. Gradient system

We will verify the conditions in Theorem 4.3 in the following Lemmas 5.1–5.3.

Lemma 5.1. *The dynamical system* $(\mathcal{H}, S(t))$ *associated to the problem* (1.1) *is gradient under the assumptions of Theorem* 2.1.

Proof. Let us take the functional Φ as the modified energy $\mathcal{E}$ defined in (5.1). Then for $z_0 = (u_0, u_1) \in \mathcal{H}$, we claim that $\Phi(S(t)z_0)$ is non-increasing.

Indeed, differentiating $\mathcal{E}(t)$ and using (1.1), we obtain

$$\begin{aligned}\mathcal{E}'(t) = &-\mu_1 \|u_t\|_2^2 - \mu_2(u_t(t-\tau), u_t) + \frac{\xi}{2}\|u_t\|_2^2 \\ &- \frac{\xi}{2}e^{-\sigma\tau}\int_\Omega u_t^2(t-\tau)\,dx - \frac{\sigma\xi}{2}\int_{t-\tau}^t e^{-\sigma(t-s)}\int_\Omega u_s^2(s)\,dxds.\end{aligned}$$

Noticing

$$-\mu_2(u_t(t-\tau), u_t) \le \frac{\mu_2}{2}\|u_t\|_2^2 + \frac{|\mu_2|}{2}\int_\Omega u_t^2(t-\tau)\,dx, \tag{5.2}$$

we deduce that

$$\begin{aligned}\mathcal{E}'(t) \le &\left(\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2}\right)\|u_t\|_2^2 + \left(\frac{|\mu_2|}{2} - \frac{\xi}{2e^{\sigma\tau}}\right)\int_\Omega u_t^2(t-\tau)\,dx \\ &- \frac{\sigma\xi}{2}\int_{t-\tau}^t e^{-\sigma(t-s)}\int_\Omega u_s^2(s)\,dxds.\end{aligned} \tag{5.3}$$

Notice that $e^{\sigma\tau} \to 1^+$ as $\sigma \to 0^+$. Then, by the continuity of the set of real number, if we choose $\sigma > 0$ sufficiently small, there exists a positive constant $\xi > 0$ such that

$$e^{\sigma\tau}|\mu_2| < \xi < \mu_1, \tag{5.4}$$

which implies that

$$\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2} < 0, \tag{5.5}$$

and

$$\frac{|\mu_2|}{2} - \frac{\xi}{2e^{\sigma\tau}} < 0. \tag{5.6}$$

Inserting (5.5) and (5.6) into (5.3), we obtain $\mathcal{E}'(t) \le 0$, which implies $\frac{d}{dt}\Phi(S(t)z_0) \le 0$, $\forall\, t > 0$.

Now, assume $\Phi(S(t)z_0) = \Phi(z_0)$ for all $t > 0$, then $\|u_t(t)\|_2^2 = 0$, $t > -\tau$, which implies $u(t) = u_0$ for all $t \ge 0$. Hence, $S(t)z_0 = (u_0, 0)$ is a stationary solution, which implies that Φ is a strict Lyapunov functional. Therefore $(\mathcal{H}, S(t))$ is gradient. $\square$

Lemma 5.2. *The Lyapunov functional Φ is bounded from above on any bounded subset of $\mathcal{H}$ and the set $\Phi_R = \{z \in \mathcal{H} \mid \Phi(z) \le R\}$ is bounded for every R.*

Proof. Let $B \ni (u, u_t)$ be any bounded subset of $\mathcal{H}$. Since Φ is defined as the modified functional given in (5.1), noticing

$$\frac{\xi}{2}\int_{t-\tau}^{t}\int_{\Omega} e^{\sigma(s-t)}u_s^2(s)\,dxds \le \tau\mu_1 C_B,$$

where we have used (5.3). It is easy to check that $\Phi(z)$ is bounded from above on bounded subset B of $\mathcal{H}$.

