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Hyper-Kloosterman Sums of Different Moduli and Their Applications to Automorphic Forms for $\mathrm{SL}_m(\mathbb{Z})$

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Abstract. Hyper-Kloosterman sums of different moduli appear naturally in Voronoi's summation formula for cusp forms for $\mathrm{GL}_m(\mathbb{Z})$. In this paper their square moment is evaluated and their bounds are proved in the case of consecutively dividing moduli. As an application, smooth sums of Fourier coefficients of a Maass form for $\mathrm{SL}_m(\mathbb{Z})$ against an exponential function $e(\alpha n)$ are estimated. These sums are proved to have rapid decay when α is a fixed rational number or a transcendental number with approximation exponent $\tau(\alpha) > m$. Non-trivial bounds are proved for these sums when $\tau(\alpha) > (m+1)/2$.

1. Introduction

Let n and q be positive integers. For $b \in \mathbb{Z}$ and $\boldsymbol{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, the n-dimensional Kloosterman sum $K_n(\boldsymbol{a}, b; q)$ is defined as follows (cf. Smith [23] and Katz [13]):

(1.1)
$$K_n(\boldsymbol{a}, b; q) = \sideset{}{^*}\sum_{x_1 (\bmod q)} \sideset{}{^*}\sum_{x_2 (\bmod q)} \cdots \sideset{}{^*}\sum_{x_n (\bmod q)} e\left(\frac{a_1x_1 + a_2x_2 + \cdots + a_nx_n + b\overline{x}_1\overline{x}_2\cdots\overline{x}_n}{q}\right).$$

Here the star in $\sum^*$ indicates that $(x_i, q) = 1$. When $n = 1$, $K_1(\boldsymbol{a}, b; q) = K(a, b; q)$ is the classical Kloosterman sum introduced by Kloosterman [14] in 1926:

$$K(a, b; q) = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e\left(\frac{ax + b\overline{x}}{q}\right). \tag{1.2}$$

The Hasse-Weil bound for $K(a, b; q)$ (Hasse [11], Weil [24] and Hooley [12]) gives

$$K(a, b; q) \ll (a, b, q)^{1/2} q^{1/2} \tau(q). \tag{1.3}$$

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This bound is sharp. Generalizations of the estimate (1.3) to $K_n(\mathbf{1}, b; q)$ for $n \geq 2$, with $\mathbf{1} = (1, 1, \ldots, 1)$ and $q = p^\alpha$, a prime power, were considered by many authors. The classical bound $K_n(\mathbf{1}, 1; p) \leq p^{(n+1)/2}$ was proved by Mordell [18] and Smith [23], while Deligne proved the optimal bound $K_n(\mathbf{1}, 1; p) \leq (n+1)p^{n/2}$. The following bound was obtained by Cochrane, Liu and Zheng in [1]: Suppose $p \nmid b$ and $p^\gamma \parallel (n+1)$. Then for any $\alpha \geq 1$ and $n \geq 2$,

$$|K_n(\mathbf{1}, b; p^\alpha)| \leq \delta_p(n+1, p-1)p^{\frac{1}{2}\min\{\gamma, \alpha-2\}}p^{\frac{n\alpha}{2}}, \tag{1.4}$$

where $\delta_p = 1$ if p is odd and $\delta_p = 2$ if $p = 2$. In Ye [25, 26] the bound (1.4) was improved in certain cases.

In this paper we consider Kloosterman sum $K_n(a, h; \boldsymbol{q})$ with different moduli:

$$\begin{aligned} K_n(a, h; \boldsymbol{q}) = &\sideset{}{^*}\sum_{t_1 (\mathrm{mod}\ q_1)} e\left(\frac{at_1}{q_1}\right) \sideset{}{^*}\sum_{t_2 (\mathrm{mod}\ q_2)} e\left(\frac{\bar{t}_1 t_2}{q_2}\right) \cdots \sideset{}{^*}\sum_{t_{n-1} (\mathrm{mod}\ q_{n-1})} e\left(\frac{\bar{t}_{n-2} t_{n-1}}{q_{n-1}}\right) \\ &\times \sideset{}{^*}\sum_{t_n (\mathrm{mod}\ q_n)} e\left(\frac{\bar{t}_{n-1} t_n + h\bar{t}_n}{q_n}\right), \end{aligned} \tag{1.5}$$

where $a, h \in \mathbb{Z}$, $n \geq 2$, q_i $(i = 1, 2, \ldots, n)$ are positive integers and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$. Note that $K_1(a, h; \boldsymbol{q}) = K(a, h; q)$ is the classical Kloosterman sum defined in (1.2). Note that for $n \geq 2$, $(h, q_n) = 1$, an obvious bound for (1.5) can be obtained by applying (1.3) to the inner sum:

$$|K_n(a, h; \boldsymbol{q})| \leq \phi(q_1) \cdots \phi(q_{n-1}) q_n^{1/2} \tau(q_n). \tag{1.6}$$

In this paper we will estimate (1.5) for consecutively dividing moduli, i.e., $q_j \mid q_{j-1}$ for $j = 2, 3, \ldots, n$. To state our result, we introduce some notations. For positive integer a, let a^{**} be the largest square-full divisor of a and write $a^* = a/a^{**}$. Then a^* is square-free and $a = a^* a^{**}$, $(a^*, a^{**}) = 1$. Here we recall that a positive integer a is square-full means that $p^2 \mid a$ for each $p \mid a$.

Theorem 1.1. *Let $n \geq 2$, $(h, q_1) = 1$. Assume $q_j \mid q_{j-1}$ for $j = 2, 3, \ldots, n$. Then we have*

$$\sum_{a=1}^{q_1} |K_n(a, h; \boldsymbol{q})|^2 = \begin{cases} \lambda_n q_1 \phi(q_1) q_n^{n-1} & \text{if } q_2^{**} = \cdots = q_n^{**}, \\ 0 & \text{otherwise}, \end{cases} \tag{1.7}$$

where for $j \geq 2$,

$$\lambda_j = \sum_{d \mid q_n^*} \frac{\mu(d)}{d\phi(d)} \theta_{j-1}(d) \quad \text{with} \quad \theta_j(d) = \begin{cases} 1 & \text{if } j = 1, \\ \sum_{u \mid d} \frac{\theta_{j-1}(u)}{u} & \text{if } j \geq 2. \end{cases}$$

Note that $\theta_j(d)$ $(j \geq 1)$ is multiplicative with $\theta_j(1) = 1$ and for any prime number p,

$$\theta_j(p) = \frac{p^j - 1}{p^{j-1}(p-1)}.$$

Thus for $j \geq 2$,

$$\lambda_j = \prod_{p \mid q_n^*} \left(1 - \frac{\theta_{j-1}(p)}{p(p-1)}\right) = \prod_{p \mid q_n^*} \left(1 - \frac{p^j - 1}{p^j (p-1)^2}\right).$$

Obviously $0 < \lambda_j < 1$ and

$$\begin{aligned} \lambda_j &> \prod_{p \mid q_n^*} \left(1 - \frac{1}{(p-1)^2}\right) = \sum_{d \mid q_n^*} \frac{\mu(d)}{\phi^2(d)} \gg 1, && \text{if } 2 \nmid q_n^*, \\ \lambda_j &\geq 2^{-j} \prod_{2 < p \mid q_n^*} \left(1 - \frac{1}{(p-1)^2}\right) \gg 2^{-j}, && \text{if } 2 \mid q_n^*, \end{aligned}$$

where the implied constant in the above inequalities are absolute. By (1.7) we have, for any integer a,

$$|K_n(a,h;\boldsymbol{q})| \leq \begin{cases} \sqrt{\lambda_n q_1 \phi(q_1)} q_n^{(n-1)/2} & \text{if } q_2^{**} = \cdots = q_n^{**}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

Apparently, this bound improves (1.6) for $n \geq 2$. For further estimate, we will prove the following.

Theorem 1.2. *Let $n \geq 2$, $(h, q_1) = 1$ and $q_j \mid q_{j-1}$ for $j = 2, 3, \ldots, n$. Write $q_1 = q' q''$ where $(q', q'') = 1$ and q'' is the largest divisor of q_1 which has the same prime divisors as q_n. Then we have $K_n(a,h;\boldsymbol{q}) = 0$ unless $q_2^{**} = \cdots = q_n^{**}$ and in this case there holds*

$$|K_n(a,h;\boldsymbol{q})| \leq 2\rho(n, q_n) \phi^{-1}\left(\frac{q'}{(a,q')}\right) \phi(q_1) q_n^{(n-1)/2}, \tag{1.9}$$

where

$$\rho(n, q_n) = n_1^{1/2} \prod_{p \mid \frac{q_n}{(n,q_n)}} (p-1, n)$$

with n_1 the largest divisor of n which has the same prime factors as q_n.

