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Multiple Solutions to a Dirichlet Problem on the Sierpinski Gasket

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Abstract. We investigate the existence of at least two nontrivial solutions to a Dirichlet problem on the Sierpinski gasket. We develop some general abstract multiplicity theorem which we apply to problem under consideration. Our approach relies on the fact that the action functional is a difference of two continuously differentiable convex functionals and therefore we can apply the ideas related to the Fenchel-Young conjugacy together to get one critical point together with the mountain pass geometry to get the other one.

1. Introduction

Let V stand for the Sierpiński gasket, V_0 be its intrinsic boundary, let Δ denote the weak Laplacian on V and let measure μ denote the restriction to V of normalized $\log N/\log 2$-dimensional Hausdorff measure, so that $\mu(V) = 1$. The aim of this paper is to consider the existence of at least two nontrivial solutions to the following boundary value problem on V

$$\begin{cases} \Delta x(y) + a(y)x(y) = \lambda g(y) f(x(y)) & \text{for a.e. } y \in V \setminus V_0, \\ x|_{V_0} = 0, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a numerical parameter and where $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Solutions to (1.1) are understood in the weak sense which we will describe in a more detail later.

Define $F\colon \mathbb{R} \to \mathbb{R}$ by $F(\xi) = \int_0^\xi f(x)\,dx$ for every $y \in \mathbb{R}$. Concerning the nonlinear term, we will employ the following conditions.

A1. $a \in L^1(V,\mu)$ and $a \leq 0$ almost everywhere in V;

A2. $g \in C(V)$ with $g \leq 0$ and such that the restriction of g to every open subset of V is not identically zero;

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A3. there exist constants $\theta > 2$, $M > 0$ such that for $v \in \mathbb{R}$ with $|v| \geq M$,

$$0 < \theta F(v) \leq v f(v);$$

A4. there are positive constants M_1, and β such that

$$\max_{y \in V, |v| \leq M_1} |g(y) f(v)| \leq \frac{M_1}{2(\beta+1)(2N+3)^2};$$

A5. function $v \to F(v)$ is convex on $\mathbb{R}$.

We note that Assumptions A1–A4 lead to the existence of a solution to (1.1) for any value of parameter $\lambda > 0$ by the Mountain Pass Theorem as suggested in [12].

On the other hand some convexity relations pertaining to the usage of a Fenchel-Young transform allow for the restriction of a numerical parameter λ to some interval with Assumptions A1, A2, A5 we get the existence of at least one solution located in some closed ball in the space $H_0^1(V)$. Coining the two approaches together we obtain the existence of at least two nontrivial solutions. To the best of author's knowledge such problem was not studied in the setting of the Sierpiński gasket before.

Remark 1.1. Note that Assumption A4 implies that $\lim_{v \to 0} \frac{|f(v)|}{|v|} = 0$, while not the other way round. This provides some difference as concerning the application our ideas to the classical Laplacian for which the behaviour around zero must be assumed.

Our approach towards multiplicity is as follows. The first critical point (which lies in the ball, perhaps on the boundary of the ball) is obtained through the Weierstrass Theorem, direct method of the calculus of variations and convexity relations with the use of a Fenchel-Young conjugacy. The second critical point, under assumption that the PS-condition is satisfied, is obtained with the aid of a general type of a Mountain Pass Lemma. The first critical point corresponds to the argument of a minimum of a functional over a closed ball which need not belong to the interior of the ball. Therefore we cannot use the classical variational tool such as Ekelenad's variational principle in order to demonstrate that the minimizer is a critical point. Ekelenad's variational principle is used in [2] for a functional satisfying the PS-condition and also considered on a closed ball. This approach towards obtaining a critical point over some not necessarily open set is sketched in [24] and further developed in several papers, see for example [15] and references therein.

The Sierpiński gasket has the origin in a paper by Sierpiński [29]. This fractal domain can be described as a subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of 1/4 of the area, removing the corresponding open triangle from each of the three constituent triangles and continuing in this way.

