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Multiplicity and Concentration of Solutions for Fractional Schrödinger Equations

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Abstract. In this paper, we study the following fractional Schrödinger equations

$$(-\Delta)^\alpha u + \lambda V(x)u = f(x,u) + \mu\xi(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N,$$

where $\lambda > 0$ is a parameter, $V \in C(\mathbb{R}^N)$ and $V^{-1}(0)$ has nonempty interior. Under some mild assumptions, we establish the existence of two different nontrivial solutions. Moreover, the concentration of these solutions is also explored on the set $V^{-1}(0)$ as $\lambda \to \infty$. As an application, we also give the similar results and concentration phenomenons for the above problem with concave and convex nonlinearities.

1. Introduction

This paper is concerned with the following fractional Schrödinger equation

$$\begin{cases} (-\Delta)^\alpha u + \lambda V(x)u = f(x,u) + \mu\xi(x)\,|u|^{p-2}\,u, & x \in \mathbb{R}^N, \\ u \in H^\alpha(\mathbb{R}^N). \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $(-\Delta)^\alpha$ is the fractional Laplacian of order α, $V \in C(\mathbb{R}^N, \mathbb{R})$, $f \in C(\mathbb{R}^N \times \mathbb{R})$, $\xi \in L^{2/(2-p)}(\mathbb{R}^N, \mathbb{R}^+)$ and $\xi(x) \not\equiv 0$, $\lambda > 0$, $\mu > 0$ and $1 < p < 2$. We need to make the following assumptions for potential V:

(V$_1$) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x) \geq 0$ on $\mathbb{R}^N$;

(V$_2$) there is $b > 0$ such that the set $V_b := \left\{x \in \mathbb{R}^N \mid V(x) < b\right\}$ has finite measure;

(V$_3$) $\Omega = \operatorname{int} V^{-1}(0)$ is nonempty and has smooth boundary $\partial\Omega$.

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In [16], the authors have proved that $(-\Delta)^\alpha$ reduces to the standard Laplacian $-\Delta$ as $\alpha \to 1$. When $\alpha = 1$ and without parameter λ, formally, Problem (1.1) reduces to the classical Schrödinger equation.

Recently, fractional Laplacian equations have concrete applications in many fields, such as thin obstacle problem, optimization, finance, phase transitions, anomalous diffusion and so on. For previous related results see [2, 8, 10, 13–15, 18–21, 24, 32, 40] and the references therein. Up to now, there have been a few results appeared in the literature for Problem (1.1). Precisely, Felmer et al. studied a similar class equations, in which $V = 1$, under suitable hypotheses on nonlinearity, using variational methods, classical positive solutions are found in [19]. Dipierro et al. studied the existence of positive and spherically symmetric solutions in [18]. The existence of bounded solutions for Problem (1.1) is proved in [15], where the potential V is unbounded. The author proved some existence results of solutions for fractional Schrödinger equations in [24, 25], under the assumption that the nonlinearity satisfies the Ambrosetti-Rabinowitz condition or is indeed of perturbative type. The author investigated the existence of radial solutions for Problem (1.1) without Ambrosetti-Rabinowitz condition in [26]. The existence of positive solutions of concave-convex Dirichlet fractional Laplacian problems in bounded domains is presented in [5].

It is known, a great attention has been devoted to the fractional and non-local integro-differential operators like (1.1), for the thought-provoking theoretical structure and their impressive applications in many fields. In fact, the fractional Laplacian $(-\Delta)^\alpha$ is a nonlocal operator in the fractional Schrödinger equation, which is obvious a difficulty. And then, Caffarelli and Silvestrein made greatest achievement in overcoming this difficulty by the extension theorem in [9]. The authors used some extension to transform the nonlocal problem into a local problem, and established some existence and nonexistence of Dirichlet problem involving the fractional Laplacian on bounded domain. Furthermore, a great deal of progress has been made to the fractional Laplacian equations after the work [9]. We refer to [11, 12, 33, 34, 38, 40–42] for the existence results and multiplicity results of solutions, and to [6, 7] for the regularity results, maximum principle, uniqueness result and other properties. Actually, for other related topics including the superquadratic singular perturbation problem and concentration phenomenon of semi-classical state, see also [29–31] and the references therein.

There are many papers taking into account potential V see for instance [35–37, 41] and the references therein. In fact, the hypotheses on potential V were first introduced by Bartsch and Wang [4] (see also [3]) in the study of a nonlinear Schrödinger equation and the potential λV with V satisfying (V_1)–(V_3) is referred as the steep well potential. It is worth mentioning that the above papers always assumed the potential V is positive ($V > 0$).

Compared with the case $V > 0$, our assumptions on V are rather weak, and perhaps more important. Generally speaking, there may exist some behaviours and phenomenons for the solutions of Problem (1.1) under condition (V_3), such as the concentration phenomenon of solutions. Besides, we are also interested in the case that the nonlinearity is a more general mixed nonlinearity involving a combination of superlinear and sublinear terms.

To the best of our knowledge, few works concern on this case up to now. Motivated by the above papers, we will consider Problem (1.1) with steep well potential, and study the existence of nontrivial solutions and investigate the concentration phenomenon of solutions on the set $V^{-1}(0)$ as $\lambda \to \infty$. In order to state our results, we need the following assumptions for superlinear term $f(x,u)$:

(F_1) $f \in C(\mathbb{R}^N \times \mathbb{R})$ and $|f(x,u)| \le c\left(1 + |u|^{q-1}\right)$ for some $q \in (2, 2_\alpha^*)$, where $2_\alpha^* = 2N/(N-2\alpha)$;

(F_2) $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly for $x \in \mathbb{R}^N$;

(F_3) there exists $\theta > 2$ such that $0 < \theta F(x,u) \le uf(x,u)$ for every $x \in \mathbb{R}^N$ and $u \neq 0$, where $F(x,u) = \int_0^u f(x,t)\,dt$.

On the existence of solutions we have the following results.

Theorem 1.1. *Assume that* (V_1)–(V_3) *and* (F_1)–(F_3) *hold, and* $\xi \in L^{2/(2-p)}(\mathbb{R}^N, \mathbb{R}^+)$ *(*$1 < p < 2$*), then there exist two positive constants* Λ_0 *and* μ_0 *such that for every* $\lambda \ge \Lambda_0$ *and* $0 < \mu < \mu_0$*, Problem* (1.1) *has at least two nontrivial solutions* $u_{\lambda,i}$ *(*$i = 1,2$*).*

On the concentration of solutions we have the following result.

Theorem 1.2. *Let* $u_{\lambda,i}$ *(*$i = 1,2$*) be the solutions of Problem* (1.1) *obtained in Theorem* 1.1*, then* $u_{\lambda,i} \to u_{0,i}$ *in* $H^\alpha(\mathbb{R}^N)$ *as* $\lambda \to \infty$*, where* $u_{0,i}$ *are solutions of the equation*

$$\begin{cases} (-\Delta)^\alpha u = f(x,u) + \mu\xi(x)\,|u|^{p-2}\,u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.2}$$

Furthermore,

$$\frac{1}{2}\int_\Omega \left|(-\Delta)^{\alpha/2}u_{0,1}\right|^2 dx - \int_\Omega F(x,u_{0,1})\,dx - \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)\,|u_{0,1}|^p\,dx > 0$$

and

$$\frac{1}{2}\int_\Omega \left|(-\Delta)^{\alpha/2}u_{0,2}\right|^2 dx - \int_\Omega F(x,u_{0,2})\,dx - \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)\,|u_{0,2}|^p\,dx \le 0.$$

A model nonlinearity is

$$g(x,u) := |u|^{q-2}\,u + \mu\xi(x)\,|u|^{p-2}\,u \tag{1.3}$$

with $1 < p < 2 < q < 2_\alpha^*$ and $\xi \in L^{2/(2-p)}(\mathbb{R}^N, \mathbb{R}^+)$. Clearly, $g(x,u)$ satisfies (F_1)–(F_3). Following [1], the nonlinear term $g(x,u)$ is called concave and convex nonlinear term. Therefore, our results can be applied to the concave and convex nonlinear term case. As a consequence, we have

Corollary 1.3. *Assume that* (V_1)–(V_3) *are satisfied and let the nonlinearity be of the form* (1.3)*, then there exist two positive constants* Λ_0 *and* μ_0 *such that for every* $\lambda \geq \Lambda_0$ *and* $0 < \mu < \mu_0$*, Problem* (1.1) *has at least two nontrivial solutions* $u_{\lambda,i}$ *(*$i = 1, 2$*).*

Corollary 1.4. *Let* $u_{\lambda,i}$ *(*$i = 1, 2$*) be the solutions of Problem* (1.1) *obtained in Corollary* 1.3*, then* $u_{\lambda,i} \to u_{0,i}$ *in* $H^\alpha(\mathbb{R}^N)$ *as* $\lambda \to \infty$*, where* $u_{0,i}$ *are solutions of the equation*

$$\begin{cases} (-\Delta)^\alpha u = |u|^{q-2} u + \mu\xi(x) |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

The rest of the present paper is organized as follows. In Section 2, we establish the variational framework associated with Problem (1.1), and we also give the proof of Theorem 1.1. In Section 3, we study the concentration of solutions and prove Theorem 1.2.