The Kloosterman sum defined in (1.5) appears naturally in theory of modular forms via Voronoi summation formula. Let f be a full-level cusp form for $\mathrm{GL}_m(\mathbb{Z})$ with Langlands' parameters $\mu_f(j)$, $j = 1, 2, \ldots, m$, and Fourier coefficients $A_f(c_{m-2}, \ldots, c_1, n)$. Let $\psi \in$

$C_c^\infty(\mathbb{R}^+)$, h, q any coprime positive integers and $h\overline{h} \equiv 1 \pmod q$. The Voronoi summation formula for f was proved by Miller and Schmid [17]:

$$\begin{aligned}
&\sum_{n\neq 0} A_f(c_{m-2}, c_{m-3}, \ldots, c_1, n) e\left(-\frac{nh}{q}\right)\psi(|n|)\\
&= q \sum_{d_1|c_1 q} \sum_{d_2\left|\frac{c_1c_2q}{d_1}\right.} \cdots \sum_{d_{m-2}\left|\frac{c_1\cdots c_{m-2}q}{d_1d_2\cdots d_{m-3}}\right.} \sum_{n\neq 0} \frac{A_f(n, d_{m-2}, \ldots, d_1)}{d_1\cdots d_{m-2}\,|n|}\\
&\quad \times S(n, \overline{h}; q, \boldsymbol{c}, \boldsymbol{d})\Psi\left(\frac{|n|}{q^m}\prod_{i=1}^{m-2}\frac{d_i^{m-i}}{c_i^{m-i-1}}\right),
\end{aligned} \tag{1.10}$$

where $\boldsymbol{c} = (c_1, \ldots, c_{m-2})$, $\boldsymbol{d} = (d_1, \ldots, d_{m-2})$, and

$$\begin{aligned}
S(n, \overline{h}; q, \boldsymbol{c}, \boldsymbol{d}) = &\sideset{}{^*}\sum_{x_1\left(\bmod \frac{c_1q}{d_1}\right)} e\left(\frac{d_1x_1n}{q}\right) \sideset{}{^*}\sum_{x_2\left(\bmod \frac{c_1c_2q}{d_1d_2}\right)} e\left(\frac{d_2x_2\overline{x}_1}{\frac{c_1q}{d_1}}\right)\cdots\\
&\times \sideset{}{^*}\sum_{x_{m-2}\left(\bmod \frac{c_1\cdots c_{m-2}q}{d_1\cdots d_{m-2}}\right)} e\left(\frac{d_{m-2}x_{m-2}\overline{x}_{m-3}}{\frac{c_1\cdots c_{m-3}q}{d_1\cdots d_{m-3}}} + \frac{\overline{h}\overline{x}_{m-2}}{\frac{c_1\cdots c_{m-2}q}{d_1\cdots d_{m-2}}}\right).
\end{aligned} \tag{1.11}$$

Here $\Psi(x)$ is given by

$$\Psi(x) = \frac{1}{2\pi i}\int_{\operatorname{Re} s = -\sigma} \widetilde{\psi}(s) x^s \frac{\widetilde{F}(1-s)}{F(s)}\, ds, \tag{1.12}$$

where $\widetilde{\psi}(s)$ is the Mellin transform of $\psi(s)$ and

$$F(s) = \pi^{-ms/2}\prod_{i=1}^m \Gamma\left(\frac{s-\mu_f(j)}{2}\right), \quad \widetilde{F}(s) = \pi^{-ms/2}\prod_{i=1}^m \Gamma\left(\frac{s-\overline{\mu}_f(j)}{2}\right).$$

In the above expression $\{\overline{\mu}_f(j)\}_{1\le j\le m} = \left\{\mu_{\widetilde{f}}(j)\right\}_{1\le j\le m}$ are the Langlands' parameters for the dual form $\widetilde{f}$ of f. A special case of (1.10) for even Maass forms for $\mathrm{SL}_m(\mathbb{Z})$ and $c_1 = c_2 = \cdots = c_{m-2} = 1$ was proved by Goldfeld and Li [6–8]. Note that for $\boldsymbol{c} = (1, 1, \ldots, 1)$, $S(n, \overline{h}; q, \boldsymbol{c}, \boldsymbol{d})$ can be rewritten as $K_{m-2}(n, h; \boldsymbol{q})$ where $\boldsymbol{q} = (q_1, q_2, \ldots, q_{m-2})$ is given by

$$q_i = \frac{q}{d_1d_2\cdots d_i}, \quad i = 1, 2, \ldots, m-2. \tag{1.13}$$

Note that $q_i \mid q_{i-1}$ for $i = 2, 3, \ldots, m-2$. A Voronoi summation formula for Rankin-Selberg products of $\mathrm{SL}_m(\mathbb{Z})$ Maass forms was proved by Czarnecki [2].

For applications of Theorem 1.1, we consider the sum

$$\sum_n A_f(1, \ldots, 1, n) e(\alpha n)\phi\left(\frac{n}{X}\right), \quad \alpha \in (0, 1], \tag{1.14}$$

where $X > 2$, $\phi(x) \in C_c^\infty(0,\infty)$ a fixed function supported on $[1,2]$ and $A_f(c_{m-2},\ldots,c_1,n)$ are Fourier coefficients for a Maass cusp form f for $\mathrm{SL}_m(\mathbb{Z})$. By Rankin-Selberg theory (cf. [7]), one has

$$\sum_{|n|\prod_{i=1}^{m-2} d_i^{m-i}\le X} |A_f(n,d_{m-2},\ldots,d_1)|^2 \ll_f X. \tag{1.15}$$

Replacing f by $\widetilde{f}$ and noting that $A_f(n,1,\ldots,1) = A_{\widetilde{f}}(1,\ldots,n)$, we see that a possible uniform bound over α for (1.14) would be $X^{1/2+\varepsilon}$. This is well-understood in the case of $\mathrm{GL}_2(\mathbb{Z})$ (Hafner and Ivić [9,10]). However it is far beyond reach at present in the case of $\mathrm{GL}_m(\mathbb{Z})$, $m\ge 3$. Actually, in the case of $\mathrm{GL}_3(\mathbb{Z})$, the best uniform bound so far is $X^{3/4+\varepsilon}$ (Miller [16] and Ren and Ye [20]), this bound is obtained by using the Hasse-Weil bound (1.3) for the classical Kloosterman sum. When $m \ge 4$, much less is known concerning the sum (1.14). Actually it seems difficult to achieve a uniform bound in these cases. One of the difficulties comes from lack of proper control of the Kloosterman sum $K_{m-2}(n,h;\boldsymbol{q})$. In this paper, we seek a nontrivial bound for (1.14) and prove the following.

Theorem 1.3. *Let f be a full-level cusp form for $\mathrm{GL}_m(\mathbb{Z})$, $m \ge 4$. Denote $\alpha = a/q+\lambda$, $(a,q)=1$ and $\lambda\in\mathbb{R}$.*

(i) *Suppose $q^m \le X$ and $|\lambda| \le 1/(2qX^{1-1/m})$, then for any integer $r > m/2$, we have*

$$\sum_{n\ne 0} A_f(1,\ldots,1,n)e(\alpha n)\phi\left(\frac{n}{X}\right) \ll_{f,\varepsilon,r} (qX)^{1/2+\varepsilon}\left(\frac{X}{q^m}\right)^{-r/m}. \tag{1.16}$$

(ii) *In other cases we have*

$$\sum_{n\ne 0} A_f(1,\ldots,1,n)e(\alpha n)\phi\left(\frac{n}{X}\right) \ll_{f,\varepsilon} q^{(m+1)/2+\varepsilon}\left((|\lambda|\, X)^{m/2}+1\right). \tag{1.17}$$

Corollary 1.4. *Let $\alpha = a/q$ be a fixed rational number with $(a,q)=1$. Then for $q^m \ll X^{1-\varepsilon}$,*

$$\sum_{n>0} A_f(1,\ldots,1,n)e\left(\frac{an}{q}\right)\phi\left(\frac{n}{X}\right) \ll_{q,f,M} X^{-M}$$

for any $M>0$.

Proof. This follows from Theorem 1.3(i) by taking $\lambda = 0$. □

Recall that an irrational number α has approximation exponent $\tau(\alpha)$ if $\tau(\alpha)$ is the smallest number such that for any $\mu > \tau(\alpha)$ the inequality $|\alpha - a/q| < q^{-\mu}$ has only finitely many solutions. By analogous argument as in the proof of Corollary 1.3 in [20], we can easily obtain the following assertion.

Corollary 1.5. *For any fixed transcendental number α with approximation exponent $\tau(\alpha) > m$, there is a sequence $X_k \to +\infty$ such that*

$$\sum_n A_f(1,\ldots,1,n)e(\alpha n)\phi\left(\frac{n}{X_k}\right) \ll_{f,M} X_k^{-M}$$

for any $M > 0$.

Proof. There are infinitely many ways to express $\alpha = a_k/q_k + \lambda_k$ with $|\lambda_k| \le \frac{1}{2}q_k^{-\tau(\alpha)+\varepsilon}$, where we can take $\varepsilon > 0$ such that $\tau(\alpha) - \varepsilon \ge m + \varepsilon$. Let $X_k = q_k^{m+\varepsilon}$. Then

$$|\lambda_k| \le \frac{1}{2q_k^{m+\varepsilon}} < \frac{1}{2q_k X_k^{1-1/m}}.$$

Using Theorem 1.3(i) we have proved the corollary. □

We pointed out that the strong decay of the sums in Corollaries 1.4 and 1.5 is a manifestation of the analytic properties of the underlying L-function twisted by such $e(\alpha n)$, due to the fact that ϕ is a smooth function.