Now let $z(t) = (u(t), u_t(t)) \in \mathcal{H}$ be any weak solution to (1.1) such that $\Phi(z(t)) \le R$. It follows from (3.6) that

$$\beta_0\left(\|\nabla u\|_2^2 + \|u_t\|_2^2\right) - M \le \Phi(z(t)) = \mathcal{E}(u(t)) \le R,$$

where $M = L_0\,|\Omega| + \frac{1}{\lambda_1\varrho}\,\|h\|_2^2$, and then $\|z(t)\|_{\mathcal{H}}^2 \le (R+M)\beta_0^{-1}$. Therefore Φ_R is bounded in $\mathcal{H}$. □

Lemma 5.3. *The set $\mathcal{N} = \{(u,0) \in \mathcal{H} \mid -\triangle u + f(\cdot\,,u) = h\}$ of the stationary solutions of the problem* (1.1) *is bounded in $\mathcal{H}$.*

Proof. The proof of this lemma is evident, see also [11]. In fact, from the first equation of (1.1) we obtain

$$\|\nabla u\|_2^2 = -\int_{\Omega} f(\cdot\,,u)u\,dx + \int_{\Omega} hu\,dx.$$

From the condition (2.6) and since $\lambda_1\,\|u\|_2^2 \le \|\nabla u\|_2^2$,

$$-\int_{\Omega} f(x,u)u\,dx \le \frac{\beta}{\lambda_1}\,\|\nabla u\|_2^2 + L_0\,|\Omega|\,,$$

and for any $\varrho > 0$,

$$\int_{\Omega} hu\,dx \le \frac{\varrho}{4}\,\|\nabla u\|_2^2 + \frac{1}{\lambda_1\varrho}\,\|h\|_2^2\,,$$

it follows that

$$\left(1 - \frac{\beta}{\lambda_1} - \frac{\varrho}{4}\right)\|\nabla u\|_2^2 \le \frac{1}{\lambda_1\varrho}\,\|h\|_2^2 + L_0\,|\Omega|\,.$$

Hence we obtain that $\mathcal{N}$ is bounded in $\mathcal{H}$ if taking $\varrho > 0$ sufficiently small. □

5.2. Asymptotic smoothness

Lemma 5.4 (Stabilizability)**.** *Suppose the assumptions of Theorem* 2.5 *hold. Given a bounded set $B \subset \mathcal{H}$, let $z^1 = (u, u_t)$ and $z^2 = (v, v_t)$ be two weak solution of problem* (1.1) *with $z^1(0) = (u_0, u_1)$ and $z^2(0) = (v_0, v_1)$ are in set B. Then*

$$\left\|z^1(t) - z^2(t)\right\|_{\mathcal{H}}^2 \le C_B e^{-\gamma t} + C_B\int_0^t e^{-\gamma(t-s)}\,\|w\|_{2(\alpha+1)}^2\,ds \tag{5.7}$$

for any $t \geq 0$, where $w = u - v$ and γ, C_B are positive constants on the size of B but not on t.

Proof. We set $w = u - v$, then $(w, w_t) = z^1(t) - z^2(t)$ solves the problem

$$w_{tt} - \triangle w + \mu_1 w_t + \mu_2 w_t(t-\tau) + f(x,u) - f(x,v) = 0, \tag{5.8}$$

in the weak sense, with the initial condition

$$(w(0), w_t(0)) = z^1(0) - z^2(0).$$

Step 1. By density we can assume the solution have higher regularity so that we can multiply equation (5.8) by w_t in $L^2(\Omega)$, and integrate over Ω. Proceeding as in the proof of Theorem 2.1, we obtain

$$\begin{aligned} G'(t) = &-\mu_1 \|w_t\|_2^2 - \mu_2(w_t(t-\tau), w_t) + \int_\Omega (f(\cdot\,,u) - f(\cdot\,,v))w_t\,dx \\ &+ \frac{\xi}{2}\|w_t\|_2^2 - \frac{\xi}{2}e^{-\sigma\tau}\int_\Omega w_t^2(t-\tau)\,dx - \frac{\sigma\xi}{2}\int_{t-\tau}^t e^{-\sigma(t-s)}\int_\Omega w_s^2(s)\,dxds, \end{aligned}$$

where we denote

$$G(t) = \frac{1}{2}\|w_t(t)\|_2^2 + \frac{1}{2}\|\nabla w(t)\|_2^2 + \frac{\xi}{2}\int_{t-\tau}^t \int_\Omega e^{\sigma(s-t)} w_s^2(s)\,dxds. \tag{5.9}$$