2. Variational setting and proof of Theorem 1.1

Below we denote by $\|\cdot\|_s$ the usual L^s-norm for $2 \leq s \leq 2_\alpha^*$ and by $\widehat{u}$ the usual Fourier transform of u, c_i, C, C_i stand for different positive constants. Now, we establish the variational setting of Problem (1.1) in fractional Sobolev spaces.

A complete introduction to fractional Sobolev spaces can be found in [16], we offer below a short review. We recall that the Sobolev spaces $W^{\alpha,p}(\mathbb{R}^N)$ is defined for any $p \in [1, +\infty)$ and $\alpha \in (0,1)$ as

$$W^{\alpha,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{\alpha p + N}}\, dxdy < +\infty \right\}.$$

This space is endowed with the Gagliardo norm

$$\|u\|_{W^{\alpha,p}} = \left(\int_{\mathbb{R}^N} |u|^p\, dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{\alpha p + N}}\, dxdy \right)^{1/p}.$$

When $p = 2$, these spaces are also denoted by $H^\alpha(\mathbb{R}^N)$.

If $p = 2$, an equivalent definition of fractional Sobolev spaces is possible, based on Fourier analysis. Indeed, it turns out that

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2\alpha})\, |\widehat{u}|^2\, d\xi < +\infty \right\},$$

and the norm can be equivalently written by

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \left(\|\widehat{u}\|_2^2 + \int_{\mathbb{R}^N} |\xi|^{2\alpha} |\widehat{u}|^2 \, d\xi\right)^{1/2}.$$

Furthermore, we know that $\|\cdot\|_{H^\alpha(\mathbb{R}^N)}$ is equivalent to the norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\alpha/2}u\right|^2 + u^2\right) dx\right)^{1/2}.$$

In this article, in view of the potential $V(x)$, we consider its subspace

$$E = \left\{u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\alpha/2}u\right|^2 + V(x)u^2\right) dx < +\infty\right\}.$$

Then, by [24], E is a Hilbert space with the inner product

$$(u,v)_E = \int_{\mathbb{R}^N} \left(|\xi|^{2\alpha}\, \widehat{u}(\xi)\widehat{v}(\xi) + \widehat{u}(\xi)\widehat{v}(\xi)\right) d\xi + \int_{\mathbb{R}^N} V(x)u(x)v(x)\, dx, \quad \forall\, u, v \in E,$$

and the norm

$$\|u\|_E = \left(\int_{\mathbb{R}^N} \left(|\xi|^{2\alpha}\, \widehat{u}^2 + \widehat{u}^2\right) d\xi + \int_{\mathbb{R}^N} V(x)u^2\, dx\right)^{1/2}, \quad u \in E.$$

Furthermore, we know that $\|\cdot\|_E$ is equivalent to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} \left(\left|(-\Delta)^{\alpha/2}u\right|^2 + V(x)u^2\right) dx\right)^{1/2}, \quad u \in E.$$

The corresponding inner product is

$$(u,v) = \int_{\mathbb{R}^N} \left((-\Delta)^{\alpha/2}u(x)(-\Delta)^{\alpha/2}v(x) + V(x)u(x)v(x)\right) dx, \quad \forall\, u, v \in E.$$

For $\lambda > 0$, we also need the following inner product

$$(u,v)_\lambda = \int_{\mathbb{R}^N} \left((-\Delta)^{\alpha/2}u(-\Delta)^{\alpha/2}v + \lambda V(x)uv\right) dx, \quad \forall\, u, v \in E,$$

and the corresponding norm $\|u\|_\lambda^2 = (u,u)_\lambda$. It is clear that $\|u\| \le \|u\|_\lambda$ for $\lambda \ge 1$.

Set $E_\lambda = (E, \|\cdot\|_\lambda)$, then E_λ is a Hilbert space. By (V_1)–(V_2) and the Sobolev inequality, we can demonstrate that there exist positive constants λ_0, γ_0 (independent of λ) such that

$$\|u\|_{H^\alpha(\mathbb{R}^N)} \le \gamma_0 \|u\|_\lambda \quad \text{for all } u \in E_\lambda,\ \lambda \ge \lambda_0.$$

In fact, by using (V_1)–(V_2) and the Sobolev inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\left|(-\Delta)^{\alpha/2}u\right|^2+u^2\right)dx\\
&=\int_{\mathbb{R}^N}\left|(-\Delta)^{\alpha/2}u\right|^2dx+\int_{V_b}u^2\,dx+\int_{\mathbb{R}^N\setminus V_b}u^2\,dx\\
&\le\int_{\mathbb{R}^N}\left|(-\Delta)^{\alpha/2}u\right|^2dx+(\mathrm{meas}(V_b))^{2\alpha/N}\left(\int_{\mathbb{R}^N}u^{2^*_\alpha}\,dx\right)^{(N-2\alpha)/N}+\int_{\mathbb{R}^N\setminus V_b}u^2\,dx\\
&\le\int_{\mathbb{R}^N}\left|(-\Delta)^{\alpha/2}u\right|^2dx+(\mathrm{meas}(V_b))^{2\alpha/N}\left(\int_{\mathbb{R}^N}u^{2^*_\alpha}\,dx\right)^{(N-2\alpha)/N}+\frac{1}{\lambda b}\int_{\mathbb{R}^N\setminus V_b}\lambda V(x)u^2\,dx\\
&\le\int_{\mathbb{R}^N}\left|(-\Delta)^{\alpha/2}u\right|^2dx+(\mathrm{meas}(V_b))^{2\alpha/N}\left(\int_{\mathbb{R}^N}u^{2^*_\alpha}\,dx\right)^{(N-2\alpha)/N}+\frac{1}{\lambda b}\int_{\mathbb{R}^N}\lambda V(x)u^2\,dx\\
&\le\int_{\mathbb{R}^N}\left|(-\Delta)^{\alpha/2}u\right|^2dx+(\mathrm{meas}(V_b))^{2\alpha/N}\left(\int_{\mathbb{R}^N}u^{2^*_\alpha}\,dx\right)^{(N-2\alpha)/N}+\frac{1}{\lambda b}\int_{\mathbb{R}^N}\lambda V(x)u^2\,dx\\
&\le\int_{\mathbb{R}^N}\left|(-\Delta)^{\alpha/2}u\right|^2dx+(\mathrm{meas}(V_b))^{2\alpha/N}C_{2^*_\alpha}^{-2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\alpha/2}u\right|^2dx+\frac{1}{\lambda b}\int_{\mathbb{R}^N}\lambda V(x)u^2\,dx\\
&\le\left[1+(\mathrm{meas}(V_b))^{2\alpha/N}C_{2^*_\alpha}^{-2}\right]\int_{\mathbb{R}^N}\left(\left|(-\Delta)^{\alpha/2}u\right|^2+\lambda V(x)u^2\right)dx,\\
&:=\gamma_0\int_{\mathbb{R}^N}\left(\left|(-\Delta)^{\alpha/2}u\right|^2+\lambda V(x)u^2\right)dx,\quad \lambda\ge\lambda_0:=\frac{1}{b}\left[1+(\mathrm{meas}(V_b))^{2\alpha/N}C_{2^*_\alpha}^{-2}\right]^{-1}.
\end{aligned}$$

