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Erratum to: Total Scalar Curvature and Harmonic Curvature

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It has been realized that the proof of Theorem 5.1 in Section 5 is imcomplete. It was pointed out by Professors Jongsu Kim and Israel Evangelista. Here we give a correct proof of Theorem 5.1. All other results of the paper remain unaffected and the equations are numbered as in the paper.

First, we list some results of the paper:

$$(n-2)\widetilde{i}_{\nabla f}\mathcal{W} = (n-1)\, df \wedge z + i_{\nabla f} z \wedge g, \tag{7}$$

$$i_{\nabla f} z = \alpha\, df, \tag{10}$$

$$\delta(i_{\nabla f} z) = -(1+f)|z|^2, \tag{12}$$

$$(1+f)|z|^2 = -\frac{s_g f}{n-1}\,\alpha + \langle \nabla\alpha, \nabla f\rangle, \tag{13}$$

$$|z|^2 = \frac{n}{n-1}\,\alpha^2 + \left(\frac{n-2}{n-1}\right)^2 |\mathcal{W}_N|^2, \tag{15}$$

where $N = \nabla f/|\nabla f|$ and $\alpha = z(N,N)$.

Now, we prove Theorem 5.1.

Theorem 5.1. *Let (g,f) be a non-trivial solution of the CPE. Assume also that (M,g) has harmonic curvature. Then $\mathcal{W}_N = 0$ on M.*

For the proof, we need the following results.

Lemma 1. *On M, we have*

$$\frac{1}{2}\nabla f(|z|^2) = 2(1+f)\,\alpha|z|^2 + \frac{sf}{n-1}\,\alpha^2 - (1+f)\ \mathrm{tr}(z^3) + \frac{sf}{n(n-1)}|z|^2. \tag{28}$$

Proof. Since (M,g) has harmonic curvature and s_g is constant, we have

$$0 = d^D z(E_i, E_j, E_k) = D_{E_i} z_{jk} - D_{E_j} z_{ik}. \tag{29}$$

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Thus, by (10) we have

$$\begin{aligned}\frac{1}{2}\nabla f(|z|^2) &= \sum_{i,j} D_{\nabla f} z(E_i,E_j)\, z(E_i,E_j) = \sum_{i,j} D_{E_i} z(\nabla f, E_j)\, z(E_i,E_j)\\ &= \sum_{i,j} \left[E_i(z(\nabla f, E_j)) - z(D_{E_i} df, E_j)\right] z_{ij}\\ &= \sum_{i,j} \left[E_i(\alpha\, df(E_j)) - (1+f) z\circ z(E_i,E_j)\right] z_{ij} + \frac{sf}{n(n-1)}|z|^2\\ &= z(\nabla\alpha, \nabla f) + (1+f)\,\alpha|z|^2 - (1+f)\ \mathrm{tr}(z^3) + \frac{sf}{n(n-1)}|z|^2.\end{aligned}$$

Here, by (13)

$$z(\nabla\alpha,\nabla f) = \langle \nabla\alpha, \nabla f\rangle\ \alpha = (1+f)\,\alpha|z|^2 + \frac{sf}{n-1}\alpha^2. \qquad \square$$

Let $a = (n-2)/(n-1)$. By (13), (15) and Lemma 1, we have

Corollary 2. *On M,*

$$\frac{a^2}{2}\nabla f(|\mathcal{W}_N|^2) = \frac{n-2}{n-1}\,(1+f)\,\alpha|z|^2 - (1+f)\ \mathrm{tr}(z^3) + \frac{a^2 sf}{n(n-1)}|\mathcal{W}_N|^2.$$

Lemma 3. *On M, we have*

$$\mathrm{div}\left(a\,|\mathcal{W}_N|^2\, df\right) = -(1+f)\ \left\langle \mathring{\mathcal{W}}z, z\right\rangle.$$