Corollary 1.6. *Let f be a full-level cusp form for $\mathrm{GL}_m(\mathbb{Z})$, $m \ge 4$. For any fixed irrational number α with approximation exponent $\tau(\alpha)$, there exists a sequence $X_k \to +\infty$ such that*

$$\sum_n A_f(1,\ldots,1,n)e(\alpha n)\phi\left(\frac{n}{X_k}\right) \ll_{f,\alpha,\varepsilon} X_k^{(m+1)/(2\tau(\alpha))+\varepsilon}.$$

Proof. There are infinitely many ways to write $\alpha = a_k/q_k + \lambda_k$ with $|\lambda_k| \le q_k^{-\tau(\alpha)+\varepsilon}$. Take $X_k = q_k^{\tau(\alpha)}$. Then $|\lambda_k|\, X_k \le q_k^{\varepsilon}$. Applying (1.17) we get

$$\sum_n A_f(1,\ldots,1,n)e(\alpha n)\phi\left(\frac{n}{X_k}\right) \ll q_k^{(m+1)/2+2\varepsilon} \ll X_k^{(m+1)/(2\tau(\alpha))+\varepsilon} \tag{1.18}$$

because $\tau(\alpha) \ge 2$. □

We remark that (1.18) is nontrivial if $\tau(\alpha) > (m+1)/2$. Similar sums were studied by Ernvall-Hytönen et al. [4,5], Ren and Ye [22] and Czarnecki [2]. For the case of $\mathrm{GL}_3(\mathbb{Z})$, we proved in [20] the bound $\ll X_k^{m/(2\tau(\alpha))+\varepsilon}$. One can see that now we have $X_k^{(m+1)/(2\tau(\alpha))+\varepsilon}$. It is interesting if one can improve (1.17) to $\ll q^{m/2+\varepsilon}\left((|\lambda|\, X)^{m/2} + 1\right)$, as we did when $m = 3$ (cf. [20, (3.18), p. 235]) which implies the uniform bound $X^{3/4+\varepsilon}$.

2. Proofs of Theorems 1.1 and 1.2

Lemma 2.1. *Let $s \ge 1$ and $r \mid s$. Let $(b,s) = 1$ and $b\overline{b} \equiv 1 \pmod{s}$. Write*

$$T(s,r;b,d) = \sum_{\substack{x=1\\(x,s)=1}}^{s} \frac{\rho(\overline{b}x,s)}{\rho(\overline{x},d)}\rho(\overline{x},r), \tag{2.1}$$

where

$$\rho(g,h)=\frac{\mu\left(\frac{h}{(g-1,h)}\right)}{\phi\left(\frac{h}{(g-1,h)}\right)}. \tag{2.2}$$

Then for $d\mid r^*$,

$$T(s,r;b,d)=\begin{cases}\frac{r}{\phi(s)}\cdot\frac{\rho(\bar b,r)}{d\rho(\bar b,d)}\sum_{u\left|\frac{r^*}{d}\right.}\frac{\mu(u)}{\rho(\bar b,u)u\phi(u)} & \text{if } r^{**}=s^{**},\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We first show that if $s=s_1s_2$ with $(s_1,s_2)=1$, then for $r\mid s$, $d\mid s$, $r_i=(r,s_i)$ and $d_i=(d,s_i)$, there holds

$$T(s_1s_2,r;b,d)=T(s_1,r_1;b,d_1)T(s_2,r_2;b,d_2). \tag{2.3}$$

In fact, for $(s_1,s_2)=1$ we have

$$T(s_1s_2,r;b,d)=\sum_{\substack{x_1=1\\(x_1,s_1)=1}}^{s_1}\sum_{\substack{x_2=1\\(x_2,s_2)=1}}^{s_2}\frac{\rho(\bar b(x_1s_2+x_2s_1),s_1s_2)}{\rho(\overline{x_1s_2+x_2s_1},d_1d_2)}\rho(\overline{x_1s_2+x_2s_1},r_1r_2),$$

where

$$b\bar b\equiv 1 \pmod{s_1s_2},\quad (x_1s_2+x_2s_1)\overline{x_1s_2+x_2s_1}\equiv 1 \pmod{s_1s_2}.$$

Note that $d_i\mid s_i$ and $r_i\mid s_i$ $(i=1,2)$ imply

$$\begin{aligned}\rho(\bar b(x_1s_2+x_2s_1),s_1s_2)&=\rho(\bar bx_1s_2,s_1)\rho(\bar bx_2s_1,s_2),\\ \rho(\overline{x_1s_2+x_2s_1},r_1r_2)&=\rho(\overline{x_1s_2},r_1)\rho(\overline{x_2s_1},r_2),\\ \rho(\overline{x_1s_2+x_2s_1},d_1d_2)&=\rho(\overline{x_1s_2},d_1)\rho(\overline{x_2s_1},d_2),\end{aligned}$$

where $(x_1s_2)\overline{x_1s_2}\equiv 1 \pmod{s_1}$ and $(x_2s_1)\overline{x_2s_1}\equiv 1 \pmod{s_2}$. Therefore

$$\begin{aligned}T(s_1s_2,r;b,d)&=\sum_{\substack{x_1=1\\(x_1,s_1)=1}}^{s_1}\frac{\rho(\bar bx_1s_2,s_1)}{\rho(\overline{x_1s_2},d_1)}\rho(\overline{x_1s_2},r_1)\sum_{\substack{x_2=1\\(x_2,s_2)=1}}^{s_2}\frac{\rho(\bar bx_2s_1,s_2)}{\rho(\overline{x_2s_1},d_2)}\rho(\overline{x_2s_1},r_2)\\ &=\sum_{\substack{x=1\\(x,s_1)=1}}^{s_1}\frac{\rho(\bar bx,s_1)}{\rho(\bar x,d_1)}\rho(\bar x,r_1)\sum_{\substack{x=1\\(x,s_2)=1}}^{s_2}\frac{\rho(\bar bx,s_2)}{\rho(\bar x,d_2)}\rho(\bar x,r_2)\\ &=T(s_1,r_1;b,d_1)T(s_2,r_2;b,d_2).\end{aligned}$$

We write $s=\prod_{p\mid s}p^{k_p}$, $k_p\ge 1$. Then $r\mid s$ and $d\mid r$ imply

$$r=\prod_{p\mid s}p^{u_p},\quad d=\prod_{p\mid s}p^{v_p},\quad 0\le v_p\le u_p\le k_p.$$

Moreover, $d \mid r^*$ implies $v_p = 0$ if $u_p > 1$; $v_p = 0$ or 1 if $u_p = 1$; $v_p = 0$ if $u_p = 0$. By (2.3), we get

$$T(s, r; b, d) = \prod_{p|s} \sigma(p^{k_p}), \tag{2.4}$$

where for $k = k_p$, $u = u_p$ and $v = v_p$,

$$\sigma(p^k) = \sum_{\substack{x=1 \\ p\nmid x}}^{p^k} \frac{\rho(\overline{b}x, p^k)}{\rho(\overline{x}, p^v)} \rho(\overline{x}, p^u).$$

We will show that for $k = 1$,

$$\sigma(p^k) = \begin{cases} \frac{1}{\phi(p)} & \text{if } u = v, \\ \frac{p}{\phi(p)} \rho(\overline{b}, p^u) - \frac{1}{\phi^2(p)} & \text{if } u \neq v, \end{cases} \tag{2.5}$$

and for $k > 1$,

$$\sigma(p^k) = \begin{cases} 0 & \text{if } u < k, \\ \frac{p}{\phi(p)} \rho(\overline{b}, p^u) & \text{if } u = k. \end{cases} \tag{2.6}$$

(i) Suppose $k = 1$. Then $0 \leq v \leq u \leq 1$ and

$$\sigma(p^k) = \sum_{x=1}^{p-1} \frac{\rho(\overline{b}x, p)}{\rho(\overline{x}, p^v)} \rho(\overline{x}, p^u).$$

If $p = 2$ then the right above is equal to 1, hence (2.5) is true. Let $p > 2$. If $u = v$ then

$$\sigma(p^k) = \sum_{x=1}^{p-1} \rho(\overline{b}x, p) = \sum_{x=1}^{p-1} \rho(x, p) = 1 - \frac{p-2}{\phi(p)} = \frac{1}{\phi(p)}.$$

If $u \neq v$, then one has $u = 1$ and $v = 0$. Therefore

$$\sigma(p^k) = \sum_{x=1}^{p-1} \rho(\overline{b}x, p) \rho(\overline{x}, p) = \rho(\overline{b}, p) - \frac{1}{\phi(p)} \sum_{x=2}^{p-1} \rho(\overline{b}x, p). \tag{2.7}$$

The last sum is equal to

$$\sum_{x=1}^{p-1} \rho(x, p) - \rho(\overline{b}, p) = \frac{1}{\phi(p)} - \rho(\overline{b}, p).$$

Putting in (2.7) we obtain (2.5).