Using Hölder's inequality, it follows that

$$\begin{aligned} G'(t) \leq &\left(\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2}\right)\|w_t\|_2^2 + \left(\frac{|\mu_2|}{2} - \frac{\xi}{2e^{\sigma\tau}}\right)\int_\Omega w_t^2(t-\tau)\,dx \\ &+ \int_\Omega (f(\cdot\,,u) - f(\cdot\,,v))w_t\,dx - \frac{\sigma\xi}{2}\int_{t-\tau}^t e^{-\sigma(t-s)}\int_\Omega w_s^2(s)\,dxds. \end{aligned} \tag{5.10}$$

Step 2. Now we consider the functions

$$G_\varepsilon(t) = G(t) + \varepsilon\phi(t), \quad \varepsilon > 0,$$

where $\varepsilon > 0$ will be fixed later and

$$\phi(t) = \int_\Omega w(t)w_t(t)\,dx. \tag{5.11}$$

It is easy to check that there exists a constant C_1 such that

$$|G_\varepsilon(t) - G(t)| \leq \varepsilon C_1 G(t), \quad \forall\, t \geq 0,\ \varepsilon > 0. \tag{5.12}$$

Step 3. In what follows we show the estimate of $\phi'(t)$. By differentiating the function in (5.11), using equation (5.8) in the weak sense, subtracting and adding $G(t)$, we get

$$\begin{aligned} \phi'(t) = &-G(t) + \frac{3}{2}\|w_t\|_2^2 - \frac{1}{2}\|\nabla w\|_2^2 - \mu_1(w_t, w) - \mu_2(w_t(t-\tau), w) \\ &- \int_\Omega (f(\cdot\,,u) - f(\cdot\,,v))w\,dx + \frac{\xi}{2}\int_{t-\tau}^t \int_\Omega e^{\sigma(s-t)} w_s^2(s)\,dxds. \end{aligned}$$

We derive from Young's inequality that

$$-\mu_1(w_t, w) \leq \frac{\delta\mu_1}{\lambda_1}\|\nabla w\|_2^2 + \frac{\mu_1}{4\delta}\|w_t\|_2^2$$

and

$$-\mu_2(w_t(t-\tau), w) \leq \frac{\delta|\mu_2|}{\lambda_1}\|\nabla w\|_2^2 + \frac{|\mu_2|}{4\delta}\int_\Omega w_t^2(t-\tau)\,dx,$$

where $\delta > 0$ is a small constant which will be chosen later. Hence, we arrive at

$$\begin{aligned}\phi'(t) \leq{}& -G(t) + \left(\frac{3}{2} + \frac{\mu_1}{4\delta}\right)\|w_t\|_2^2 - \left(\frac{1}{2} - \frac{\delta\mu_1}{\lambda_1} - \frac{\delta|\mu_2|}{\lambda_1}\right)\|\nabla w\|_2^2 + \frac{|\mu_2|}{4\delta}\int_\Omega w_t^2(t-\tau)\,dx\\ &- \int_\Omega (f(\cdot,u) - f(\cdot,v))w\,dx + \frac{\xi}{2}\int_{t-\tau}^t\int_\Omega e^{\sigma(s-t)}w_s^2(s)\,dxds.\end{aligned}$$