Proof. First we claim that $\delta\mathring{\mathcal{W}}z(\nabla f) = 0$. Let $\{E_i\}$, $i = 1,2,\ldots,n$, be a local geodesic frame field. By (29), since $\delta\mathcal{W} = 0$,

$$\begin{aligned}-\delta\mathring{\mathcal{W}}z(\nabla f) &= \sum_{i,j} D_{E_i}\mathring{\mathcal{W}}z(E_i,E_j)\,\langle\nabla f, E_j\rangle = \sum_{i,j} E_i(\mathring{\mathcal{W}}z(E_i,E_j))\,\langle\nabla f, E_j\rangle\\ &= -\sum_{k,l}\delta\mathcal{W}(E_k,\nabla f, E_l) z_{kl} + \sum_{i,k,l}\mathcal{W}(E_i,E_k,\nabla f,E_l)\, D_{E_i} z_{kl}\\ &= \sum_{i,k,l}\mathcal{W}(E_k,E_i,E_l,\nabla f)\, D_{E_i} z_{kl}\\ &= \frac{n-1}{n-2}\left(\sum_{i,k,l} df(E_k) z_{il} D_{E_i} z_{kl} - \frac{1}{2}\nabla f(|z|^2)\right) = 0.\end{aligned}$$

Here, the fifth equality comes from (7) with $\delta z = 0$, and the last equation from (29).

Therefore, we have

$$\begin{aligned}0 = -\delta\mathring{\mathcal{W}}z(\nabla f) &= \sum_i D_{E_i}(\mathring{\mathcal{W}}z)(E_i,\nabla f)\\ &= \mathrm{div}(\mathring{\mathcal{W}}z(\nabla f,\cdot)) - \sum_i \mathring{\mathcal{W}}z(E_i, D_{E_i} df)\\ &= \mathrm{div}(\mathring{\mathcal{W}}z(\nabla f,\cdot)) - (1+f)\left\langle\mathring{\mathcal{W}}z, z\right\rangle.\end{aligned}$$

Now, by (7) we have

$$\begin{aligned}\mathring{\mathcal{W}}z(\nabla f,\xi) &= \mathcal{W}(\nabla f, E_i, \xi, E_k)z(E_i,E_k)\\ &= \mathcal{W}(E_k,\xi,E_i,\nabla f)z(E_i,E_k)\\ &= \frac{n-1}{n-2}\left(z\circ z(\nabla f,\xi) - df(\xi)|z|^2\right) + \frac{\alpha}{n-2}\, z(\xi,\nabla f).\end{aligned}$$

Thus, by (10) and (15)

$$i_{\nabla f}\mathring{\mathcal{W}}z = -a\,|\mathcal{W}_N|^2 df,$$

proving our lemma. □

Now, we will prove Theorem 5.1. Let $Q = \left\langle \mathring{\mathcal{W}}z, z\right\rangle$. By Lemma 3,

$$a\nabla f(|\mathcal{W}_N|^2) - \frac{asf}{n-1}\,|\mathcal{W}_N|^2 = -(1+f)\,Q. \tag{30}$$

By combining Corollary 2 and (30),

$$\frac{a}{2}(1+f)\,Q = (1+f)\left(\mathrm{tr}(z^3) - a\alpha\,|z|^2\right) + \frac{a^3 sf}{2n}\,|\mathcal{W}_N|^2. \tag{31}$$

In particular, $|\mathcal{W}_N|^2 = 0$ on $B = f^{-1}(-1)$. As a result, $Q = 0$ on B; by Cauchy-Schwarz inequality with the fact that $|z|^2 = \frac{n}{n-1}\,\alpha^2$,

$$z_{ii} = -\frac{\alpha}{n-1}$$

and $z_{ij} = 0$ for $i \neq j$, $z_{in} = 0$ with $z_{nn} = \alpha$, which implies that

$$Q = -\frac{\alpha}{n-1}\sum_{i=1}^{n-1}\mathcal{W}(E_i,E_j,E_i,E_j)\,z_{jj} = -\frac{\alpha}{n-1}\sum_{i=1}^{n}\mathcal{W}(E_i,E_j,E_i,E_j)\,z_{jj} = 0.$$