(ii) Suppose $k > 1$. Note that $\rho(\overline{b}x, p^k) = 0$ unless $(\overline{b}x - 1, p^k) = p^k$ or p^{k-1}, that is

$$x \equiv b \pmod{p^k} \quad \text{or} \quad x = x_h \equiv b(1 + hp^{k-1}) \pmod{p^k} \quad \text{with} \quad h = 1, 2, \ldots, p-1.$$

This shows that

$$\sigma(p^k) = \frac{\rho(\overline{b}, p^u)}{\rho(\overline{b}, p^v)} - \frac{1}{\phi(p)} \sum_{h=1}^{p-1} \frac{\rho(\overline{x}_h, p^u)}{\rho(\overline{x}_h, p^v)}. \tag{2.8}$$

It is easy to see that $\overline{x}_h \equiv \overline{b}(1 - hp^{k-1}) \pmod{p^k}$. If $u < k$, then $\rho(\overline{x}_h, p^u) = \rho(\overline{b}, p^u)$, $\rho(\overline{x}_h, p^v) = \rho(\overline{b}, p^v)$, hence $\sigma(p^k) = 0$. If $u = k$, then $v = 0$ and (2.8) becomes

$$\sigma(p^k) = \rho(\overline{b}, p^k) - \frac{1}{\phi(p)} \sum_{h=1}^{p-1} \rho(\overline{x}_h, p^k). \tag{2.9}$$

Let $p^j \parallel \overline{b} - 1$. If $j \geq k$, then $(\overline{x}_h - 1, p^k) = p^{k-1}$, therefore $\rho(\overline{b}, p^k) = 1$ and $\rho(\overline{x}_h, p^k) = -\frac{1}{\phi(p)}$. This gives

$$\sigma(p^k) = 1 + \frac{p-1}{\phi^2(p)} = \frac{p}{\phi(p)} \rho(\overline{b}, p^k).$$

If $j \leq k - 2$, then $(\overline{x}_h - 1, p^k) = p^j$, and hence $\rho(\overline{b}, p^k) = \rho(\overline{x}_h, p^k) = 0$. Therefore $\sigma(p^k) = 0$.

If $j = k - 1$, then one can write $\overline{b} = 1 + t_0 p^{k-1}$ for some t_0 with $p \nmid t_0$. Now $(\overline{x}_h - 1, p^k) = p^{k-1}(t_0 - h, p)$. Thus $\rho(\overline{x}_h, p^k) = \rho(t - h, p)$ and $\rho(\overline{b}, p^k) = -\frac{1}{\phi(p)}$. Therefore

$$\sigma(p^k) = -\frac{1}{\phi(p)} - \frac{1}{\phi(p)} \sum_{h=1}^{p-1} \rho(t_0 - h, p).$$

For $p = 2$ this gives $\sigma(2^k) = -2$ which verifies (2.6). For $p > 2$, it gives

$$\sigma(p^k) = -\frac{2}{\phi(p)} + \frac{p-2}{\phi^2(p)} = -\frac{p}{\phi^2(p)} = \frac{p}{\phi(p)} \rho(\overline{b}, p^k).$$

By (2.4)–(2.6) we see that $T(s, r; b, d) = 0$ unless $u_p = k_p$ for each $p \mid s$, that is, $r^{**} = s^{**}$. In this case we have $r^* \mid s^*$, and for $d \mid r^*$,

$$\begin{aligned} T(s, r; b, d) &= \prod_{p \mid \frac{s^*}{r^*}} \frac{1}{\phi(p)} \prod_{p \mid d} \frac{1}{\phi(p)} \prod_{p \mid \frac{r^*}{d}} \left\{ \frac{p}{\phi(p)} \rho(\overline{b}, p) - \frac{1}{\phi^2(p)} \right\} \prod_{p^k \parallel r^{**}} \frac{p}{\phi(p)} \rho(\overline{b}, p^k) \\ &= \frac{1}{\phi(\frac{s^*}{r^*}) \phi(d)} \frac{r^{**}}{\phi(r^{**})} \rho(\overline{b}, r^{**}) \frac{\frac{r^*}{d}}{\phi(\frac{r^*}{d})} \rho\left(\overline{b}, \frac{r^*}{d}\right) \prod_{p \mid \frac{r^*}{d}} \left(1 - \frac{1}{\rho(\overline{b}, p) p \phi(p)}\right) \\ &= \frac{r}{\phi(s)} \frac{\rho(\overline{b}, r)}{d \rho(\overline{b}, d)} \sum_{u \mid \frac{r^*}{d}} \frac{\mu(u)}{\rho(\overline{b}, u) u \phi(u)}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 1.1. By definition,

$$\sum_{1\le a\le q_1}|K_n(a,h;\boldsymbol{q})|^2=\sideset{}{^*}\sum_{t_1,t_1'(\mathrm{mod}\ q_1)}\sum_{a\le q_1}e\left(\frac{at_1-at_1'}{q_1}\right)\sideset{}{^*}\sum_{t_2,t_2'(\mathrm{mod}\ q_2)}e\left(\frac{\bar t_1t_2-\bar t_1't_2'}{q_2}\right)\cdots$$
$$\times\sideset{}{^*}\sum_{t_{n-1},t_{n-1}'(\mathrm{mod}\ q_{n-1})}e\left(\frac{\bar t_{n-2}t_{n-1}-\bar t_{n-2}'t_{n-1}'}{q_{n-1}}\right)$$
$$\times\sideset{}{^*}\sum_{t_n,t_n'(\mathrm{mod}\ q_n)}e\left(\frac{\bar t_{n-1}t_n-\bar t_{n-1}'t_n'}{q_n}\right)e\left(\frac{h(\bar t_n-\bar t_n')}{q_n}\right).$$

The sum over a is equal to 0 unless $t_1\equiv t_1'\pmod{q_1}$ in which case it equals q_1. Write $t_i'\equiv t_ia_i\pmod{q_i}$ for $i=2,3,\ldots,n$, we get

$$\sum_{1\le a\le q_1}|K_n(a,h;\boldsymbol{q})|^2$$
$$=q_1\sideset{}{^*}\sum_{t_1(\mathrm{mod}\ q_1)}\sideset{}{^*}\sum_{t_2,a_2(\mathrm{mod}\ q_2)}e\left(\frac{\bar t_1t_2(1-a_2)}{q_2}\right)\sideset{}{^*}\sum_{t_3,a_3(\mathrm{mod}\ q_3)}e\left(\frac{\bar t_2t_3(1-\bar a_2a_3)}{q_3}\right)\cdots$$
$$\times\sideset{}{^*}\sum_{t_n,a_n(\mathrm{mod}\ q_n)}e\left(\frac{\bar t_{n-1}t_n(1-\bar a_{n-1}a_n)}{q_n}\right)e\left(\frac{h\bar t_n(1-\bar a_n)}{q_n}\right).$$

Changing the order of the first two sums, then for $(t_2,q_2)=1$ the sum over t_1 is equal to

$$\sideset{}{^*}\sum_{t_1(\mathrm{mod}\ q_1)}e\left(\frac{\bar t_1t_2(1-a_2)q_1/q_2}{q_1}\right)$$
$$=\mu\left(\frac{q_1}{(t_2(a_2-1)q_1/q_2,q_1)}\right)\phi(q_1)\phi^{-1}\left(\frac{q_1}{(t_2(a_2-1)q_1/q_2,q_1)}\right)$$
$$=\phi(q_1)\mu\left(\frac{q_2}{(a_2-1,q_2)}\right)\phi^{-1}\left(\frac{q_2}{(a_2-1,q_2)}\right)=\phi(q_1)\rho(a_2,q_2).$$

Hence

$$\sum_{1\le a\le q_1}|K_n(a,h;\boldsymbol{q})|^2=q_1\phi(q_1)\sideset{}{^*}\sum_{a_2(\mathrm{mod}\ q_2)}\rho(a_2,q_2)\sideset{}{^*}\sum_{t_2(\mathrm{mod}\ q_2)}\sideset{}{^*}\sum_{t_3,a_3(\mathrm{mod}\ q_3)}e\left(\frac{\bar t_2t_3(1-\bar a_2a_3)}{q_3}\right)\cdots$$
$$\times\sideset{}{^*}\sum_{t_n,a_n(\mathrm{mod}\ q_n)}e\left(\frac{\bar t_{n-1}t_n(1-\bar a_{n-1}a_n)}{q_n}\right)e\left(\frac{h\bar t_n(1-\bar a_n)}{q_n}\right).$$