Further, since $\frac{\alpha}{2(\alpha+1)} + \frac{1}{2(\alpha+1)} + \frac{1}{2} = 1$, by the generalized Hölder's inequality, assumptions (2.4) and (2.5), there exists a constant C_B (may be different from line to line)

$$\begin{aligned}&\left|\int_\Omega (f(\cdot,u) - f(\cdot,v))w_t\,dx\right|\\ &\leq k_0\int_\Omega (1 + |u|^\alpha + |v|^\alpha)|w||w_t|\,dx\\ &\leq k_0\left(|\Omega|^{\alpha/[2(\alpha+1)]} + \|u\|_{2(\alpha+1)}^\alpha + \|v\|_{2(\alpha+1)}^\alpha\right)\|w\|_{2(\alpha+1)}\|w_t\|_2\\ &\leq C_B\|w\|_{2(\alpha+1)}\|w_t\|_2\\ &\leq \frac{\varepsilon}{2}\|w_t\|_2^2 + C_B\|w\|_{2(\alpha+1)}^2,\end{aligned} \tag{5.13}$$

since $L^{2(\alpha+1)}(\Omega) \hookrightarrow L^2(\Omega)$, similarly,

$$\left|\int_\Omega (f(\cdot,u) - f(\cdot,v))w\,dx\right| \leq C_B\|w\|_{2(\alpha+1)}^2.$$

Combining this estimates with (5.10), we obtain

$$\begin{aligned}G_\varepsilon'(t) \leq{}& -\varepsilon G(t) + \left[\frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2} + \varepsilon\left(2 + \frac{\mu_1}{4\delta}\right)\right]\|w_t\|_2^2 - \varepsilon\left(\frac{1}{2} - \frac{\delta\mu_1}{\lambda_1} - \frac{\delta|\mu_2|}{\lambda_1}\right)\|\nabla w\|_2^2\\ &+ \left(\frac{|\mu_2|}{2} - \frac{\xi}{2e^{\sigma\tau}} + \varepsilon\frac{|\mu_2|}{4\delta}\right)\int_\Omega w_t^2(t-\tau)\,dx + C_B\|w\|_{2(\alpha+1)}^2\\ &- \frac{\xi}{2}(\sigma - \varepsilon)\int_{t-\tau}^t\int_\Omega e^{\sigma(s-t)}w_s^2(s)\,dxds.\end{aligned}$$

First, we fix $\delta > 0$ sufficiently small such that

$$\frac{1}{2} - \frac{\delta\mu_1}{\lambda_1} - \frac{\delta|\mu_2|}{\lambda_1} \geq 0,$$

then we choose $\varepsilon_1 \leq \sigma$ sufficiently small, thanks to (5.5) and (5.6), it follows

$$G_\varepsilon'(t) \leq -\varepsilon G(t) + C_B\|w\|_{2(\alpha+1)}^2, \quad \forall\, t \geq 0,\ \forall\, \varepsilon \in (0, \varepsilon_1]. \tag{5.14}$$

Step 4. Now we take $\varepsilon_0 = \min\{1/(2C_1), \varepsilon_1\}$ and select $\varepsilon \le \varepsilon_0$. Then (5.12) implies that

$$\frac{1}{2}G(t) \le G_\varepsilon(t) \le \frac{3}{2}G(t), \quad \forall\, t \ge 0. \tag{5.15}$$

By a standard combination of (5.14) and (5.15), we conclude

$$G'_\varepsilon(t) \le G_\varepsilon(0)e^{-\frac{2\varepsilon}{3}t} + C_B \int_0^t e^{-\frac{2\varepsilon}{3}(t-s)} \|w(s)\|^2_{2(\alpha+1)}\, ds.$$

Using (5.15) again we obtain

$$G(t) \le 3G(0)e^{-\frac{2\varepsilon}{3}t} + 2C_B \int_0^t e^{-\frac{2\varepsilon}{3}(t-s)} \|w(s)\|^2_{2(\alpha+1)}\, ds.$$