This shows that $E_\lambda\hookrightarrow H^\alpha(\mathbb{R}^N)$ for $\lambda\ge\lambda_0$. By [16], $H^\alpha(\mathbb{R}^N)\hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $s\in[2,2^*_\alpha]$ and $H^\alpha(\mathbb{R}^N)\hookrightarrow L^s_{\mathrm{loc}}(\mathbb{R}^N)$ is compact for $s\in[2,2^*_\alpha)$, therefore the embedding $E_\lambda\hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $s\in[2,2^*_\alpha]$ and $E_\lambda\hookrightarrow L^s_{\mathrm{loc}}(\mathbb{R}^N)$ is compact for $s\in[2,2^*_\alpha)$, i.e., there are constants $\gamma_s,\gamma_0>0$ such that

$$\|u\|_s\le\gamma_s\|u\|_{H^\alpha(\mathbb{R}^N)}\le\gamma_s\gamma_0\|u\|_\lambda\quad\text{for all } u\in E_\lambda,\ 2\le s\le 2^*_\alpha. \tag{2.1}$$

Let

$$\Phi_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(\left|(-\Delta)^{\alpha/2}u\right|^2+\lambda V(x)u^2\right)dx-\Psi(u), \tag{2.2}$$

where

$$\Psi(u)=\int_{\mathbb{R}^N}F(x,u)\,dx+\frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)\,|u|^p\,dx.$$

By a standard argument and the Hölder inequality, it is easy to verify that $\Phi_\lambda\in C^1(E_\lambda,\mathbb{R})$ and

$$\left\langle\Phi_\lambda'(u),v\right\rangle=\int_{\mathbb{R}^N}\left((-\Delta)^{\alpha/2}u(-\Delta)^{\alpha/2}v+\lambda V(x)uv\right)dx-\left\langle\Psi'(u),v\right\rangle \tag{2.3}$$

for all $u,v\in E_\lambda$, where

$$\left\langle\Psi'(u),v\right\rangle=\int_{\mathbb{R}^N}f(x,u)v\,dx+\mu\int_{\mathbb{R}^N}\xi(x)\,|u|^{p-2}\,uv\,dx.$$

We say that $I \in C^1(X,\mathbb{R})$ satisfies (PS) condition if any sequence $\{u_n\}$ such that $I(u_n) \to d$, $I'(u_n) \to 0$ has a convergent subsequence. To prove our result, we need the following Mountain Pass Theorem.

Theorem 2.1. [23, Theorem 2.2] *Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$ satisfying* (PS) *condition. Suppose $I(0) = 0$ and*

(I$_1$) *there are constants $\rho, \eta > 0$ such that $I_{\partial B_\rho(0)} \geq \eta$,*

(I$_2$) *there is an element $e \in X \setminus \overline{B}_\rho(0)$ such that $I(e) \leq 0$,*

then I possesses a critical value $\beta \geq \eta$.

Lemma 2.2. *Assume that* (F$_1$), (F$_2$) *are satisfied, and $\xi \in L^{2/(2-p)}(\mathbb{R}^N,\mathbb{R}^+)$. Then there exist three positive constants μ_0, ρ and η such that $\Phi_\lambda(u)\big|_{\|u\|_\lambda=\rho} \geq \eta > 0$ for all $\mu \in (0,\mu_0)$.*

Proof. For any $\varepsilon > 0$, it follows from the conditions (F$_1$) and (F$_2$) that there exists $C_\varepsilon > 0$ such that

$$|F(x,t)| \leq \frac{\varepsilon}{2}|t|^2 + \frac{C_\varepsilon}{q}|t|^q \quad \text{for all } t \in \mathbb{R}. \tag{2.4}$$

Thus, from (2.1), (2.4) and the Sobolev inequality, we have for all $u \in E_\lambda$,

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u)\,dx &\leq \frac{\varepsilon}{2}\int_{\mathbb{R}^N} u^2\,dx + \frac{C_\varepsilon}{q}\int_{\mathbb{R}^N}|u|^q\,dx \\ &\leq \frac{\gamma_2^2\gamma_0^2\varepsilon}{2}\|u\|_\lambda^2 + \frac{C_\varepsilon\gamma_q^q\gamma_0^q}{q}\|u\|_\lambda^q,\end{aligned}$$

which implies

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,u)\,dx - \frac{\mu}{p}\int_{\mathbb{R}^N}\xi(x)|u|^p\,dx \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \frac{\gamma_2^2\gamma_0^2\varepsilon}{2}\|u\|_\lambda^2 - \frac{C_\varepsilon\gamma_q^q\gamma_0^q}{q}\|u\|_\lambda^q - \frac{\mu\gamma_2^p\gamma_0^p}{p}\|\xi\|_{2/(2-p)}\|u\|_\lambda^p \\ &= \|u\|_\lambda^p\left[\frac{1}{2}(1-\gamma_2^2\gamma_0^2\varepsilon)\|u\|_\lambda^{2-p} - \frac{C_\varepsilon\gamma_q^q\gamma_0^q}{q}\|u\|_\lambda^{q-p} - \frac{\mu\gamma_2^p\gamma_0^p}{p}\|\xi\|_{2/(2-p)}\right].\end{aligned} \tag{2.5}$$

Take $\varepsilon = 1/(2\gamma_2^2\gamma_0^2)$ and define

$$g(t) = \frac{1}{4}t^{2-p} - \frac{C_\varepsilon\gamma_q^q\gamma_0^q}{q}t^{q-p} \quad \text{for } t \geq 0.$$

It is easy to prove that there exists $\rho > 0$ such that

$$\max_{t\geq 0} g(t) = g(\rho) = \frac{q-2}{4(q-p)}\left[\frac{(2-p)q}{4C_\varepsilon\gamma_q^q\gamma_0^q(q-p)}\right]^{(2-p)/(q-2)}.$$

Then it follows from (2.5) that there exist two positive constants μ_0 and η such that $\Phi_\lambda(u)\big|_{\|u\|_\lambda=\rho} \geq \eta$ for all $\mu \in (0,\mu_0)$. $\square$

Lemma 2.3. *Assume that* (F_1), (F_2) *and* (F_3) *are satisfied, and* $\xi \in L^{2/(2-p)}(\mathbb{R}^N, \mathbb{R}^+)$. *Let* ρ *be as in Lemma 2.2. Then there exists* $e \in E_\lambda$ *with* $\|e\|_\lambda > \rho$ *such that* $\Phi_\lambda(e) < 0$ *for all* $\mu \geq 0$.

Proof. By (2.4) and (F_3), there exists $c > 0$ such that

$$F(x,u) \geq c\left(|u|^\theta - |u|^2\right), \quad \forall\,(x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus, for $t > 0$, $u \in E_\lambda$, we have

$$\begin{aligned}\Phi_\lambda(tu) &= \frac{t^2}{2}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,tu)\,dx - \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)\,|tu|^p\,dx \\ &\leq \frac{t^2}{2}\|u\|_\lambda^2 - ct^\theta \int_{\mathbb{R}^N} |u|^\theta\,dx + ct^2\int_{\mathbb{R}^N} |u|^2\,dx - \frac{\mu}{p}t^p \int_{\mathbb{R}^N} \xi(x)\,|u|^p\,dx,\end{aligned}$$

which implies that $\Phi_\lambda(tu) \to -\infty$ as $t \to \infty$. Therefore, there exist $t_0 > 0$ and $e := t_0 u$ with $\|e\|_\lambda > \rho$ such that $\Phi_\lambda(e) < 0$. This completes the proof. □

To find critical points of Φ_λ, we shall show that Φ_λ satisfies the (PS) condition, i.e., any (PS) sequence $\{u_n\}$ has a convergent subsequence in E_λ. Since there is no compactness of the Sobolev embedding, the situation is more difficult. To overcome this difficulty, we need the following convergence results.

Lemma 2.4. *Suppose that* $u_n \rightharpoonup u_0$ *in* E_λ. *Then, passing to a subsequence*

$$\Phi_\lambda(u_n) = \Phi_\lambda(u_n - u_0) + \Phi_\lambda(u_0) + o(1) \tag{2.6}$$

and

$$\Phi'_\lambda(u_n) = \Phi'_\lambda(u_n - u_0) + \Phi'_\lambda(u_0) + o(1). \tag{2.7}$$

In particular, if $\{u_n\}$ *is a* (PS) *sequence such that* $\Phi_\lambda(u_n) \to d$ *for some* $d \in \mathbb{R}$, *then*

$$\Phi_\lambda(u_n - u_0) \to d - \Phi_\lambda(u_0) \quad \text{and} \quad \Phi'_\lambda(u_n - u_0) \to 0 \tag{2.8}$$

after passing to a subsequence.