The study of the Laplacian on fractals started in physical sciences in [1, 26, 27]. The Laplacian on the Sierpiński gasket was first constructed in [16, 19]. Among the contributions to the theory of nonlinear elliptic equations on fractals we mention [6, 10, 12, 18, 30]. Concerning some recent results by variational methods and critical point theory pertaining to the existence and the multiplicity of solutions by the recently developed variational tools we must mention the following sources [4, 5, 7, 23].

Let us mention [2, 25] for some recent results concerning a general type of critical point theorem on a bounded set (with the PS-condition which we do not need as concerns one critical point). Note that in [25] the bounded critical point theorem due to Schechter is investigated, so the setting is in a Hilbert space, while in [2] it is a Banach space. The application of another type of critical point on closed sets has just been developed by Marano, see [21], and to [22] for applications to differential inclusions, and also some earlier result [20].

2. Remarks on the abstract fractal setting

Concerning the Sierpiński gasket we follow remarks collected in [5]. Let $N \geq 2$ be a natural number and let $p_1, \dots, p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, \dots, N\}$, the map $S_i \colon \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}x + \frac{1}{2}p_i.$$

Let $\mathcal{S} := \{S_1, \dots, S_N\}$ and denote by $G \colon \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$G(A) = \bigcup_{i=1}^{N} S_i(A).$$

It is known that there is a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the attractor of the family $\mathcal{S}$, such that $G(V) = V$ (see, [11, Theorem 9.1]).

The set V is called the *Sierpiński gasket* in $\mathbb{R}^{N-1}$. It can be constructed inductively as follows:

Put $V_0 := \{p_1, \dots, p_N\}$ which is called the *intrinsic boundary* of V and define $V_m := G(V_{m-1})$, for $m \geq 1$, and put $V_* := \bigcup_{m \geq 0} V_m$. Since $p_i = S_i(p_i)$ for $i \in \{1, \dots, N\}$, we have $V_0 \subseteq V_1$, hence $G(V_*) = V_*$. Taking into account that the maps S_i, $i \in \{1, \dots, N\}$, are homeomorphisms, we conclude that $\overline{V_*}$ is a fixed point of G. On the other hand, denoting by C the convex hull of the set $\{p_1, \dots, p_N\}$, we observe that $S_i(C) \subseteq C$ for $i = 1, \dots, N$. Thus $V_m \subseteq C$ for every $m \in \mathbb{N}$, so $\overline{V_*} \subseteq C$. It follows that $\overline{V_*}$ is non-empty and compact, hence $V = \overline{V_*}$.

V is considered to be endowed with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$.

Denote by $C(V)$ the space of real-valued continuous functions on V and by

$$C_0(V) := \{u \in C(V) \mid u|_{V_0} = 0\}.$$

The spaces $L^2(V,\mu)$, $C(V)$ and $C_0(V)$ are endowed with the usual norms, i.e., the norm induced by the product

$$\langle v, h\rangle = \int_V v(y)h(y)\, d\mu$$

and supremum norm $\|\cdot\|_\infty$, respectively.

For a function $u\colon V \to \mathbb{R}$ and for $m \in \mathbb{N}$, let

$$W_m(u) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m\\ |x-y|=2^{-m}}} (u(x)-u(y))^2. \tag{2.1}$$

Since $W_m(u) \leq W_{m+1}(u)$ for every natural m, we can put

$$W(u) = \lim_{m\to\infty} W_m(u).$$

Define now

$$H_0^1(V) := \{u \in C_0(V) \mid W(u) < \infty\}.$$

$H_0^1(V)$ is a dense linear subset of $L^2(V,\mu)$ equipped with the $\|\cdot\|_2$ norm. We now endow $H_0^1(V)$ with the norm

$$\|u\| = \sqrt{W(u)}.$$

There is an inner product defining this norm: for $u, v \in H_0^1(V)$ and $m \in \mathbb{N}$ let

$$\mathcal{W}_m(u,v) = \left(\frac{N+2}{N}\right)^m \sum_{\substack{x,y\in V_m\\ |x-y|=2^{-m}}} (u(x)-u(y))(v(x)-v(y)).$$