By the definition (5.9), we obtain (5.7) with $\gamma = 2\varepsilon/3$. □

Lemma 5.5. *Under the assumptions of Theorem* 2.5*, the dynamical system* $(\mathcal{H}, S(t))$ *corresponding to problem* (1.1) *is asymptotically smooth.*

Proof. We apply Theorem 4.2. Let B be a bounded positively invariant subsect of $\mathcal{H}$ with respect to $S(t)$. For initial data z_0^1, z_0^2 in set B, we write

$$S(t)z_0^i = (u^i(t), u_t^i(t)), \quad i = 1, 2.$$

Given $\varepsilon > 0$, we choose T sufficiently large such that $C_B e^{-\frac{\gamma}{2}T} < \varepsilon$, where C_B is given in Lemma 5.4.

We claim that there exists a constant $C_{BT} > 0$ such that

$$\left\|S(T)z_0^1 - S(T)z_0^2\right\|_{\mathcal{H}} \le \varepsilon + \phi_T(z_0^1, z_0^2), \quad \forall\, z_0^1,\ z_0^2 \text{ in } B, \tag{5.16}$$

where

$$\phi_T(z_0^1, z_0^2) = C_{BT}\left(\int_0^T \left\|u^1(s) - u^2(s)\right\|_2^{\vartheta}\right)^{1/4} \tag{5.17}$$

for some constant $\vartheta > 0$.

Indeed, applying Gagliardo-Nirenberg interpolation inequality we obtain

$$\left\|u^1(t) - u^2(t)\right\|_{2(\alpha+1)} \le C_\theta \left\|\nabla u^1(t) - \nabla u^2(t)\right\|_2^{\theta} \left\|u^1(t) - u^2(t)\right\|_2^{1-\theta}$$

with $\theta = \frac{N}{2}(1 - \frac{1}{\alpha+1})$.

Then we can rewrite (5.7) as

$$\begin{aligned} &\left\|z^1(t) - z^2(t)\right\|_{\mathcal{H}}^2 \\ &\le C_B e^{-\gamma t} + C_B \left(\int_0^T \left(\left\|\nabla u^1(s)\right\|_2 + \left\|\nabla u^2(s)\right\|_2\right)^{4\theta} dsl\right)^{1/2} \left(\int_0^T \left\|u^1(t) - u^2(t)\right\|_2^{4(1-\theta)} ds\right)^{1/2} \end{aligned}$$

for $t < T$. Since $u^1, u^2 \in L^\infty_{\text{loc}}((0,\infty); H^1_0(\Omega)$, we deduce that there exists $C_{BT} > 0$ such that

$$\left\|z^1(T) - z^2(T)\right\|_{\mathcal{H}} \le C_B e^{-\frac{\gamma}{2}T} + C_{BT}\left(\int_0^T \left\|u^1(t) - u^2(t)\right\|_2^{4(1-\theta)} ds\right)^{1/4},$$

which implies that (5.16) and (5.17) hold.

It remains to show that ϕ_T satisfies (4.1). Indeed, given a sequence of initial data (z_0^n) in B, we denote $S(t)(z_0^n) = (u^n(t), u_t^n(t))$. Since $(u^n(t), u^n(t))$ is bounded in $C([0,T], H^1_0(\Omega)\times L^2(\Omega))$, $T > 0$, then from the compact embedding of $H^1_0(\Omega) \subset L^2(\Omega)$, the Aubin's lemma implies that there exists a subsequence (u^{n_k}) that converges strongly in $C([0,T], L^2(\Omega))$. Therefore we see that

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_0^T \|u^{n_k}(s) - u^{n_l}(s)\|_2^{\vartheta}\, ds = 0,$$

which shows (4.1) holds. The asymptotic smoothness property of $(\mathcal{H}, S(t))$ follows from Theorem 4.2. □

5.3. Proof of Theorem 2.5

Lemmas 5.1 and 5.4 show that $(\mathcal{H}, S(t))$ is a gradient dynamical system which, moreover, is asymptotically smooth. Then the global existence of a global attractor $\mathcal{A} = \mathcal{M}^u(\mathcal{N})$ to problem (1.1) follows from Theorem 4.3, Lemmas 5.2 and 5.3.