Proof. Since $u_n \rightharpoonup u_0$ in E_λ, we have

$$(u_n, u_0)_\lambda \to (u_0, u_0)_\lambda,$$

which yields

$$\begin{aligned}\|u_n\|_\lambda^2 &= (u_n - u_0, u_n - u_0)_\lambda + (u_0, u_n)_\lambda + (u_n - u_0, u_0)_\lambda \\ &= \|u_n - u_0\|_\lambda^2 + \|u_0\|_\lambda^2 + o(1).\end{aligned}$$

It is clear that

$$(u_n, \phi)_\lambda = (u_n - u_0, \phi)_\lambda + (u_0, \phi)_\lambda \quad \text{for all } \phi \in E_\lambda.$$

Hence, to obtain (2.6) and (2.7), it suffices to check that

$$\int_{\mathbb{R}^N} [F(x, u_n) - F(x, u_n - u_0) - F(x, u_0)]\, dx = o(1), \tag{2.9}$$

$$\int_{\mathbb{R}^N} \xi(x) [|u_n|^p - |u_n - u_0|^p - |u_0|^p]\, dx = o(1), \tag{2.10}$$

$$\int_{\mathbb{R}^N} (f(x, u_n) - f(x, u_n - u_0) - f(x, u_0))\, \phi\, dx = o(1) \quad \text{for all } \phi \in E_\lambda \tag{2.11}$$

and

(2.12)

$$\int_{\mathbb{R}^N} \xi(x) \left(|u_n|^{p-2} u_n - |u_n - u_0|^{p-2} (u_n - u_0) - |u_0|^{p-2} u_0 \right) \phi\, dx = o(1) \quad \text{for all } \phi \in E_\lambda.$$

Here, we only prove (2.9) and (2.10), the verifications of (2.11) and (2.12) are similar. Take $\omega_n := u_n - u_0$, we have $\omega_n \rightharpoonup 0$ in E_λ and $\omega_n(x) \to 0$ a.e. $x \in \mathbb{R}^N$. It follows from (F_1) and (F_2) that

$$|f(x, u)| \le \varepsilon |u| + C_\varepsilon |u|^{q-1}, \quad \forall\, (x, u) \in \mathbb{R}^N \times \mathbb{R} \tag{2.13}$$

and

$$|F(x, u)| \le \int_0^1 |f(x, tu)|\, |u|\, dt \le \varepsilon |u|^2 + C_\varepsilon |u|^q, \quad \forall\, (x, u) \in \mathbb{R}^N \times \mathbb{R}, \tag{2.14}$$

then

$$\begin{aligned} |F(x, \omega_n + u_0) - F(x, \omega_n)| &\le \int_0^1 |f(x, \omega_n + \zeta u_0)|\, |u_0|\, d\zeta \\ &\le \int_0^1 \left(\varepsilon |\omega_n + \zeta u_0|\, |u_0| + C_\varepsilon |\omega_n + \zeta u_0|^{q-1} |u_0| \right) d\zeta \\ &\le c_1 \left(\varepsilon |\omega_n|\, |u_0| + \varepsilon |u_0|^2 + C_\varepsilon |\omega_n|^{q-1} |u_0| + C_\varepsilon |u_0|^q \right). \end{aligned}$$

By Young's inequality, we have

$$|F(x, \omega_n + u_0) - F(x, \omega_n)| \le c_2 \left(\varepsilon |\omega_n|^2 + \varepsilon |u_0|^2 + \varepsilon |\omega_n|^q + C_\varepsilon |u_0|^q \right),$$

so that, using (2.14), we get

$$\begin{aligned} &|F(x, \omega_n + u_0) - F(x, \omega_n) - F(x, u_0)| \\ &\le c_3 \left(\varepsilon |\omega_n|^2 + \varepsilon |u_0|^2 + \varepsilon |\omega_n|^q + C_\varepsilon |u_0|^q \right), \quad n \in \mathbb{N}. \end{aligned}$$

Let

$$H_n(x) := \max \left\{ |F(x, \omega_n + u_0) - F(x, \omega_n) - F(x, u_0)| - c_3 \varepsilon \left(|\omega_n|^2 + |\omega_n|^q \right), 0 \right\}.$$

It follows that

$$0 \le H_n(x) \le c_3 \left(\varepsilon |u_0|^2 + C_\varepsilon |u_0|^q \right) \in L^1(\mathbb{R}^N).$$

Thus, using Lebesgue dominated convergence theorem,

$$\int_{\mathbb{R}^N} H_n(x)\, dx \to 0 \quad \text{as } n \to \infty. \tag{2.15}$$

From the definition of $H_n(x)$, we have

$$|F(x, \omega_n + u_0) - F(x, \omega_n) - F(x, u_0)| \le c_3 \varepsilon \left(|\omega_n|^2 + |\omega_n|^q \right) + H_n(x), \quad \forall\, n \in \mathbb{N},$$

which, together with (2.15) and (2.1), we get

$$\int_{\mathbb{R}^N} |F(x, \omega_n + u_0) - F(x, \omega_n) - F(x, u_0)|\, dx \le c_3 \varepsilon \left(\|\omega_n\|_2^2 + \|\omega_n\|_q^q \right) + \varepsilon \le c_4 \varepsilon,$$

for n sufficiently large, hence

$$\int_{\mathbb{R}^N} [F(x, u_n) - F(x, u_n - u_0) - F(x, u_0)]\, dx = o(1)$$

that is, (2.9) holds.

Observe that $\xi \in L^{2/(2-p)}(\mathbb{R}^N, \mathbb{R}^+)$, thus, for any $\epsilon > 0$ we can choose $R_\epsilon > 0$ such that

$$\left(\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |\xi(x)|^{2/(2-p)}\, dx \right)^{(2-p)/2} < \epsilon. \tag{2.16}$$

By Sobolev's embedding theorem, $u_n \rightharpoonup u_0$ in E_λ implies

$$u_n \to u_0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N),$$

and hence,

$$\lim_{n \to \infty} \int_{B_{R_\epsilon}} |u_n - u_0|^2\, dx = 0. \tag{2.17}$$

By (2.17), there exists $N_0 \in \mathbb{N}$ such that

$$\int_{B_{R_\epsilon}} |u_n - u_0|^2\, dx < \epsilon^2 \quad \text{for } n \ge N_0. \tag{2.18}$$

Hence, by (2.1), (2.18) and the Hölder inequality, for any $n \ge N_0$, we have

$$\begin{aligned} &\frac{\mu}{p} \int_{B_{R_\epsilon}} \xi(x) |u_n - u_0|^p\, dx \\ &\le \frac{\mu}{p} \left(\int_{B_{R_\epsilon}} |\xi(x)|^{2/(2-p)}\, dx \right)^{(2-p)/2} \left(\int_{B_{R_\epsilon}} |u_n - u_0|^2\, dx \right)^{p/2} \\ &\le \frac{\mu}{p} \epsilon^p \|\xi(x)\|_{2/(2-p)}\,. \end{aligned} \tag{2.19}$$

On the other hand, by (2.1) and (2.16), we have

$$
\begin{aligned}
&\frac{\mu}{p}\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} \xi(x)\,|u_n-u_0|^p\,dx \\
&\le \frac{\mu}{p}\left(\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |\xi(x)|^{2/(2-p)}\,dx\right)^{(2-p)/2}\left(\int_{\mathbb{R}^N\setminus B_{R_\epsilon}} |u_n-u_0|^2\,dx\right)^{p/2} \\
&\le \frac{\mu}{p}\epsilon c_5.
\end{aligned}
\tag{2.20}
$$

Since ϵ is arbitrary, combining (2.19) with (2.20), we have

$$
\frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)\,|u_n-u_0|^p\,dx = o(1) \tag{2.21}
$$

and

$$
\frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)\,(|u_n|^p-|u_0|^p)\,dx \le \frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)\,|u_n-u_0|^p\,dx,
$$

thus,

$$
\frac{\mu}{p}\int_{\mathbb{R}^N} \xi(x)\,(|u_n|^p-|u_n-u_0|^p-|u_0|^p) = o(1),
$$

that is, (2.10) holds.