Put

$$\mathcal{W}(u,v) = \lim_{m\to\infty} \mathcal{W}_m(u,v).$$

$\mathcal{W}(u,v) \in \mathbb{R}$ and the space $H_0^1(V)$, equipped with the inner product $\mathcal{W}$, which induces the norm $\|\cdot\|$, become real Hilbert spaces. Moreover,

$$\|u\|_\infty \leq (2N+3)\|u\|, \quad \text{for every } u \in H_0^1(V), \tag{2.2}$$

and the embedding

$$(H_0^1(V), \|\cdot\|) \hookrightarrow (C_0(V), \|\cdot\|_\infty)$$

is compact, see also, [14] for further details.

Note that $(H_0^1(V), \|\cdot\|)$ is a Hilbert space which is dense in $L^2(V,\mu)$, that $\mathcal{W}$ is a Dirichlet form on $L^2(V,\mu)$. Let Z be a linear subset of $H_0^1(V)$ which is dense in $L^2(V,\mu)$.

Then, in [12] it is defined a linear self-adjoint operator $\Delta\colon Z \to L^2(V,\mu)$, the *(weak) Laplacian* on V, such that

$$-\mathcal{W}(u,v) = \int_V \Delta u \cdot v \, d\mu, \quad \text{for every } (u,v) \in Z \times H_0^1(V).$$

Let $H^{-1}(V)$ be the closure of $L^2(V,\mu)$ with respect to the pre-norm

$$\|u\|_{-1} = \sup_{\substack{h \in H_0^1(V) \\ \|h\|=1}} |\langle u, h\rangle| ,$$

$v \in L^2(V,\mu)$ and $h \in H_0^1(V)$. Then $H^{-1}(V)$ is a Hilbert space. Then the relation

$$-\mathcal{W}(u,v) = \langle \Delta u, v\rangle \quad \forall\, v \in H_0^1(V),$$

uniquely defines a function $\Delta u \in H^{-1}(V)$ for every $u \in H_0^1(V)$.

While we mainly work with the weak Laplacian, there is also a directly defined version. We say that Δ_s is the *standard Laplacian* of u if $\Delta_s u\colon V \to \mathbb{R}$ is continuous and

$$\lim_{m\to\infty} \sup_{x \in V\setminus V_0} |(N+2)^m (H_m u)(x) - \Delta_s u(x)| = 0,$$

where

$$(H_m u)(x) := \sum_{\substack{y \in V_m \\ |x-y|=2^{-m}}} (u(y) - u(x)),$$

for $x \in V_m$. We say that $u \in C_0(V)$ is a *strong solution* of (1.1) if $\Delta_s u$ exists and is continuous for all $x \in V \setminus V_0$, and

$$\Delta u(x) + a(x)u(x) = \lambda g(x) f(u(x)), \quad \forall\, x \in V \setminus V_0.$$

The existence of the standard Laplacian of a function $u \in H_0^1(V)$ implies the existence of the weak Laplacian Δ (see, [12]).

3. Abstract critical point theorems

In this section we are concerned with the abstract multiplicity tools. Let us introduce the space setting and the structure condition required on the action functional under consideration. We assume that

- H1. $\Phi\colon E \to \mathbb{R}$ is a convex, continuously Fréchet differentiable functional with derivative $\varphi\colon E \to E^*$;
- H2. $H\colon E \to \mathbb{R}$ is a continuously Fréchet differentiable functional with derivative $h\colon E \to E^*$;

H3. operator $h\colon E \to E^*$ is compact;

H4. there exist constants $\alpha, \alpha_1 > 1$, $\gamma > 0$ such that

$$\gamma \|v\|_E^\alpha \leq \langle \varphi(v), v\rangle \quad \text{for all } v \in E$$

and

$$\limsup_{\|x\|\to\infty} \frac{\Phi(x)}{\|x\|^{\alpha_1}} = +\infty;$$

H5. H is a convex functional.