6. Decay property

In this section, we study the decay property of solution to (1.1) with $h \equiv 0$.

Proof of Theorem 2.6. Similar to the proof of (3.6), it follows

$$E(u(t)) \ge \beta_0\left(\|\nabla u\|_2^2 + \|u_t\|_2^2\right). \tag{6.1}$$

As in (5.1), we define the modified energy functional as

$$\mathcal{E}(t) := \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{\xi}{2}\int_{t-\tau}^t\int_\Omega e^{\sigma(s-t)}u_s^2(s)\, dxds + \int_\Omega F(\cdot, u(t))\, dx.$$

Now we consider the functions

$$\mathcal{E}_\varepsilon(t) = \mathcal{E}(t) + \varepsilon\varphi(t), \quad \varepsilon > 0,$$

where $\varepsilon > 0$ will be fixed later and

$$\varphi(t) = \int_\Omega u(t)u_t(t)\, dx. \tag{6.2}$$

Similarly, there exists a constant $C_2 > 0$ such that

$$|\mathcal{E}_\varepsilon(t) - \mathcal{E}(t)| \le \varepsilon C_2 \mathcal{E}(t), \quad \forall\, t \ge 0,\ \varepsilon > 0. \tag{6.3}$$

Now we show that there exists a constant $\varepsilon_1 > 0$ such that

$$\mathcal{E}'_\varepsilon(t) \le -\varepsilon \mathcal{E}(t), \quad \forall\, t \ge 0,\ \forall\, \varepsilon \in (0, \varepsilon_1]. \tag{6.4}$$

Indeed, taking derivative of (6.2), using the equation in (1.1), adding and subtracting $\mathcal{E}(t)$ into the expression, we obtain

$$\begin{aligned}\varphi'(t) = {}& -\mathcal{E}(t) + \frac{3}{2}\|u_t\|_2^2 - \frac{1}{2}\|\nabla u\|_2^2 - \mu_1(u_t, u) - \mu_2(u_t(t-\tau), u)\\ &+ \int_\Omega (F(\cdot\,, u) - f(\cdot\,, u)u)\,dx + \frac{\xi}{2}\int_{t-\tau}^t \int_\Omega e^{\sigma(s-t)} w_s^2(s)\,dxds.\end{aligned}$$

From (2.7), we have

$$\int_\Omega (F(\cdot\,, u) - f(\cdot\,, u)u)\,dx \le 0.$$

Hence, as in the proof of Lemma 5.4, for any $\delta > 0$, we see that

$$\begin{aligned}\varphi'(t) \le {}& -\mathcal{E}(t) + \left(\frac{3}{2} + \frac{\mu_1}{4\delta}\right)\|u_t\|_2^2 - \left(\frac{1}{2} - \frac{\delta\mu_1}{\lambda_1} - \frac{\delta\,|\mu_2|}{\lambda_1}\right)\|\nabla u\|_2^2\\ &+ \frac{|\mu_2|}{4\delta}\int_\Omega u_t^2(t-\tau)\,dx + \frac{\xi}{2}\int_{t-\tau}^t \int_\Omega e^{\sigma(s-t)} u_s^2(s)\,dxds.\end{aligned}$$

Once the positive constants σ and ξ being fixed such that (5.4)–(5.6) are satisfied. Here we denote the positive constants C_3, C_4 by

$$-C_3 = \frac{|\mu_2|}{2} - \mu_1 + \frac{\xi}{2}, \quad -C_4 = \frac{|\mu_2|}{2} - \frac{\xi}{2e^{\sigma\tau}}.$$