Now, we are going to prove that $Q = 0$ on M. Let $M_0 = \{x \in M \mid f(x) < -1\}$. If M_0 is empty, Theorem 5.1 still holds by Lemma 1 of [2]. Thus, we may assume that M_0 is not empty. Consider a sufficiently small neighborhood U of $B = f^{-1}(-1)$ and take the intersection $V = U \cap M_0$. Thus, $x \in V$ satisfies $f(x) = -1-\epsilon$ for small $\epsilon > 0$. We may assume that $\mathcal{W}_N \neq 0$ in V, otherwise $\mathcal{W}_N = 0$ on M since g is analytic on M by [1].

First, we claim that $Q \geq 0$ on V. For an arbitrary $\epsilon > 0$, by (30)

$$\mathrm{div}\left((\epsilon + |\mathcal{W}_N|^2)\,df\right) = (\epsilon + |\mathcal{W}_N|^2)\left(\left\langle \nabla \log(\epsilon + |\mathcal{W}_N|^2), \nabla f\right\rangle - \frac{sf}{n-1}\right). \tag{32}$$

Note that $|\mathcal{W}_N|^2(x)$ is constant on each level set of f and is decreasing to 0 as x tends to B. Thus, $\left\langle \nabla f, \nabla |\mathcal{W}_N|^2\right\rangle_x$ and $\left\langle \nabla \log(\epsilon + |\mathcal{W}_N|^2), \nabla f\right\rangle_x$ go to 0 as x tends to B. Therefore, by (32), $\mathrm{div}\left((\epsilon + |\mathcal{W}_N|^2)\,df\right)_x$ goes to $\frac{s}{n-1}\,\epsilon$ as x tends to B, and so

$$\mathrm{div}\left((\epsilon + |\mathcal{W}_N|^2)\,df\right) > 0$$

on V. This implies that $\operatorname{div}\left(|\mathcal{W}_N|^2 df\right) \geq 0$; otherwise, $\operatorname{div}\left(|\mathcal{W}_N|^2 df\right) < 0$ for some $f^{-1}(-1-\epsilon)$ and so for a sufficiently small $\epsilon' > 0$,

$$\operatorname{div}\left((\epsilon' + |\mathcal{W}_N|^2)\, df\right)_x = \epsilon' \nabla f + \operatorname{div}(|\mathcal{W}_N|^2\, df)_x < 0$$

for $x \in f^{-1}(-1-\epsilon) \subset V$, which is a contradiction. This implies that $Q \geq 0$ on V.

Now, since $\mathcal{W}_N = 0$ on B, by Lemma 3, for an arbitrary small $\epsilon > 0$

$$\int_{-1-\epsilon < f < -1} (1+f)\, Q = \int_{f=-1-\epsilon} a|\mathcal{W}_N|^2 |\nabla f| \geq 0,$$

which goes to zero as ϵ tends to 0. Since $(1+f)\, Q \leq 0$ by the previous claim, we may conclude that $Q = 0$ on V. Hence, for an arbitrary small $\epsilon > 0$

$$a \int_{f^{-1}(-1-\epsilon)} |\mathcal{W}_N|^2\, |\nabla f| = \int_{-1-\epsilon < f < -1} (1+f)\, Q = 0$$

by Stokes's theorem. In other words, $\mathcal{W}(\cdot, \nabla f, \cdot, \nabla f) = 0$ on V. Since the metric g and f are analytic in harmonic coordinates on M by [1], we may conclude that $\mathcal{W}(\cdot, \nabla f, \cdot, \nabla f) = 0$ on M. So, $\mathcal{W}_N = 0$ on M. This completes the proof of Theorem 5.1. □

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