$\langle \cdot, \cdot \rangle$ denotes the action of a derivative on a suitable element or else a duality pairing. We will determine such a value $\lambda^* > 0$ that for each $\lambda \in (0, \lambda^*]$ the corresponding Euler action functional $J\colon E \to \mathbb{R}$

$$J(u) = \Phi(u) - \lambda H(u)$$

has a critical point on B_ρ, where B_ρ is a arbitrarily chosen closed ball centered at 0 with radius ρ. This implies the solvability of

$$\varphi(u) = \lambda h(u), \quad u \in E \tag{3.1}$$

in sense of equality in E^*, i.e.,

$$\langle \varphi(u) - \lambda h(u), x\rangle = 0 \quad \text{for any } x \in E. \tag{3.2}$$

This equation may be viewed as the Euler-Lagrange equation for J. In the sequel when we write equation of type (3.1) we mean it is satisfied in a sense provided in relation (3.2).

We see that Conditions H1–H4 can be regarded as a kind of generic assumptions that are typically satisfied in problems considered by the critical point theory method. They easily hold for our model problem as we shall show later. The only demanding one is H5. Note that convexity of H is related to monotonicity of h.

We begin with some general result following only by convexity. This result involves only basic convexity calculations and it is close to its counterpart from [15] but since we provide a shorter and more clear proof, we decided to place it here. Also the abstract setting in [15] is somehow different. Now, we provide necessary mathematical prerequisites which are needed for the proof. The Fenchel-Young dual for a convex continuously Fréchet differentiable function $H\colon E \to \mathbb{R}$, see [9], reads

$$H^*(v) = \sup_{u\in E} \left\{ \langle v, u\rangle_{E^*,E} - H(u) \right\}, \quad H^*\colon E^* \to \mathbb{R}.$$

We have the following relations

$$H(u) + H^*(v) = \langle v, u\rangle \iff v = h(u),$$

where h stands for the Fréchet derivative, and Fenchel-Young inequality

$$\langle p, u \rangle_{E^*,E} \leq H(u) + H^*(p)$$

is valid for any $p \in E^*$, $u \in E$.

A functional $J \in C^1(E, \mathbb{R})$ satisfies the Palais-Smale condition (PS-condition for short) if every sequence (u_n) such that $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$, has a convergent subsequence.

Lemma 3.1 (Mountain Pass Lemma, MPL Lemma)**.** [17] *Let E be a Banach space and assume that $J \in C^1(E, \mathbb{R})$ satisfies the PS-condition. Let S be a closed subset of E which disconnects E. Let x_0 and x_1 be points of E which are in distinct connected components of $E \setminus S$. Suppose that J is bounded below in S, and in fact the following condition is verified for some b*

$$\inf_{x \in S} J(x) \geq b \quad \text{and} \quad \max\{J(x_0), J(x_1)\} < b. \tag{3.3}$$

If we denote by Γ the family of continuous paths $\gamma\colon [0,1] \to E$ joining x_0 and x_1, then

$$c := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} J(\gamma(s)) \geq \max\{J(x_0), J(x_1)\} > -\infty$$

is a critical value and J has a non-zero critical point x at level c.

Now our abstract critical point theorem reads.

Theorem 3.2. *Assume that* H1, H2, H5 *are satisfied. Fix some $\lambda^* > 0$ and let $u, v \in E$ be such that*

$$J(u) \leq J(v) \quad \text{and} \quad \varphi(v) = \lambda^* h(u). \tag{3.4}$$

Then u is a critical point to J, and thus it solves (3.1).

Proof. We assume for clarity of notation that $\lambda^* = 1$. If this is not the case, we put $H_1 = \lambda^* H$ and replace H with H_1 in our reasoning.