Moreover, noticing the estimate $\mathcal{E}'(t)$ in (5.3), we have

$$\begin{aligned}\mathcal{E}'_\varepsilon(t) \le {}& -\varepsilon\mathcal{E}(t) - \left[C_3 - \varepsilon\left(\frac{3}{2} + \frac{\mu_1}{4\delta}\right)\right]\|u_t\|_2^2 - \left(\frac{1}{2} - \frac{\delta\mu_1}{\lambda_1} - \frac{\delta\,|\mu_2|}{\lambda_1}\right)\|\nabla u\|_2^2\\ &- \left(C_4 - \varepsilon\frac{|\mu_2|}{4\delta}\right)\int_\Omega u_t^2(t-\tau)\,dx - \frac{\xi}{2}(\sigma - \varepsilon)\int_{t-\tau}^t \int_\Omega e^{\sigma(s-t)} u_s^2(s)\,dxds.\end{aligned}$$

Choosing $\delta > 0$ sufficiently small such that

$$\frac{1}{2} - \frac{\delta\mu_1}{\lambda_1} - \frac{\delta\,|\mu_2|}{\lambda_1} \ge 0,$$

then we choose $\varepsilon_1 \le \sigma$ sufficiently small such that $C_3 - \varepsilon_1\left(\frac{3}{2} + \frac{\mu_1}{4\delta}\right) \ge 0$ and $C_4 - \varepsilon\frac{|\mu_2|}{4\delta} \ge 0$. This ends the proof of (6.4).

Now let us put $\varepsilon_0 = \min\{1/(2C_2), \varepsilon_1\}$, for all $\varepsilon \le \varepsilon_0$, follows from (6.3)

$$\frac{1}{2}\mathcal{E}(t) \le \mathcal{E}_\varepsilon(t) \le \frac{3}{2}\mathcal{E}(t). \tag{6.5}$$

Combining (6.4) and (6.5), we obtain

$$\mathcal{E}(t) \le 3\mathcal{E}(0)e^{-\frac{2\varepsilon}{3}t}, \quad t \ge 0. \tag{6.6}$$

This finishes the proof of Theorem 2.6 with $\kappa = 2\varepsilon/3$ and $K = 3\mathcal{E}(0)$. □

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References

[1] R. de O. Araújo, T. F. Ma and Y. Qin, *Long-time behavior of a quasilinear viscoelastic equation with past history*, J. Differential Equations **254** (2013), no. 10, 4066–4087. https://doi.org/10.1016/j.jde.2013.02.010

[2] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992. https://doi.org/10.1016/s0168-2024(08)x7020-1

[3] J. M. Ball, *Global attractors for damped semilinear wave equations*, Discrete Contin. Dyn. Syst. **10** (2004), no. 1-2, 31–52. https://doi.org/10.3934/dcds.2004.10.31

[4] I. Chueshov and I. Lasiecka, *Long-time behavior of second order evolution equations with nonlinear damping*, Mem. Amer. Math. Soc. **195** (2008), no. 912, 183 pp. https://doi.org/10.1090/memo/0912

[5] ______, *Von Karman Evolution Equations*, Springer Monographs in Mathematics, Springer, New York, 2010. https://doi.org/10.1007/978-0-387-87712-9

[6] Q. Dai and Z. Yang, *Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay*, Z. Angew. Math. Phys. **65** (2014), no. 5, 885–903. https://doi.org/10.1007/s00033-013-0365-6

[7] R. Datko, *Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks*, SIAM J. Control Optim. **26** (1988), no. 3, 697–713. `https://doi.org/10.1137/0326040`

[8] R. Datko, J. Lagnese and M. P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim. **24** (1986), no. 1, 152–156. `https://doi.org/10.1137/0324007`

[9] A. Eden, A. J. Milani and B. Nicolaenko, *Finite-dimensional exponential attractors for semilinear wave equations with damping*, J. Math. Anal. Appl. **169** (1992), no. 2, 408–419. `https://doi.org/10.1016/0022-247x(92)90087-t`

[10] L. C. Evans, *Partial Differential Equations*, Graduates Studies in Mathematics **19**, American Mathematical Society, Providence, RI, 1998.