Now, we consider the case $\{u_n\}$ is a (PS) sequence such that $\Phi_\lambda(u_n)\to d$ and $\Phi'_\lambda(u_n)\to 0$. It follows from (2.6) and (2.7) that

$$
\Phi_\lambda(u_n-u_0) = d-\Phi_\lambda(u_0)+o(1) \quad\text{and}\quad \Phi'_\lambda(u_n-u_0) = -\Phi'_\lambda(u_0)+o(1), \tag{2.22}
$$

we show that $\Phi'_\lambda(u_0)=0$. For every $\psi\in C_0^\infty(\mathbb{R}^N)$, it follows from (2.13) and the fact that $u_n\to u_0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ that

$$
\int_{\mathbb{R}^N} (f(x,u_n)-f(x,u_0))\,\psi\,dx = \int_{\operatorname{supp}\psi} (f(x,u_n)-f(x,u_0))\,\psi\,dx = o(1)
$$

and

$$
\begin{aligned}
\mu\int_{\mathbb{R}^N} \xi(x)\left(|u_n|^{p-2}u_n-|u_0|^{p-2}u_0\right)\psi\,dx &= \mu\int_{\operatorname{supp}\psi} \xi(x)\left(|u_n|^{p-2}u_n-|u_0|^{p-2}u_0\right)\psi\,dx \\
&= o(1)
\end{aligned}
$$

which implies that

$$
\left\langle \Phi'_\lambda(u_0),\psi\right\rangle = \lim_{n\to\infty}\left\langle \Phi'_\lambda(u_n),\psi\right\rangle = 0.
$$

Hence, $\Phi'_\lambda(u_0)=0$, which together with the second equation of (2.22) shows that $\Phi'_\lambda(u_n-u_0)\to 0$ as $n\to\infty$. Consequently, (2.8) holds and the proof is complete. □

Lemma 2.5. *Let* (V$_1$)–(V$_3$) *and* (F$_1$)–(F$_3$) *be satisfied, there exists* $\Lambda_0>0$, *any* (PS) *sequence of* Φ_λ *has a convergent subsequence for all* $\lambda\ge\Lambda_0$.

Proof. We adapt an argument in [17]. Let $\{u_n\}$ be a sequence such that

$$\Phi_\lambda(u_n) \to d \quad \text{and} \quad \Phi'_\lambda(u_n) \to 0 \quad \text{for some } d \in \mathbb{R},$$

thus

$$\begin{aligned} 1 + d + \|u_n\|_\lambda &\geq \Phi_\lambda(u_n) - \frac{1}{\theta} \left\langle \Phi'_\lambda(u_n), u_n \right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \left[\frac{1}{\theta} u_n f(x, u_n) - F(x, u_n)\right] dx \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} - \frac{1}{p}\right) \mu \xi(x) \left|u_n\right|^p dx, \end{aligned}$$

hence

$$\begin{aligned} &1 + d + \|u_n\|_\lambda + \left(\frac{1}{p} - \frac{1}{\theta}\right) \mu \int_{\mathbb{R}^N} \xi(x) \left|u_n\right|^p dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \left[\frac{1}{\theta} u_n f(x, u_n) - F(x, u_n)\right] dx. \end{aligned}$$

Since

$$\begin{aligned} &\left(\frac{1}{p} - \frac{1}{\theta}\right) \mu \int_{\mathbb{R}^N} \xi(x) \left|u_n\right|^p dx \\ &\leq \left(\frac{1}{p} - \frac{1}{\theta}\right) \mu \left(\int_{\mathbb{R}^N} \left|\xi(x)\right|^{2/(2-p)} dx\right)^{(2-p)/2} \left(\int_{\mathbb{R}^N} \left|u_n\right|^2 dx\right)^{p/2} \\ &= \left(\frac{1}{p} - \frac{1}{\theta}\right) \mu \left\|\xi\right\|_{2/(2-p)} \|u_n\|_2^p \\ &\leq \left(\frac{1}{p} - \frac{1}{\theta}\right) \mu \gamma_2^p \gamma_0^p \left\|\xi\right\|_{2/(2-p)} \|u_n\|_\lambda^p . \end{aligned}$$

Hence,

$$\begin{aligned} &1 + d + \|u_n\|_\lambda + \left(\frac{1}{p} - \frac{1}{\theta}\right) \mu \gamma_2^p \gamma_0^p \left\|\xi\right\|_{2/(2-p)} \|u_n\|_\lambda^p \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \left[\frac{1}{\theta} u_n f(x, u_n) - F(x, u_n)\right] dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_n\|_\lambda^2 . \end{aligned}$$

This proves that $\{u_n\}$ is bounded in E_λ. Then, passing to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in E_λ, then $u_n \to u_0$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq q < 2^*_\alpha$. Taking $\omega_n := u_n - u_0$, we have

$$\begin{aligned} \|\omega_n\|_2^2 &\leq \frac{1}{\lambda b} \int_{\{x \in \mathbb{R}^N : V(x) > b\}} \lambda V(x) \omega_n^2 \, dx + \int_{V_b} \omega_n^2 \, dx \\ &\leq \frac{1}{\lambda b} \|\omega_n\|_\lambda^2 + o(1), \end{aligned} \tag{2.23}$$

since $\omega_n \rightharpoonup 0$ in E_λ and $V(x) < b$ on a set of finite measure. Combining this with (2.1) and the Hölder inequality, we obtain for $2 < q < \sigma < 2_\alpha^*$. Given $\nu \in (\sigma, 2_\alpha^*)$, we have

$$\begin{aligned} \|\omega_n\|_\sigma^\sigma &\le \|\omega_n\|_2^{2(\nu-\sigma)/(\nu-2)} \|\omega_n\|_\nu^{\nu(\sigma-2)/(\nu-2)} \\ &\le \left(\frac{1}{\lambda b}\right)^{(\nu-\sigma)/(\nu-2)} \|\omega_n\|_\lambda^{2(\nu-\sigma)/(\nu-2)} (\gamma_\nu\gamma_0 \|\omega_n\|_\lambda)^{\nu(\sigma-2)/(\nu-2)} + o(1) \\ &\le (\gamma_\nu\gamma_0)^{\nu(\sigma-2)/(\nu-2)} \left(\frac{1}{\lambda b}\right)^{(\nu-\sigma)/(\nu-2)} \|\omega_n\|_\lambda^\sigma + o(1). \end{aligned} \tag{2.24}$$

For convenience, let $\mathcal{F}(x,u) = \frac{1}{2}f(x,u)u - F(x,u)$. It follows from Lemma 2.4 and (2.21) that

$$\begin{aligned} \int_{\mathbb{R}^N} \mathcal{F}(x,\omega_n)\,dx &= \Phi_\lambda(\omega_n) - \frac{1}{2}\langle \Phi_\lambda'(\omega_n), \omega_n\rangle - \left(\frac{1}{2} - \frac{1}{p}\right)\mu \int_{\mathbb{R}^N} \xi(x)\,|\omega_n|^p\,dx \\ &\to d - \Phi_\lambda(u_0). \end{aligned} \tag{2.25}$$

Therefore, there exists $M > 0$ such that

$$\left|\int_{\mathbb{R}^N} \mathcal{F}(x,\omega_n)\,dx\right| \le M. \tag{2.26}$$

Now we note that $\frac{q}{q-2} > \max\left\{1, \frac{N}{2\alpha}\right\}$ because $q \in (2, 2_\alpha^*)$. Fix $\tau \in \left(\max\left\{1, \frac{N}{2\alpha}\right\}, \frac{q}{q-2}\right)$, from (2.13), we know if $|u| \ge 1$, then $|f(x,u)| \le c_6\,|u|^{q-1}$. Choose R_1 so large that $\frac{1}{\theta} \le \frac{1}{2} - \frac{c_6^{\tau-1}}{|u|^{q-(q-2)\tau}}$, whenever $|u| \ge R_1$. Then, for $|u|$ large enough, we have

$$0 \le F(x,u) \le \frac{1}{\theta}uf(x,u) \le \left[\frac{1}{2} - \frac{c_6^{\tau-1}}{|u|^{q-(q-2)\tau}}\right]uf(x,u) \le \left[\frac{1}{2} - \frac{|f(x,u)|^{\tau-1}}{|u|^{\tau+1}}\right]uf(x,u),$$

which implies that

$$\frac{|f(x,u)|^\tau}{|u|^\tau} \le \frac{1}{2}uf(x,u) - F(x,u) = \mathcal{F}(x,u). \tag{2.27}$$