We define $p = \varphi(v) = h(u)$. Since $\frac{d}{du}\Phi = \varphi$ and $\frac{d}{du}H = h$, we have by the definition of p and by the properties of the Fenchel-Young transform

$$\Phi(v) = \langle p, v \rangle - \Phi^*(p) \quad \text{and} \quad H(u) = \langle u, p \rangle - H^*(p). \tag{3.5}$$

By the Fenchel-Young inequality $-H(v) \leq H^*(p) - \langle p, v \rangle$ and by the first relation in (3.5) we have

$$\begin{aligned}\Phi(u) - H(u) = J(u) \leq J(v) &= \Phi(v) - H(v) \\ &= \langle p, v \rangle - \Phi^*(p) - H(v) \leq H^*(p) - \Phi^*(p).\end{aligned}$$

So by the Fenchel-Young inequality

$$\langle u, p\rangle \leq \Phi(u) + \Phi^*(p) \leq H(u) + H^*(p) = \langle u, p\rangle .$$

Thus $\langle u, p\rangle = \Phi(u) + \Phi^*(p)$, and so, recalling the definition of p we see that

$$p = \frac{d}{du}\Phi(u) = \varphi(u) = h(u). \qquad \square$$

Remark 3.3. Theorem 3.2 generalizes the main result from [15] in that we do not require J to be minimized over some set. Instead we impose relations (3.4) to be satisfied. Note that merely Gateaux differentiability of both Φ and H would suffice for the proof. No growth conditions are required as well.

In order to apply Theorem 3.2 we make precise assumptions which lead to have relations (3.4) satisfied. Thus a special case of Theorem 3.2 can now be stated as follows.

Theorem 3.4. *Let E be an infinite dimensional reflexive Banach space and let B_ρ be fixed. Assume that* H1–H5 *are satisfied. Then there exists $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*]$ there exists $u \in B_\rho$ with*

$$J(u) = \inf_{x\in B_\rho} J(x) \tag{3.6}$$

and such that u is a critical point to J, and thus it solves (3.1)*.*

Proof. By Assumption H3 we can chose $\beta_1 > 0$ such that $\|y\|_{E^*} \leq \beta_1$ for all $y \in h(B_\rho)$. Put $\lambda^* = \frac{\gamma\rho^{\alpha-1}}{\beta_1}$ and fix some $0 < \lambda \leq \lambda^*$. Consider J on B_ρ. Observe that J is sequentially weakly l.s.c. on B_ρ. Indeed, Φ has this property as a convex, continuous functional, while H is weakly continuous since it has a compact derivative. Since B_ρ is weakly compact some u exists for which (3.6) holds.

Now consider on E functional $J_1\colon E \to \mathbb{R}$ given by the formula

$$J_1(w) = \Phi(w) - \lambda \langle h(u), w\rangle .$$

Since Φ is weakly l.s.c. and is coercive, so is J_1 and therefore we get the existence of an argument of a minimum to J_1 over E, which we denote by v. Obviously v is a critical point to J_1 and so, having calculated a derivative, we obtain

$$\langle \varphi(v), x\rangle - \lambda \langle h(u), x\rangle = 0 \quad \text{for any } x \in E. \tag{3.7}$$

This means that v solves

$$\varphi(v) = \lambda h(u) \tag{3.8}$$

in the weak sense. Observe that v belongs to B_ρ. Indeed, put $x = v$ in (3.7). Thus from H3, the definitions of β_1 and λ^* and since $\lambda \le \lambda^*$, we see that

$$\gamma \|v\|_E^\alpha \le \langle \varphi(v), v\rangle = \lambda \langle h(u), v\rangle \le \lambda \|h(u)\|_{E^*} \|v\|_E \le \lambda\beta_1 \|v\|_E .$$

Therefore $\|h\|_E^{\alpha-1} \le \lambda\frac{\beta_1}{\gamma} \le \rho^{\alpha-1}$ and $h \in B_\rho$.

The proof that u is a critical point now follows from Theorem 3.2 since $J(u) = \inf_{x\in B_\rho} J(x) \le J(v)$ and since (3.8) holds. □

Some comments on the assumptions are also in order.

1. Instead of assuming that h is a compact operator, we could assume that for all $\lambda > 0$ functional J is weakly l.s.c. on E since we assume that h is compact only to get the weak continuity of H.
2. In the setting of a Hilbert space, i.e., when E is a Hilbert space, instead of assuming that Φ is coercive, we could impose that φ is a bounded operator, i.e., bounded on bounded sets. Then it follows that Φ is coercive, compare with [8, Lemma 6.2.18].