[11] L. H. Fatori, M. A. Jorge Silva, T. F. Ma and Z. Yang, *Long-time behavior of a class of thermoelastic plates with nonlinear strain*, J. Differential Equations **259** (2015), no. 9, 4831–4862. `https://doi.org/10.1016/j.jde.2015.06.026`

[12] E. Feireisl, *Asymptotic behavior and attractors for a semilinear damped wave equation with supercritical exponent*, Proc. Roy. Soc. Edinburgh Sect. A **125** (1995), no. 5, 1051–1062. `https://doi.org/10.1017/s0308210500022630`

[13] J.-M. Ghidaglia and R. Temam, *Attractors for damped nonlinear hyperbolic equations*, J. Math. Pures Appl. (9) **66** (1987), no. 3, 273–319.

[14] A. Guesmia, *Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay*, IMA J. Math. Control Inform. **30** (2013), no. 4, 507–526. `https://doi.org/10.1093/imamci/dns039`

[15] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs **25**, American Mathematical Society, Providence, RI, 1988.

[16] M. A. Jorge Silva and T. F, Ma, *Long-time dynamics for a class of Kirchhoff models with memory*, J. Math. Phys. **54** (2013), no. 2, 021505, 15 pp. `https://doi.org/10.1063/1.4792606`

[17] J.-L. Lions, *Quelques Méthodes de Résolution des Problémes aux Limites non Linéaries*, Dunod-Gauthier Villars, Paris, 1969.

[18] W. Liu, *General decay of the solution for a viscoelastic wave equation with a time-varying delay term in the internal feedback*, J. Math. Phys. **54** (2013), no. 4, 043504, 9 pp. `https://doi.org/10.1063/1.4799929`

[19] ———, *General decay rate estimate for the energy of a aweak viscoelastic equation with an internal time-varying delay term*, Taiwanese J. Mathematics **17** (2013), no. 6, 2101–2115. https://doi.org/10.11650/tjm.17.2013.2968

[20] G. Liu and H. Zhang, *Well-posedness for a class of wave equation with past history and a delay*, Z. Angew. Math. Phys. **67** (2016), no. 1, 14 pp. https://doi.org/10.1007/s00033-015-0593-z

[21] M. Nakao, *Global attractors for nonlinear wave equations with nonlinear dissipative terms*, J. Differential Equations **227** (2006), no. 1, 204–229. https://doi.org/10.1016/j.jde.2005.09.013

[22] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim. **45** (2006), no. 5, 1561–1585. https://doi.org/10.1137/060648891

[23] ———, *Stabilization of the wave equation with boundary or internal distributed delay*, Differential Integral Equations **21** (2008), no. 9-10, 935–958.

[24] ———, *Interior feedback stabilization of wave equations with time dependent delay*, Electron. J. Differential Equations **2011** (2011), no. 41, 20 pp.

[25] ———, *Exponential stability of abstract evolution equations with time delay*, J. Evol. Equ. **15** (2015), no. 1, 107–129. https://doi.org/10.1007/s00028-014-0251-5

[26] S. Nicaise, J. Valein and E. Fridman, *Stability of the heat and of the wave equations with boundary time-varying delays*, Discrete Contin. Dyn. Syst. Ser. S **2** (2009), no. 3, 559–581. https://doi.org/10.3934/dcdss.2009.2.559

[27] S. H. Park, *Decay rate estimates for a weak viscoelastic beam equation with time-varying delay*, Appl. Math. Lett. **31** (2014), 46–51. https://doi.org/10.1016/j.aml.2014.02.005

[28] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences **68**, Springer-Verlag, New York, 1988. https://doi.org/10.1007/978-1-4684-0313-8

[29] G. Q. Xu, S. P. Yung and L. K. Li, *Stabilization of wave systems with input delay in the boundary control*, ESAIM Control Optim. Calc. Var. **12** (2006), no. 4, 770–785. https://doi.org/10.1051/cocv:2006021

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