Similarly, the sum over t_j $(2\le j\le t_{n-1})$ is equal to $\phi(q_j)\rho(\bar a_ja_{j+1},q_{j+1})$, and finally the sum over t_n is equal to $\phi(q_n)\rho(\bar a_n,q_n)$. Therefore we get

$$\sum_{1\le a\le q_1}|K_n(a,h;\boldsymbol{q})|^2$$
$$=q_1\phi(q_1)\phi(q_2)\cdots\phi(q_m)\sideset{}{^*}\sum_{a_2(\mathrm{mod}\ q_2)}\rho(a_2,q_2)\sideset{}{^*}\sum_{a_3(\mathrm{mod}\ q_3)}\rho(\bar a_2a_3,q_3)\cdots\tag{2.10}$$
$$\times\sideset{}{^*}\sum_{a_{n-1}(\mathrm{mod}\ q_{n-1})}\rho(\bar a_{n-2}a_{n-1},q_{n-1})\sideset{}{^*}\sum_{a_n(\mathrm{mod}\ q_n)}\rho(\bar a_{n-1}a_n,q_n)\rho(\bar a_n,q_n).$$

When $n = 2$, we get

$$\sum_{1\le a\le q_1} |K_2(a,h;\boldsymbol{q})|^2 = q_1\phi(q_1)\phi(q_2) \sum_{a_2 (\mathrm{mod}\ q_2)}^{*} \rho(a_2,q_2)\rho(\overline{a}_2,q_2).$$

The sum over a_2 is $T(q_2,q_2;1,1)$ which, by Lemma 2.1, is equal to

$$\frac{q_2}{\phi(q_2)} \sum_{u|q_2^*} \frac{\mu(u)}{u\phi(u)}.$$

Denote by λ_2 the sum over u, then we get

$$\sum_{1\le a\le q_1} |K_2(a,h;\boldsymbol{q})|^2 = \lambda_2 q_1\phi(q_1)q_2.$$

When $n \ge 3$, the sum over a_n in (2.10) is $T(q_n,q_n;a_{n-1},1)$ which, by Lemma 2.1, is equal to

$$\frac{q_n}{\phi(q_n)}\rho(\overline{a}_{n-1},q_n) \sum_{u|q_n^*} \frac{\mu(u)}{\rho(\overline{a}_{n-1},u)u\phi(u)}.$$

Thus (2.10) becomes

$$\begin{aligned}
&\sum_{1\le a\le q_1} |K_n(a,h;\boldsymbol{q})|^2 \\
&= q_1\phi(q_1)\phi(q_2)\cdots\phi(q_{n-1})q_n \sum_{a_2 (\mathrm{mod}\ q_2)}^{*} \rho(a_2,q_2) \sum_{a_3 (\mathrm{mod}\ q_3)}^{*} \rho(\overline{a}_2 a_3,q_3)\cdots \\
&\quad\times \sum_{a_{n-2} (\mathrm{mod}\ q_{n-2})}^{*} \rho(\overline{a}_{n-3}a_{n-2},q_{n-2}) \sum_{d|q_n^*} \frac{\mu(d)}{d\phi(d)} T(q_{n-1},q_n;a_{n-2},d).
\end{aligned} \tag{2.11}$$

By Lemma 2.1, $T(q_{n-1},q_n;a_{n-2},d) = 0$ unless $q_{n-1}^{**} = q_n^{**}$, in this case it is equal to

$$\frac{q_n}{\phi(q_{n-1})} \cdot \frac{\rho(\overline{a}_{n-2},q_n)}{d\rho(\overline{a}_{n-2},d)} \sum_{u\left|\frac{q_n^*}{d}\right.} \frac{\mu(u)}{\rho(\overline{a}_{n-2},u)u\phi(u)}.$$

This gives

$$\begin{aligned}
\sum_{d|q_n^*} \frac{\mu(d)}{d\phi(d)} T(q_{n-1},q_n;a_{n-2},d) &= \frac{q_n\rho(\overline{a}_{n-2},q_n)}{\phi(q_{n-1})} \sum_{d|q_n^*} \frac{\mu(d)}{\rho(\overline{a}_{n-2},d)d^2\phi(d)} \sum_{u\left|\frac{q_n^*}{d}\right.} \frac{\mu(u)}{\rho(\overline{a}_{n-2},u)u\phi(u)} \\
&= \frac{q_n\rho(\overline{a}_{n-2},q_n)}{\phi(q_{n-1})} \sum_{v|q_n^*} \frac{\mu(v)}{\rho(\overline{a}_{n-2},v)v\phi(v)} \sum_{d|v} \frac{1}{d}.
\end{aligned}$$

Write $\theta_2(v) = \sum_{t|v} t^{-1}$ and back to (2.11) we obtain

$$\begin{aligned}
\sum_{1\le a\le q_1} |K_n(a,h;\boldsymbol{q})|^2 &= q_1\phi(q_1)\phi(q_2)\cdots\phi(q_{n-2})q_n^2 \sum_{a_2 (\mathrm{mod}\ q_2)}^{*} \rho(a_2,q_2) \sum_{a_3 (\mathrm{mod}\ q_3)}^{*} \rho(\overline{a}_2 a_3,q_3)\cdots \\
&\quad\times \sum_{a_{n-3} (\mathrm{mod}\ q_{n-3})}^{*} \rho(\overline{a}_{n-4}a_{n-3},q_{n-3}) \sum_{d|q_n^*} \frac{\mu(d)\theta_2(d)}{d\phi(d)} T(q_{n-2},q_n;a_{n-3},d).
\end{aligned}$$

Continuing this process one yields either $\sum_{1\le a\le q_1} |K_n(a,h;\boldsymbol{q})|^2 = 0$ or $q_3^{**} = \cdots = q_n^{**}$ and

$$\sum_{1\le a\le q_1} |K_n(a,h;\boldsymbol{q})|^2 = q_1\phi(q_1)\phi(q_2)q_n^{n-2}\sum_{d|q_n^*}\frac{\mu(d)\theta_{n-2}(d)}{d\phi(d)}T(q_2,q_n;1,d), \tag{2.12}$$

where $\theta_1(d) = 1$ and $\theta_j(d) = \sum_{t|d}\frac{\theta_{j-1}(t)}{t}$ for $j \ge 2$. Applying Lemma 2.1 again we have $T(q_2, q_n; 1, d) = 0$ unless $q_2^{**} = q_n^{**}$ and

$$T(q_2,q_n;1,d) = \frac{q_n}{\phi(q_2)}\cdot\frac{1}{d}\sum_{u\left|\frac{q_n^*}{d}\right.}\frac{\mu(u)}{u\phi(u)}.$$

Back to (2.12) we finally obtain either $\sum_{1\le n\le q_1} |K_n(a,h;\boldsymbol{q})|^2 = 0$ or $q_2^{**} = q_3^{**} = \cdots = q_n^{**}$ and

$$\sum_{1\le a\le q_1} |K_n(a,h;\boldsymbol{q})|^2 = \lambda_n q_1\phi(q_1)q_n^{n-1},$$

where

$$\lambda_n = \sum_{d|q_n^*}\frac{\mu(d)\theta_{n-1}(d)}{d\phi(d)}. \qquad \square$$

Proof of Theorem 1.2. The first assertion follows immediately from Theorem 1.1. To prove (1.9) we write $\bar{t}_{i-1}t_i \equiv x_i \pmod{q_i}$ for $i = 2,3,\ldots,n$, and then write x_1 for t_1, to get

$$\begin{aligned} K_n(a,h;\boldsymbol{q}) = &\sideset{}{^*}\sum_{x_1 (\mathrm{mod}\ q_1)} e\left(\frac{ax_1}{q_1}\right)\sideset{}{^*}\sum_{x_2 (\mathrm{mod}\ q_2)} e\left(\frac{x_2}{q_2}\right)\cdots\sideset{}{^*}\sum_{x_{n-1} (\mathrm{mod}\ q_{n-1})} e\left(\frac{x_{n-1}}{q_{n-1}}\right) \\ &\times\sideset{}{^*}\sum_{x_n (\mathrm{mod}\ q_n)} e\left(\frac{x_n}{q_n}\right)e\left(\frac{h\overline{x}_1\cdots\overline{x}_{n-1}\overline{x}_n}{q_n}\right). \end{aligned} \tag{2.13}$$