Combining this with (2.24), (2.26) with $\sigma = \frac{2\tau}{\tau-1} \in (2, 2_\alpha^*)$ and the Hölder inequality, we obtain for large n

$$\begin{aligned} \int_{|\omega_n|\ge R_1} f(x,\omega_n)\omega_n\,dx &\le \left(\int_{|\omega_n|\ge R_1} \left|\frac{f(x,\omega_n)}{\omega_n}\right|^\tau dx\right)^{1/\tau} \left(\int_{|\omega_n|\ge R_1} |\omega_n|^\sigma\,dx\right)^{2/\sigma} \\ &\le \left(\int_{|\omega_n|\ge R_1} \mathcal{F}(x,\omega_n)\,dx\right)^{1/\tau} \|\omega_n\|_\sigma^2 \\ &\le M^{1/\tau}(\gamma_\nu\gamma_0)^{2\nu(\sigma-2)/[(\nu-2)\sigma]} \left(\frac{1}{\lambda b}\right)^{2(\nu-\sigma)/[(\nu-2)\sigma]} \|\omega_n\|_\lambda^2 + o(1) \\ &= c_7\left(\frac{1}{\lambda b}\right)^{\theta_1} \|\omega_n\|_\lambda^2 + o(1). \end{aligned} \tag{2.28}$$

where $c_7 = M^{1/\tau}(\gamma_\nu\gamma_0)^{2\nu(\sigma-2)/[(\nu-2)\sigma]} > 0$, $\theta_1 = \frac{2(\nu-\sigma)}{\sigma(\nu-2)} > 0$. In addition, using (2.13) and (2.24), we have

$$\begin{aligned}\int_{|\omega_n|\le R_1} f(x,\omega_n)\omega_n\,dx &\le \int_{|\omega_n|\le R_1}\left(\epsilon + C_\epsilon R_1^{q-2}\right)\omega_n^2\,dx \\ &\le \frac{C_\epsilon R_1^{q-2}}{\lambda b}\,\|\omega_n\|_\lambda^2 + o(1) \\ &= \frac{c_8}{\lambda b}\,\|\omega_n\|_\lambda^2 + o(1),\end{aligned} \tag{2.29}$$

where $c_8 = C_\epsilon R_1^{q-2}$. Consequently, combining (2.21), (2.28) with (2.29), we get

$$\begin{aligned} o(1) &= \left\langle \Phi_\lambda'(\omega_n), \omega_n\right\rangle \\ &= \|\omega_n\|_\lambda^2 - \int_{\mathbb{R}^N} f(x,\omega_n)\omega_n\,dx - \mu\int_{\mathbb{R}^N}\xi(x)\,|\omega_n|^p\,dx \\ &\ge \left[1 - \frac{c_8}{\lambda b} - c_7\left(\frac{1}{\lambda b}\right)^{\theta_1}\right]\|\omega_n\|_\lambda^2 + o(1).\end{aligned}$$

Choosing $\Lambda_0 > 0$ large enough such that the term in the brackets above is positive when $\lambda \ge \Lambda_0$, we get $\omega_n \to 0$ in E_λ, thus $u_n \to u_0$ in E_λ. This completes the proof. □

Define

$$d_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{0\le t\le 1}\Phi_\lambda(\gamma(t))$$

where

$$\Gamma_\lambda = \{\gamma \in C([0,1], E_\lambda) : \gamma(0) = 0, \gamma(1) = e\}.$$

Proof of Theorem 1.1. By Theorem 2.1, Lemmas 2.2 and 2.3, we obtain that, for each $\lambda \ge \Lambda_0$, $0 < \mu < \mu_0$, there exists (PS) sequence $\{u_n\} \subset E_\lambda$ for Φ_λ on E_λ. Then, by Lemma 2.5, we can conclude that there exist a subsequence $\{u_n\} \subset E_\lambda$ and $u_{\lambda,1} \in E_\lambda$ such that $u_n \to u_{\lambda,1}$ in E_λ. Moreover, $\Phi_\lambda(u_{\lambda,1}) = d_\lambda \ge \eta > 0$.

The second solution of Problem (1.1) will be constructed through the local minimization.

By virtue of (2.5), let $\rho > 0$ define as in Lemma 2.2, then it is easy to see that

$$\inf_{u\in\overline{B}_\rho}\Phi_\lambda(u) > -\infty \quad\text{and}\quad \inf_{u\in\partial B_\rho}\Phi_\lambda(u) \ge \eta > 0,$$

where B_ρ is the open ball in E_λ with radius ρ and ∂B_ρ denotes its boundary. Since $\xi \in L^{2/(2-p)}(\mathbb{R}^N, \mathbb{R}^+)$ and $\xi(x) \ne 0$, we can choose a function $\phi \in E_\lambda$ such that

$$\int_{\mathbb{R}^N}\xi(x)\,|\phi|^p\,dx > 0.$$

Thus, by (F_3) we have

$$\begin{aligned}\Phi_\lambda(l\phi) &= \frac{l^2}{2}\|\phi\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,l\phi)\,dx - \frac{\mu l^p}{p}\int_{\mathbb{R}^N}\xi(x)\,|\phi|^p\,dx \\ &\le \frac{l^2}{2}\|\phi\|_\lambda^2 - \frac{\mu l^p}{p}\int_{\mathbb{R}^N}\xi(x)\,|\phi|^p\,dx \\ &< 0\end{aligned} \tag{2.30}$$

for $l > 0$ small enough. Hence,

$$-\infty < \inf_{u\in\overline{B}_\rho}\Phi_\lambda(u) < 0.$$

For $n \in \mathbb{N}$ sufficiently large, set $\frac{1}{n} \in \left(0, \inf_{u\in\partial B_\rho}\Phi_\lambda(u) - \inf_{u\in\overline{B}_\rho}\Phi_\lambda(u)\right)$, there is $w_n \in \overline{B}_\rho$ such that

$$\Phi_\lambda(w_n) \le \inf_{u\in\overline{B}_\rho}\Phi_\lambda(u) + \frac{1}{n}. \tag{2.31}$$

By the Ekeland's variational principle, there exists $v_n \in \overline{B}_\rho$ such that

$$\Phi_\lambda(v_n) \le \Phi_\lambda(w_n) \quad \text{and} \quad \|w_n - v_n\| \le 1,$$

and

$$\Phi_\lambda(v_n) \le \Phi_\lambda(u) + \frac{1}{n}\|u - v_n\| \quad \text{for all } u \in \overline{B}_\rho, \tag{2.32}$$

while

$$\Phi_\lambda(v_n) \le \inf_{u\in\overline{B}_\rho}\Phi_\lambda(u) + \frac{1}{n} < \inf_{u\in\partial B_\rho}\Phi_\lambda(u).$$

So $v_n \in B_\rho$. Define $\Psi_n\colon E_\lambda \mapsto \mathbb{R}$ by

$$\Psi_n(u) = \Phi_\lambda(u) + \frac{1}{n}\|u - v_n\|.$$

By (2.32), we have $v_n \in B_\rho$ minimizes Ψ_n on $\overline{B}_\rho$. Therefore, for all $\phi \in E_\lambda$ with $\|\phi\| = 1$, take $t > 0$ such that $v_n + t\phi \in \overline{B}_\rho$, then

$$\frac{\Psi_n(v_n + t\phi) - \Psi_n(v_n)}{t} \ge 0. \tag{2.33}$$

(2.33) implies

$$\frac{\Phi_\lambda(v_n + t\phi) - \Phi_\lambda(v_n)}{t} + \frac{1}{n} \ge 0,$$

which implies

$$\left\langle \Phi_\lambda'(v_n), \phi \right\rangle \ge -\frac{1}{n}.$$

Hence,

$$\left\|\Phi_\lambda'(v_n)\right\| \leq \frac{1}{n}. \tag{2.34}$$

Passing to the limit in (2.32) and (2.34), we conclude that $\Phi_\lambda(v_n) \to \inf_{u\in\overline{B}_\rho} \Phi_\lambda(u)$ and $\Phi_\lambda'(v_n) \to 0$ as $n \to \infty$. Hence, Lemma 2.5 implies that there exists a nontrivial solution $u_{\lambda,2}$ of Problem (1.1) satisfying

$$\Phi_\lambda(u_{\lambda,2}) < 0 \quad \text{and} \quad \|u_{\lambda,2}\|_\lambda \leq \rho.$$

Therefore, we can conclude that

$$\Phi_\lambda(u_{\lambda,2}) < 0 < \eta \leq d_\lambda = \Phi_\lambda(u_{\lambda,1})$$

for all $\lambda \geq \Lambda_0$ and $0 < \mu < \mu_0$. This completes the proof of Theorem 1.1. □

3. Concentration of solutions

In the following, we investigate the concentration of solutions and give the proof of Theorem 1.2. First, we introduce some fractional spaces, for more details see [27] and [28].