In this section we are concerned with the existence of multiple solutions.

Theorem 3.5. *Let E be an infinite dimensional reflexive Banach space. Assume that* H1–H5 *are satisfied. Take some $\rho > 0$. Then there exists $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*]$ there exists $u \in B_\rho$ with*

$$J(u) = \inf_{x\in B_\rho} J(x)$$

and such that u is a critical point to J, and thus it solves (3.1). *If for some $v \in B_\rho$ it holds that $J(v) < 0$ and $J(0) = 0$ or else $h(0) \neq 0$ and $J(0) = 0$, then u is non-trivial.*

Assume additionally there exists $\lambda_1^* \le \lambda^*$ that for all $\lambda \in (0, \lambda_1^*[$

(a) *J satisfies the PS-condition,*

(b) *$J(0) < \inf_{x\in\partial B_{\rho_1}} J(x)$ for some $\rho_1 > \|u\|_E$,*

(c) *there exists $w \in E \setminus B_{\rho_1}$ with $J(w) \le 0$.*

Then for all $\lambda \in (0, \lambda_1^*[$ functional J has two nontrivial critical points, namely u and another non-zero critical point z different from u.

Proof. In order to get the second critical point, we use Lemma 3.1 taking $x_0 = u$ and $x_1 = w$. Note that condition (3.3) follows by (b) and (c). The existence of a second non-zero critical point readily follows by the mountain pass argument. The solutions are distinct since otherwise we would have that J is constant on level c and so there are infinitely many solutions in fact. □

Symbol $(0, \lambda_1^*[$ means that the interval is either right open or closed.

4. Multiple solutions for (1.1)

In this section we apply our abstract results to problem (1.1). Firstly we observe that by (2.2) for every $y \in V$

$$|x(y)| \le \|x\|_\infty \le (2N+3)\,\|x\|_{H_0^1(V)}\,. \tag{4.1}$$

Using the first inequality in (4.1) and the fact that $\mu(V)=1$ we get

$$\|x\|_{L^2(V,\mu)} \le \|x\|_\infty \le (2N+3)\,\|x\|_{H_0^1(V)}$$

for any $x \in H_0^1(V)$.

We say that a function $x \in H_0^1(V)$ is called a *weak solution* of (1.1) if

$$\mathcal{W}(x,v) - \int_V a(y)x(y)v(y)\,d\mu + \lambda\int_V g(y)f(x(y))v(x)\,d\mu = 0$$

for every $v \in H_0^1(V)$. Further on whenever we write that we obtain a solution to (1.1) we mean the weak one. The functional $J\colon H_0^1(V) \to \mathbb{R}$ given by

$$J(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}\int_V a(y)x^2(y)\,d\mu + \lambda\int_V g(y)F(x(y))\,d\mu, \quad \forall\, x \in H_0^1(V), \tag{4.2}$$

is the Euler action functional attached to problem (1.1).

Lemma 4.1. *Assume that* A1, A2 *holds. Then, the functional $J\colon H_0^1(V) \to \mathbb{R}$ defined by relation* (4.2) *is a $C^1(H_0^1(V),\mathbb{R})$ functional. Moreover,*

$$J'(x)(w) = \mathcal{W}(u,w) - \int_V a(y)x(y)w(x)\,d\mu + \lambda\int_V g(y)f(x(y))\,d\mu, \quad \forall\, w \in H_0^1(V)$$

for each point $x \in H_0^1(V)$. In particular, $x \in H_0^1(V)$ is a weak solution of problem (1.1) *if and only if x is a critical point of J. J is also weakly l.s.c.*

Proof. From results in [23] we see that $J \in C^1(H_0^1(V),\mathbb{R})$. Proposition 4.5 from [6] states that $x \to \int_V g(y)F(x(y))\,d\mu$ is weakly continuous on $H_0^1(V)$. So all assertions follow. □

With Assumptions A1–A4 and Theorem 3.5 from [12] we have the following

Proposition 4.2. *Suppose that* A1–A4 *hold. Then for any $\lambda > 0$ problem* (1.1) *has at least one nontrivial solution.*

Concerning the multiple solutions we have the main result of this section where we need only assume that F is convex in addition to assumptions leading to a mountain pass solution. Recall that constant β is defined in A4.