For $2 \le i \le n$, we write $r_i = q_i/q_n$, then r_i is square-free and $(r_i, q_n) = 1$, since $q_i^{**} = q_n^{**}$. So we can express x_i as $f_iq_n+g_ir_i$ where f_i, g_i run through reduced residue systems modulo r_i and q_n respectively. Moreover, we express x_1 as $f_1q''+g_1q'$, where $q'q'' = q_1$, $(q', q'') = 1$ and q'' is the largest factor of q_1 which has the same prime divisors as q_n. Note that $r_n = 1$ and $q_n \mid q''$. One has

$$x_1x_2\cdots x_n \equiv q'g_1\prod_{i=2}^{n} g_ir_i \pmod{q_n}.$$

Therefore

$$K_n(a,h;\boldsymbol{q}) = \sideset{}{^*}\sum_{f_1 (\bmod q')} e\left(\frac{af_1}{q'}\right) \sideset{}{^*}\sum_{g_1 (\bmod q'')} e\left(\frac{ag_1}{q''}\right) \sideset{}{^*}\sum_{f_2 (\bmod r_2)} e\left(\frac{f_2}{r_2}\right) \sideset{}{^*}\sum_{g_2 (\bmod q_n)} e\left(\frac{g_2}{q_n}\right) \cdots$$
$$\times \sideset{}{^*}\sum_{f_{n-1} (\bmod r_{n-1})} e\left(\frac{f_{n-1}}{r_{n-1}}\right) \sideset{}{^*}\sum_{g_{n-1} (\bmod q_n)} e\left(\frac{g_{n-1}}{q_n}\right)$$
$$\times \sideset{}{^*}\sum_{g_n (\bmod q_n)} e\left(\frac{g_n}{q_n}\right) e\left(\frac{h\overline{q' g_1 \prod_{i=2}^n (g_i r_i)}}{q_n}\right)$$

which in turn equals

$$\phi(q')\phi^{-1}\left(\frac{q'}{(a,q')}\right)\mu\left(\frac{q'}{(a,q')}\right)\mu(r_2)\cdots\mu(r_{n-1}) \sideset{}{^*}\sum_{g (\bmod q'')} e\left(\frac{ag}{q''}\right) H(h, q' g r_2 \cdots r_{n-1}; q_n),$$

where

$$H(h,y;d) = \sideset{}{^*}\sum_{x_1 (\bmod d)} \sideset{}{^*}\sum_{x_2 (\bmod d)} \cdots \sideset{}{^*}\sum_{x_{n-1} (\bmod d)} e\left(\frac{x_1 + \cdots + x_{n-1} + h\overline{y}\overline{x}_1 \cdots \overline{x}_{n-1}}{d}\right).$$

For $H(h,y;d)$, we have the following factorization: If $d = d_1 d_2$, $(d_1,d_2) = 1$, then for $(y,d) = 1$,

$$H(h,y;d) = H(h, y d_2^n; d_1) H(h, y d_1^n; d_2). \tag{2.14}$$

In fact, if we write $x_i = a_i d_1 + b_i d_2 \pmod{d_1 d_2}$ where a_i, b_i run through reduced residue systems modulo d_2 and d_1, respectively, then

$$y x_1 \cdots x_{n-1} \equiv y a_1 \cdots a_{n-1} d_1^{n-1} + y b_1 \cdots b_{n-1} d_2^{n-1} \pmod{d_1 d_2}.$$

Let

$$\overline{y x_1 \cdots x_{n-1}} \equiv u d_1 + v d_2 \pmod{d_1 d_2}.$$

Then

$$y a_1 \cdots a_{n-1} d_1^n u + y b_1 \cdots b_{n-1} d_2^n v \equiv 1 \pmod{d_1 d_2}$$

from which we find

$$u \equiv \overline{y a_1 \cdots a_{n-1} d_1^n} \pmod{d_2}, \quad v \equiv \overline{y b_1 \cdots b_{n-1} d_2^n} \pmod{d_1}.$$

Thus

$$H(h,y,d_1 d_2)$$
$$= \sideset{}{^*}\sum_{a_1 (\bmod d_2)} \sideset{}{^*}\sum_{b_1 (\bmod d_1)} e\left(\frac{b_1}{d_1}\right) e\left(\frac{a_1}{d_2}\right) \cdots \sideset{}{^*}\sum_{a_{n-1} (\bmod d_2)}$$
$$\times \sideset{}{^*}\sum_{b_{n-1} (\bmod d_1)} e\left(\frac{b_{n-1}}{d_1}\right) e\left(\frac{a_{n-1}}{d_2}\right) e\left(\frac{h\overline{y b_1 \cdots b_{n-1} d_2^n}}{d_1}\right) e\left(\frac{h\overline{y a_1 \cdots a_{n-1} d_1^n}}{d_2}\right)$$
$$= H(h, y d_1^n; d_2) H(h, y d_2^n; d_1).$$

Let $q_n = \prod_{j=1}^{k} p_j^{\alpha_j}$ be the canonical decomposition of q_n, and write $z_j = q_n p_j^{-\alpha_j}$. Then by (2.14) we get

$$H(h, q' g r_2 \cdots r_{n-1}; q_n) = \prod_{j=1}^{k} H(h, q' g r_2 \cdots r_{n-1} z_j^n; p_j^{\alpha_j}).$$

Therefore

$$\begin{aligned} K_n(a, h; \boldsymbol{q}) &= \phi(q')\phi^{-1}\left(\frac{q'}{(a,q')}\right)\mu\left(\frac{q'}{(a,q')}\right)\mu(r_2)\cdots\mu(r_{n-1}) \\ &\quad \times \sideset{}{^*}\sum_{g (\mathrm{mod}\ q'')} e\left(\frac{ag}{q''}\right)\prod_{j=1}^{k} H(h, q' g r_2 \cdots r_{n-1} z_j^n; p_j^{\alpha_j}). \end{aligned} \tag{2.15}$$

Write $q'' = \prod_{j=1}^{k} p_j^{\beta_j}$. Then $\beta_j \geq \alpha_j$ since $q_n \mid q''$. Let $m_i = \prod_{j=1}^{i} p_j^{\beta_j}$ and $w_i = q'' p_i^{-\beta_i}$ for $i = 1, 2, \ldots, k$. Note that $m_k = q''$ and $m_{k-1} = w_k$. Write $g \equiv u m_{k-1} + v p_k^{\beta_k} \pmod{q''}$. Then

$$\begin{aligned} &\sideset{}{^*}\sum_{g (\mathrm{mod}\ q'')} e\left(\frac{ag}{q''}\right)\prod_{j=1}^{k} H(h, q' g r_2 \cdots r_{n-1} z_j^n; p_j^{\alpha_j}) \\ &= \sideset{}{^*}\sum_{u (\mathrm{mod}\ p_k^{\beta_k})} e\left(\frac{au}{p_k^{\beta_k}}\right) H(h, q' u w_k r_2 \cdots r_{n-1} z_k^n; p_k^{\alpha_k}) \\ &\quad \times \sideset{}{^*}\sum_{v (\mathrm{mod}\ m_{k-1})} e\left(\frac{av}{m_{k-1}}\right)\prod_{j=1}^{k-1} H(h, q' v p_k^{\beta_k} r_2 \cdots r_{n-1} z_j^n; p_j^{\alpha_j}). \end{aligned} \tag{2.16}$$

Write $v \equiv u m_{k-2} + v p_{k-1}^{\beta_{k-1}} \pmod{m_{k-1}}$. Then the above sum over v is equal to

$$\begin{aligned} &\sideset{}{^*}\sum_{u (\mathrm{mod}\ p_{k-1}^{\beta_{k-1}})} e\left(\frac{au}{p_{k-1}^{\beta_{k-1}}}\right) H(h, q' u w_{k-1} r_2 \cdots r_{n-1} z_{k-1}^n; p_{k-1}^{\alpha_{k-1}}) \\ &\quad \times \sideset{}{^*}\sum_{v (\mathrm{mod}\ m_{k-2})} e\left(\frac{av}{m_{k-2}}\right)\prod_{j=1}^{k-2} H(h, q' v p_k^{\beta_k} p_{k-1}^{\beta_{k-1}} r_2 \cdots r_{n-1} z_j^n; p_j^{\alpha_j}). \end{aligned} \tag{2.17}$$

In this way one finally obtains

$$\begin{aligned} K_n(a, h; \boldsymbol{q}) &= \phi(q')\phi^{-1}\left(\frac{q'}{(a,q')}\right)\mu\left(\frac{q'}{(a,q')}\right)\mu(r_2)\cdots\mu(r_{n-1}) \\ &\quad \times \prod_{j=1}^{k} \sideset{}{^*}\sum_{u_j (\mathrm{mod}\ p_j^{\beta_j})} e\left(\frac{a u_j}{p_j^{\beta_j}}\right) H(h, q' u_j r_2 \cdots r_{n-1} w_j z_j^n; p_j^{\alpha_j}). \end{aligned}$$

Note that

$$H(h, y; p^\alpha) = H(h\overline{y}, 1; p^\alpha) = K_{n-1}(\mathbf{1}, h\overline{y}, p^\alpha),$$

where $K_n(\boldsymbol{a}, b; q)$ is defined in (1.1). Applying (1.4) we get

$$
\begin{aligned}
|K_n(a,h;\boldsymbol{q})| &\le 2\phi(q')\phi^{-1}\left(\frac{q'}{(a,q')}\right)\phi(q'')\prod_{\substack{j=1\\ p_j^{\gamma_j}\|n}}^{k}(n,p_j-1)p_j^{\frac{(n-1)\alpha_j}{2}+\frac{1}{2}\min\{\gamma_j,\alpha_j-2\}}\\
&\le 2\phi(q_1)\phi^{-1}\left(\frac{q'}{(a,q')}\right)q_n^{(n-1)/2}\left(\prod_{\substack{j=1\\ p_j^{\gamma_j}|n,\gamma_j\ge 1}}^{k}p_j^{\gamma_j/2}\right)\left(\prod_{\substack{j=1\\ p_j\nmid n}}^{k}(n,p_j-1)\right)\\
&= 2\rho(n,q_n)\phi^{-1}\left(\frac{q'}{(a,q')}\right)\phi(q_1)q_n^{(n-1)/2},
\end{aligned}
$$

which proves the theorem. □

3. Proof of Theorem 1.3

To prove Theorem 1.3 we need estimates for $\Psi(x)$. To this end we record the following lemma which can be found in Ren and Ye [21]. The case $m = 3$ of the lemma can also be found in Li [15] and Ren and Ye [19]. The Rankin-Selberg case was proved by Czarnecki [2].