Let $\alpha \in (0,1)$ fixed, $n > 2\alpha$, $\Omega \subset \mathbb{R}^N$ be an open bounded set with smooth boundary. In the sequel we denote $\mathcal{Q} = \mathbb{R}^{2N} \setminus \mathcal{O}$, where

$$\mathcal{O} = (\Omega^c \times \Omega^c) \subset \mathbb{R}^{2N} \quad \text{and} \quad \Omega^c = \mathbb{R}^N \setminus \Omega.$$

The fractional space X is defined by

$$X = \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{(2\alpha+N)/2}} \in L^2(\mathcal{Q}) \right\},$$

endowed with the norm defined as

$$\|u\|_X = \left(\int_\Omega |u|^2 \, dx + \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x-y|^{2\alpha+N}} \, dxdy \right)^{1/2}. \tag{3.1}$$

Let

$$X_0 = \left\{ u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.$$

Then, by [27], there exists a constant $\mathfrak{R} = \mathfrak{R}(N, \alpha, \Omega) > 1$, such that for any $u \in X_0$

$$\int_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x-y|^{2\alpha+N}} \, dxdy \leq \|u\|_X^2 \leq \mathfrak{R} \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x-y|^{2\alpha+N}} \, dxdy,$$

thus,

$$\|u\|_{X_0} = \left(\int_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x-y|^{2\alpha+N}} \, dxdy \right)^{1/2}$$

is a norm on X_0 equivalent to the usual one defined in (3.1). Furthermore, X_0 is a Hilbert space.

Let Ψ be the restriction of Φ_λ on X_0, then,

$$\Psi(u) = \Phi_\lambda\big|_{X_0}(u) = \frac{1}{2}\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{2\alpha+N}}\,dxdy - \int_\Omega F(x,u)\,dx - \frac{\mu}{p}\int_\Omega \xi(x)\,|u|^p\,dx.$$

Define

$$d_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{0\le t\le 1}\Phi_\lambda(\gamma(t)) \quad \text{and} \quad d_0 = \inf_{\gamma\in\widetilde{\Gamma}}\max_{0\le t\le 1}\Psi(\gamma(t)),$$

where

$$\Gamma_\lambda = \{\gamma\in C([0,1],E_\lambda) : \gamma(0)=0, \Phi_\lambda(\gamma(1))<0\}$$

and

$$\widetilde{\Gamma} = \{\gamma\in C([0,1],X_0) : \gamma(0)=0, \Psi(\gamma(1))<0\}.$$

It is obvious that d_0 is independent of λ. From the above arguments, we can conclude that Ψ has a mountain pass type solution $\widetilde{u}$ such that $\Psi(\widetilde{u}) = d_0$. Since $X_0\subset E_\lambda$ for all $\lambda>0$, it is easy to see that $0<\eta\le d_\lambda<d_0$ for all $\lambda\ge\Lambda_0$ and $0<\mu<\mu_0$.

Now, we claim that $\Psi(u)$ is bounded from above. For all $u\in X_0$, it follows from (F_3) and Fatou's lemma that

$$\begin{aligned}
\lim_{t\to\infty}\frac{\Psi(tu)}{t^2} &= \frac{1}{2}\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{2\alpha+N}}\,dxdy - \lim_{t\to\infty}\frac{1}{t^2}\int_\Omega F(x,tu)\,dx - \lim_{t\to\infty}\frac{\mu t^{p-2}}{p}\int_\Omega \xi(x)\,|u|^p\,dx\\
&\le \frac{1}{2}\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{2\alpha+N}}\,dxdy - \liminf_{t\to\infty}\frac{1}{t^2}\int_\Omega F(x,tu)\,dx - \lim_{t\to\infty}\frac{\mu t^{p-2}}{p}\int_\Omega \xi(x)\,|u|^p\,dx\\
&\le \frac{1}{2}\int_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{2\alpha+N}}\,dxdy - \int_\Omega \liminf_{t\to\infty}\frac{F(x,tu)}{t^2}\,dx - \lim_{t\to\infty}\frac{\mu t^{p-2}}{p}\int_\Omega \xi(x)\,|u|^p\,dx\\
&= -\infty,
\end{aligned}$$

therefore, $\Psi(u)$ is bounded from above. Take $C_0>d_0$, thus

$$0<\eta\le d_\lambda\le d_0<C_0$$

for all $\lambda\ge\Lambda_0$ and $0<\mu<\mu_0$.

Proof of Theorem 1.2. We follow the argument in [3]. For any sequence $\lambda_n\to\infty$, let $u_{n,i} := u_{\lambda_n,i}$ be the critical points of Φ_{λ_n} obtained in Theorem 1.1 for $i=1,2$. Since

$$\Phi_{\lambda_n}(u_{n,2}) < 0 < \eta \le d_{\lambda_n} = \Phi_{\lambda_n}(u_{n,1}) \tag{3.2}$$

and

$$\begin{aligned}
C_0 + \frac{C_0}{\theta} \|u_{n,i}\|_{\lambda_n} &\geq \Phi_{\lambda_n}(u_{n,i}) - \frac{1}{\theta} \left\langle \Phi'_{\lambda_n}(u_{n,i}), u_{n,i} \right\rangle \\
&= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_{n,i}\|^2_{\lambda_n} + \int_{\mathbb{R}^N} \left(\frac{1}{\theta} f(x, u_{n,i}) u_{n,i} - F(x, u_{n,i})\right) dx \\
&\quad - \left(\frac{\mu}{p} - \frac{\mu}{\theta}\right) \int_{\mathbb{R}^N} \xi(x) |u_{n,i}|^p \, dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u_{n,i}\|^2_{\lambda_n} - \left(\frac{\mu}{p} - \frac{\mu}{\theta}\right) \int_{\mathbb{R}^N} \xi(x) |u_{n,i}|^p \, dx,
\end{aligned}$$

which implies

$$\|u_{n,i}\|_{\lambda_0} \leq \|u_{n,i}\|_{\lambda_n} \leq c_0 \quad \text{for large } n, \tag{3.3}$$

where the constant c_0 is independent of λ_n. Therefore, for large n we may assume that $u_{n,i} \rightharpoonup u_{0,i}$ in E_{λ_0} and $u_{n,i} \to u_{0,i}$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \leq q < 2^*_\alpha$. From Fatou's lemma, we have

$$\int_{\mathbb{R}^N} V(x) |u_{0,i}|^2 \, dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N} V(x) |u_{n,i}|^2 \, dx \leq \liminf_{n\to\infty} \frac{\|u_{n,i}\|^2_{\lambda_n}}{\lambda_n} = 0,$$

which implies that $u_{0,i} = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$. Now for any $\varphi \in C_0^\infty(\Omega)$, since $\left\langle \Phi'_{\lambda_n}(u_{n,i}), \varphi \right\rangle = 0$, it is easy to verify that

$$\int_\Omega \left((-\Delta)^{\alpha/2} u_{0,i} (-\Delta)^{\alpha/2} \varphi\right) dx - \int_\Omega f(x, u_{0,i}) \varphi \, dx - \mu \int_\Omega \xi(x) |u_{0,i}|^{p-2} u_{0,i} \varphi \, dx = 0,$$

which implies that $u_{0,i}$ is a weak solution of Problem (1.2) by the density of $C_0^\infty(\Omega)$ in X_0.