Theorem 4.3. *Assume that* A1, A2, A5 *are satisfied and that* $f(0) \neq 0$. *Then there exist* $\lambda_1^* > 0$, $\lambda^* < \beta$, *such that for all* $0 < \lambda \leq \lambda_1^*$ *problem* (1.1) *has at least two nontrivial solutions.*

Proof. We need to show that Assumptions H1–H5 of Theorem 3.5 are satisfied. We must also define constants λ^* and λ_1^*. We put $E = H_0^1(V)$, $Z = L^2(V, \mu)$ and $\Phi, H \colon H_0^1(V) \to \mathbb{R}$ given by

$$\Phi(x) = \frac{1}{2}\|x\|^2 - \frac{1}{2}\int_V a(y)x^2(y)\,d\mu, \quad H(x) = -\int_V g(y)F(x(y))\,d\mu.$$

We see that

$$\langle \varphi(x), v\rangle = \mathcal{W}(x,v) - \int_V a(y)x(y)v(y)\,d\mu \quad \text{and} \quad \langle h(x), v\rangle = \int_V g(y)f(x(y))v(y)\,d\mu.$$

Now by Assumptions A1, A2, A5 and Lemma 4.1, Assumptions H1, H2, H5 hold. By Proposition 4.5 from [6] it follows that H has a compact derivative, so that H3 is satisfied. We see that α, $\alpha_1 = 2$, $\gamma = 1$ and $c = (2N+3)$. Thus we also have H4 satisfied. Theorem 3.5 from [12] suggest that we should take such a ball B_ρ on which the first relation in mountain geometry conditions (3.3) is verified. Indeed by formula (3.8) from [12] we see that for any $\lambda < \beta$ it follows that

$$J(x) \geq \frac{(\beta - \lambda)M_1^2}{2(\beta+1)(2N+3)^2}$$

for any x from ∂B_ρ and where $\rho = \frac{M_1}{(2N+3)}$. Note that $|x(y)| \leq M_1$ for $y \in V$ and any $x \in B_\rho$. By A4 and since $\mu(V) = 1$, we have for any $x \in B_\rho$

$$\int_V g(y)F(x(y))\,d\mu \leq \frac{M_1}{2(\beta+1)(2N+3)^2} := \beta_1$$

where β_1 is defined at the beginning of the proof of Theorem 3.4. Now, we see that

$$\lambda^* = \frac{\gamma\rho^{\alpha-1}}{\beta_1} = \frac{\rho}{\beta_1} = 2(\beta+1)(2N+3) > \beta.$$

Thus we can fix $\lambda_1^* < \beta$ and use Theorem 3.5 to get the assertion. □

Note that without Assumptions A3, A4 we do not have mountain geometry, but still Theorem 3.4 provides us with non-trivial critical point. Indeed, we have the following

Theorem 4.4. *Assume that* A1, A2 *and* A5 *are satisfied and that* $f(0) \neq 0$. *Then there exists* $\lambda^* > 0$ *such that for all* $0 < \lambda \leq \lambda^*$ *problem* (1.1) *has at least one nontrivial solution.*

Concerning the applicability of our results for Theorem 4.4 any convex function is sufficient. As for Theorem 4.3 we are also a bit restricted by the AR condition from Assumption A3 thus not every convex function is eligible. For example we may take function $F(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2 + x + \sin x$ in which case we take $\theta = 3$ and M sufficiently large so that $2x+3\sin x - x\cos x + \frac{1}{2}x^2 - \frac{1}{4}x^4 \leq 0$ for $|x| \geq M$. Note that $f(x) = x+\cos x + x^3+1$ does not satisfy $\lim_{v\to 0} \frac{|f(v)|}{|v|} = 0$.

We end this note with some comment concerning regularity of solutions. If we assume additionally that $a \in C(V)$, then by Lemma 2.16 from [12], it follows that every weak solution of the problem (1.1) is also a strong solution.

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