Lemma 3.1. *Let f be a full-level cusp form for $\mathrm{GL}_m(\mathbb{Z})$. Let $m \ge 3$ be an integer. Let $\psi(y) = \phi(y/X)$, where $\phi(x) \ll 1$ is a fixed smooth function of compact support on $[a, b]$ with $b > a > 0$. Then for $x > 0$, $xX \gg 1$ and $r > m/2$, we have*

$$
\begin{aligned}
\Psi(x) = x\sum_{k=0}^{r}c_k\int_0^\infty &(xy)^{1/(2m)-1/2-k/m}\psi(y)\\
&\times\left\{i^{k+(m-1)/2}e\left(m(xy)^{1/m}\right)+(-i)^{k+(m-1)/2}e\left(-m(xy)^{1/m}\right)\right\}dy\\
&+O\left((xX)^{-r/m+1/2+\varepsilon}\right),
\end{aligned}
\tag{3.1}
$$

where c_k $(k = 0, 1, \ldots, r)$ are constants depending on m and $\{\mu_f(j)\}$ with $c_0 = -1/\sqrt{m}$, and the implied constant depends at most on f, ϕ, r, a, b and ε.

The following lemma gives an upper bound estimate for $\Psi(x)$ without the restriction $xX \gg 1$ in Lemma 3.1.

Lemma 3.2. *Suppose that $\psi(y) = \phi(y/X)$ where ϕ is a fixed smooth function of compact support on the interval $[a, b]$ where $b > a > 0$. Let $x, X > 0$. Then for $\sigma > 1/4-1/(2(m^2+1))$ and any integer $h \ge 2m\sigma - m/2 + 1$, we have*

$$
\Psi(x) \ll_\sigma (\pi^m xX)^{-2\sigma+1}\sup_{t\in\mathbb{R}}\left|\int_a^b g_h(v)v^{-2(\sigma+it)}\,dv\right|,
\tag{3.2}
$$

where $g_0(v) = \phi(v)$ *and*

$$g_h(v) = \frac{d(vg_{h-1}(v))}{dv} \quad \text{for } h \geq 1.$$

Proof. By (1.12) we have

$$\Psi(x) = \frac{1}{2\pi i} \int_{\operatorname{Re} s = -\sigma} \widetilde{\psi}(s) x^s \frac{\widetilde{F}(1-s)}{F(s)}\, ds,$$

where

$$\widetilde{\psi}(s) = \int_0^\infty \phi\left(\frac{y}{X}\right) y^{s-1}\, dy = X^s \widetilde{\phi}(s).$$

Changing s to $2s-1$ we get

$$\Psi(x) = i\pi^{-m/2-1} \int_{\operatorname{Re} s = \sigma} G(s)(\pi^m x X)^{-2s+1} \widetilde{\phi}(-2s+1)\, ds, \tag{3.3}$$

where $\sigma > \sigma_0 = 1/4 - 1/(2(m^2+1))$ and

$$G(s) = \prod_{j=1}^m \frac{\Gamma\left(s - \frac{\overline{\mu}_f(j)}{2}\right)}{\Gamma\left(-s + \frac{1-\mu_f(j)}{2}\right)}.$$

By integrating by parts,

$$\widetilde{\phi}(-2s+1) = \int_0^\infty \phi(v) v^{-2s}\, dv = \frac{1}{(2s)^h} \int_a^b g_h(v) e^{-2s \log v}\, dv,$$

where

$$g_0(v) = \phi(v), \quad g_h(v) = \frac{d(vg_{h-1}(v))}{dv}, \quad h = 1, 2, \ldots.$$

Therefore

$$\left|\widetilde{\phi}(-2s+1)\right| \leq \frac{1}{|2s|^h} \left| \int_a^b g_h(v) e^{-2s \log v}\, dv \right|. \tag{3.4}$$

By Stirling's formula, for $|t| \geq t_0 = 2 + \sigma + \max_{1 \leq j \leq m} \{|\mu_f(j)|\}$, one has

$$\log G(s) = \left(ms - \frac{m}{2}\right) \log s - 2ms + ms \log(-s) + O(|s|^{-1}).$$

Hence

$$|G(s)| \ll |s|^{2m\sigma - m/2} e^{-2m\sigma + m|t|(\pi - 2|\arg s|)} \ll |s|^{2m\sigma - m/2}, \quad |t| \geq t_0.$$

Back to (3.3) we get

$$\begin{aligned} \Psi(x) \ll{}& (\pi^m x X)^{-2\sigma+1} \int_{\substack{\operatorname{Re} s = \sigma \\ |t| \geq t_0}} |s|^{2m\sigma - m/2} \left|\widetilde{\phi}(-2s+1)\right| |ds| \\ &+ (\pi^m x X)^{-2\sigma+1} \int_{\substack{\operatorname{Re} s = \sigma \\ |t| \leq t_0}} |G(s)| \left|\widetilde{\phi}(-2s+1)\right| |ds|. \end{aligned} \tag{3.5}$$

Applying (3.4) and choosing $h > 2m\sigma - m/2 + 1$, the first quantity on the right side above is bounded by

$$(\pi^m xX)^{-2\sigma+1} \sup_{t\in\mathbb{R}} \left| \int_a^b g_h(v) v^{-2(\sigma+it)}\, dv \right|. \tag{3.6}$$

Note that $G(s) \ll_{\sigma_0} 1$ on the segment $\operatorname{Re} s = \sigma > \sigma_0$ and $|t| \le t_0$. Applying (3.4) again we see that the second term in the right of (3.5) is also dominated by (3.6). This finishes the proof of Lemma 3.2. □

Proof of Theorem 1.3. To prove Theorem 1.3, we let $\psi(n) = e(\lambda n)\phi(n/X)$ and $c_1 = \cdots = c_{m-2} = 1$ in (1.10). Then

$$\begin{aligned}
&\sum_{n\neq 0} A_f(1,\ldots,1,n) e(\alpha n)\phi\left(\frac{n}{X}\right) \\
&= \sum_{n\neq 0} A_f(1,1,\ldots,1,n) e\left(\frac{a}{q} n\right)\psi(n) \\
&= q\sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}}\sum_{n\neq 0}\frac{A_f(n,d_{m-2},\ldots,d_1)}{d_1\cdots d_{m-2}\,|n|} K_{m-2}(n,-\bar{a};\boldsymbol{q})\Psi\left(\frac{|n|\,h(\boldsymbol{d})}{q^m}\right),
\end{aligned}$$

where $h(\boldsymbol{d}) = \prod_{i=1}^{m-2} d_i^{m-i}$, $K_{m-2}(n,-\bar{a};\boldsymbol{q})$ is as defined in (1.5) with $\boldsymbol{q} = (q_1,q_2,\ldots,q_{m-2})$ and q_i defined by (1.13), that is

$$q_i = \frac{q}{d_1 d_2 \cdots d_i}.$$

By Theorem 1.1, $K_{m-2}(n,-\bar{a};\boldsymbol{q}) = 0$ unless $q_2^{**} = \cdots = q_{m-2}^{**}$, and in this case

$$K_{m-2}(n,-\bar{a};\boldsymbol{q}) \ll q_1 q_{m-2}^{(m-3)/2} = \frac{q^{(m-1)/2}}{d_1(d_1d_2\cdots d_{m-2})^{(m-3)/2}}. \tag{3.7}$$

Therefore

$$\begin{aligned}
&\sum_{n\neq 0} A_f(1,\ldots,1,n) e(\alpha n)\phi\left(\frac{n}{X}\right) \\
&\ll q^{(m+1)/2}\sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}}\frac{1}{d_1(d_1d_2\cdots d_{m-2})^{(m-1)/2}} \\
&\quad\times\sum_{n\neq 0}\frac{|A_f(n,d_{m-2},\ldots,d_1)|}{|n|}\left|\Psi\left(\frac{|n|\,h(\boldsymbol{d})}{q^m}\right)\right|.
\end{aligned} \tag{3.8}$$

Write

$$x = \frac{|n|\,h(\boldsymbol{d})}{q^m},\quad n_0(\boldsymbol{d}) = \frac{q^m}{h(\boldsymbol{d})X},\quad n_1(\boldsymbol{d}) = \frac{(2\,|\lambda|\,qX)^m}{h(\boldsymbol{d})X}. \tag{3.9}$$