Next, we show that $u_{n,i} \to u_{0,i}$ in $L^q(\mathbb{R}^N)$ for $2 \leq q < 2^*_\alpha$. Otherwise, by Lions vanishing lemma [22, 39], there exist $\delta > 0$, $R_0 > 0$ and $x_n \in \mathbb{R}^N$ such that

$$\int_{B_{R_0}(x_n)} |u_{n,i} - u_{0,i}|^2 \, dx \geq \delta.$$

Since $u_{n,i} \to u_{0,i}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, $|x_n| \to \infty$. Hence $\mathrm{meas}\,(B_{R_0}(x_n) \cap V_b) \to 0$. By the Hölder inequality, we have

$$\begin{aligned}
&\int_{B_{R_0}(x_n) \cap V_b} |u_{n,i} - u_{0,i}|^2 \, dx \\
&\leq (\mathrm{meas}(B_{R_0}(x_n) \cap V_b))^{(2^*_\alpha - 2)/2^*_\alpha} \left(\int_{\mathbb{R}^N} |u_{n,i} - u_{0,i}|^{2^*_\alpha}\right)^{2/2^*_\alpha} \to 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|u_{n,i}\|^2_{\lambda_n} &\geq \lambda_n b \int_{B_{R_0}(x_n) \cap \{x \in \mathbb{R}^N : V(x) \geq b\}} |u_{n,i}|^2 \, dx \\
&= \lambda_n b \int_{B_{R_0}(x_n) \cap \{x \in \mathbb{R}^N : V(x) \geq b\}} |u_{n,i} - u_{0,i}|^2 \, dx
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_n b \left(\int_{B_{R_0}(x_n)} |u_{n,i} - u_{0,i}|^2 \, dx - \int_{B_{R_0}(x_n) \cap V_b} |u_{n,i} - u_{0,i}|^2 \, dx + o(1) \right) \\
&\to \infty,
\end{aligned}
$$

which contradicts (3.3). Next, we show that $u_{n,i} \to u_{0,i}$ in $H^\alpha(\mathbb{R}^N)$. Recall that $\|u\|_{H^\alpha(\mathbb{R}^N)} \le \gamma_0 \|u\|_\lambda$ for all $u \in E_\lambda$, $\lambda \ge \lambda_0$, therefore it suffices to show that $u_{n,i} \to u_{0,i}$ in E_{λ_0}.

By virtue of $\langle \Phi'_{\lambda_n}(u_{n,i}), u_{n,i} \rangle = \langle \Phi'_{\lambda_n}(u_{n,i}), u_{0,i} \rangle = 0$, we have

$$
\begin{aligned}
\|u_{n,i}\|_{\lambda_0}^2 &\le \int_{\mathbb{R}^N} \left(\left| (-\Delta)^{\alpha/2} u_{n,i} \right|^2 + \lambda_n V(x) u_{n,i}^2 \right) dx \\
&= \int_{\mathbb{R}^N} f(x, u_{n,i}) u_{n,i} \, dx + \mu \int_{\mathbb{R}^N} \xi(x) |u_{n,i}|^p \, dx.
\end{aligned} \tag{3.4}
$$

Similarly, we obtain

$$
\begin{aligned}
\|u_{0,i}\|_{\lambda_0}^2 &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left((-\Delta)^{\alpha/2} u_{n,i} (-\Delta)^{\alpha/2} u_{0,i} + \lambda_0 V(x) u_{n,i} u_{0,i} \right) dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \left((-\Delta)^{\alpha/2} u_{n,i} (-\Delta)^{\alpha/2} u_{0,i} + \lambda_n V(x) u_{n,i} u_{0,i} \right) dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} f(x, u_{n,i}) u_{0,i} \, dx + \mu \int_{\mathbb{R}^N} \xi(x) |u_{n,i}|^{p-2} u_{n,i} u_{0,i} \, dx.
\end{aligned} \tag{3.5}
$$

Next we prove

$$
\int_{\mathbb{R}^N} f(x, u_{n,i})(u_{n,i} - u_{0,i}) \, dx = o(1) \tag{3.6}
$$

and

$$
\mu \int_{\mathbb{R}^N} \xi(x) \left(|u_{n,i}|^p - |u_{n,i}|^{p-2} u_{n,i} u_{0,i} \right) dx = o(1). \tag{3.7}
$$

From (F_1) and (F_2), for any $\epsilon > 0$, there exists C_ϵ such that

$$
|f(x,t)| \le \epsilon |t| + C_\epsilon |t|^{q-1} \quad \text{for all } t \in \mathbb{R}.
$$

Since the embedding $E_\lambda \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $s \in [2, 2_\alpha^*]$ and $u_{n,i} \to u_{0,i}$ in $L^r(\mathbb{R}^N)$ for $2 \le r < 2_\alpha^*$, then by the Hölder inequality, we have

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^N} f(x, u_{n,i})(u_{n,i} - u_{0,i}) \, dx \right| \\
&\le \epsilon \int_{\mathbb{R}^N} |u_{n,i}| \, |u_{n,i} - u_{0,i}| \, dx + C_\epsilon \int_{\mathbb{R}^N} |u_{n,i}|^{q-1} |u_{n,i} - u_{0,i}| \, dx \\
&\le \epsilon \|u_{n,i}\|_2 \|u_{n,i} - u_{0,i}\|_2 + C_\epsilon \|u_{n,i}\|_q^{q-1} \|u_{n,i} - u_{0,i}\|_q \, .
\end{aligned}
$$

Taking the limit in the above inequality and using the arbitrariness of ϵ, conclusion (3.6) follows. Analogously, we deduce

$$\begin{aligned}
&\left|\mu \int_{\mathbb{R}^N} \xi(x) \left(|u_{n,i}|^p - |u_{n,i}|^{p-2} u_{n,i} u_{0,i}\right) dx\right| \\
&= \left|\mu \int_{\mathbb{R}^N} \xi(x) \, |u_{n,i}|^{p-2} u_{n,i}(u_{n,i} - u_{0,i}) \, dx\right| \\
&\leq \mu \int_{\mathbb{R}^N} |\xi(x)| \, |u_{n,i}|^{p-1} \, |u_{n,i} - u_{0,i}| \, dx \\
&\leq \mu \, \|\xi\|_{2/(2-p)} \left(\int_{\mathbb{R}^N} |u_{n,i}|^{2(p-1)/p} \, |u_{n,i} - u_{0,i}|^{2/p} \, dx\right)^{p/2} \\
&\leq \mu \, \|\xi\|_{2/(2-p)} \, \|u_{n,i}\|_2^{p-1} \, \|u_{n,i} - u_{0,i}\|_2 \, .
\end{aligned}$$

This shows that (3.7) holds. Therefore, it follows from (3.4), (3.5), (3.6) and (3.7) that

$$\limsup_{n\to\infty} \|u_{n,i}\|_{\lambda_0}^2 \leq \|u_{0,i}\|_{\lambda_0}^2 \, .$$

On the other hand, the weakly lower semi-continuity of norm yields

$$\|u_{0,i}\|_{\lambda_0}^2 \leq \liminf_{n\to\infty} \|u_{n,i}\|_{\lambda_0}^2 \leq \limsup_{n\to\infty} \|u_{n,i}\|_{\lambda_0}^2 \, .$$

Thus, $u_{n,i} \to u_{0,i}$ in E_{λ_0}, and so

$$u_{n,i} \to u_{0,i} \quad \text{in } H^\alpha(\mathbb{R}^N). \tag{3.8}$$

Using (3.2), (3.8) and the fact that constant η is independent of λ_n, we have

$$\frac{1}{2} \int_\Omega \left|(-\Delta)^{\alpha/2} u_{0,1}\right|^2 dx - \int_\Omega F(x, u_{0,1}) \, dx - \frac{\mu}{p} \int_{\mathbb{R}^N} \xi(x) \, |u_{0,1}|^p \, dx \geq \eta > 0$$

and

$$\frac{1}{2} \int_\Omega \left|(-\Delta)^{\alpha/2} u_{0,2}\right|^2 dx - \int_\Omega F(x, u_{0,2}) \, dx - \frac{\mu}{p} \int_{\mathbb{R}^N} \xi(x) \, |u_{0,2}|^p \, dx \leq 0,$$

which implies that $u_{0,1} \neq u_{0,2}$. This completes the proof. □

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