Note that for $|n| \ge n_0(\boldsymbol{d})$ one has $xX \ge 1$. Hence we can use Lemma 3.1 to bound $\Psi(x)$. For $1 \le |n| < n_0(\boldsymbol{d})$ one has $xX < 1$. We will use Lemma 3.2 to bound $\Psi(x)$. Denote by $S(X)$ and $T(X)$ the sums in (3.8) corresponding to $1 \le |n| < n_0(\boldsymbol{d})$ and $|n| \ge n_0(\boldsymbol{d})$, respectively. Then

$$\sum_{n\neq 0} A_f(1,\dots,1,n)e(\alpha n)\phi\left(\frac{n}{X}\right) = S(X) + T(X). \tag{3.10}$$

Suppose that $q^m \le X$. Then $n_0(\boldsymbol{d}) \le 1$ for all $\boldsymbol{d}$. Therefore we get

$$S(X) = 0. \tag{3.11}$$

To bound $T(X)$, we apply Lemma 3.1 and changing variable $y = Xt$ to get

$$\Psi(x) = \Psi_1(x) + O((xX)^{-r/m+1/2+\varepsilon}), \tag{3.12}$$

where $r > m/2$ and

$$\Psi_1(x) = \sum_{k=0}^{r} c_k (xX)^{1/(2m)+1/2-k/m}\left\{I_+(x) + I_-(x)\right\} \tag{3.13}$$

with

$$I_\pm(x) = (\pm i)^{k+(m-1)/2}\int_0^\infty t^{1/(2m)-1/2-k/m}\phi(t)e(\lambda Xt \pm m(xXt)^{1/m})\,dt. \tag{3.14}$$

Let $f(t) = \lambda Xt \pm m(xXt)^{1/m}$. Then for $|n| \ge n_1(\boldsymbol{d})$, one has

$$f'(t) = \lambda X \pm (xX)^{1/m}t^{1/m-1} \gg (xX)^{1/m}.$$

By repeated integrating by parts and using the fact that ϕ is supported on $[1,2]$ and $\phi^{(k)}(t) \ll 1$, we get

$$I_\pm(x) \ll (xX)^{-\ell/m} \quad \text{for any integer } \ell \ge 0.$$

Choosing $\ell = r+1$ one obtains $\Psi_1(x) \ll_r (xX)^{-r/m+1/2}$ for $|n| \ge n_1(\boldsymbol{d})$. Therefore

$$T(X) \ll \mathfrak{T}_1(X) + \mathfrak{T}_2(X), \tag{3.15}$$

where

$$\mathfrak{T}_1(X) = q^{(m+1)/2}\sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}}\frac{1}{d_1(d_1\cdots d_{m-2})^{(m-1)/2}} \times \sum_{1\le|n|<n_1(\boldsymbol{d})}\frac{|A_f(n,d_{m-2},\dots,d_1)|}{|n|}|\Psi_1(x)|$$

and

$$\mathfrak{T}_2(X) = \sqrt{qX}\left(\frac{X}{q^m}\right)^{-r/m+\varepsilon}\sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}}\frac{(h(\boldsymbol{d}))^{1/2-r/m+\varepsilon}}{d_1(d_1\cdots d_{m-2})^{(m-1)/2}} \times \sum_{|n|\neq 0}\frac{|A_f(n,d_{m-2},\dots,d_1)|}{|n|^{1/2+r/m+\varepsilon}}. \tag{3.16}$$

By (1.15) and Cauchy's inequality, for $\theta > 1$ and $Y > 1$,

$$\sum_{Y<|n|\le 2Y} \frac{|A_f(n, d_{m-2}, \dots, d_1)|}{|n|^\theta} \ll Y^{1-\theta}(h(\boldsymbol{d}))^{1/2}.$$

Thus for $r > m/2$, the sum over n in (3.16) is $\ll \sqrt{h(\boldsymbol{d})} = d_1^{(m-1)/2} d_2^{(m-2)/2} \cdots d_{m-2}$. This shows

$$\mathfrak{T}_2(X) \ll (qX)^{1/2+\varepsilon} \left(\frac{X}{q^m}\right)^{-r/m}. \tag{3.17}$$

To bound $\mathfrak{T}_1(X)$, we distinguish two cases according to $(2|\lambda| qX)^m \le X$ or not.

(a) Suppose $(2|\lambda| qX)^m \le X$, then $\mathfrak{T}_1(X)$ disappears since now $n_1(\boldsymbol{d}) \le 1$ for all $\boldsymbol{d}$. In this case we obtain

$$\sum_{n\neq 0} A_f(1, \dots, 1, n) e(\alpha n) \phi\left(\frac{n}{X}\right) \ll (qX)^{1/2+\varepsilon} \left(\frac{X}{q^m}\right)^{-r/m}. \tag{3.18}$$

(b) Suppose $(2|\lambda| qX)^m > X$. Then $n_1(\boldsymbol{d}) > 1$ when $h(\boldsymbol{d}) < (2|\lambda| qX)^m X^{-1}$. One has $I_\pm(x) \ll (xX)^{-1/(2m)}$, by the second derivative test. Hence $\Psi_1(x) \ll (xX)^{1/2}$ and

$$\mathfrak{T}_1(X) \ll (qX)^{1/2} \sum_{d_1|q} \sum_{d_2|q_1} \cdots \sum_{d_{m-2}|q_{m-3}} \frac{\sqrt{h(\boldsymbol{d})}}{d_1(d_1 \cdots d_{m-2})^{(m-1)/2}} \times \sum_{1\le|n|\le n_1(\boldsymbol{d})} \frac{|A_f(n, d_{m-2}, \dots, d_1)|}{\sqrt{|n|}}.$$

By (1.15), the above sum over n is

$$\ll \sqrt{h(\boldsymbol{d}) n_1(\boldsymbol{d})} \ll (|\lambda| qX)^{m/2} X^{-1/2}.$$

Thus we get

$$\mathfrak{T}_1(X) \ll q^{1/2+\varepsilon}(|\lambda| qX)^{m/2}.$$

This together with (3.10), (3.11), (3.15) and (3.17) shows that

$$\sum_{n\neq 0} A_f(1, \dots, 1, n) e(\alpha n) \phi\left(\frac{n}{X}\right) \ll q^{1/2+\varepsilon}(|\lambda| qX)^{m/2}.$$

Suppose $q^m > X$, then $n_0(\boldsymbol{d}) > 1$ whenever $h(\boldsymbol{d}) < q^m X^{-1}$. By Lemma 3.2, and choosing $\sigma = 1/4 - 1/(2(m^2+1)) + \varepsilon$, $h = 1$, we get

$$\Psi(x) \ll (xX)^{-2\sigma+1} \sup_{t\in\mathbb{R}} \left| \int_1^2 (v\phi(v) e(\lambda X v))' v^{-2(\sigma+it)}\, dv \right| \ll_m (1 + |\lambda| X)(xX)^{-2\sigma+1}.$$

This gives

$$S(X) \ll (qX)^{1/2}(1+|\lambda|\,X)\left(\frac{X}{q^m}\right)^{1/(m^2+1)-2\varepsilon} \times \sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}} \frac{h(\boldsymbol{d})^{1/2+1/(m^2+1)-2\varepsilon}}{d_1(d_1\cdots d_{m-2})^{(m-1)/2}} \sum_{1<|n|\le n_0(\boldsymbol{d})} \frac{|A_f(n,d_{m-2},\ldots,d_1)|}{|n|^{1/2-1/(m^2+1)+2\varepsilon}}.$$

Note that $n_0(\boldsymbol{d})h(\boldsymbol{d}) = q^m X^{-1}$. By (1.15), the last sum is

$$\ll n_0(\boldsymbol{d})^{1/2+1/(m^2+1)-2\varepsilon} h(\boldsymbol{d})^{1/2} = \left(\frac{q^m}{X}\right)^{1/2+1/(m^2+1)-2\varepsilon} h(\boldsymbol{d})^{-1/(m^2+1)+2\varepsilon}.$$

Thus

$$\begin{aligned} S(X) &\ll q^{(m+1)/2}(1+|\lambda|\,X)\sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}} \frac{\sqrt{h(\boldsymbol{d})}}{d_1(d_1\cdots d_{m-2})^{(m-1)/2}} \\ &\ll q^{(m+1)/2+\varepsilon}(1+|\lambda|\,X). \end{aligned} \tag{3.19}$$

To bound $T(X)$ we follow the argument from (3.10) to (3.15) to obtain

$$T(X) \ll \mathfrak{R}_1(X) + \mathfrak{R}_2(X)$$

where

$$\mathfrak{R}_1(X) = q^{(m+1)/2}\sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}} \frac{1}{d_1(d_1\cdots d_{m-2})^{(m-1)/2}} \times \sum_{n_0(\boldsymbol{d})\le|n|<n_1(\boldsymbol{d})} \frac{|A_f(n,d_{m-2},\ldots,d_1)|}{|n|}\,|\Psi_1(x)|$$

and

$$\mathfrak{R}_2(X) = (qX)^{1/2}\left(\frac{X}{q^m}\right)^{-r/m+\varepsilon}\sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}} \frac{(h(\boldsymbol{d}))^{1/2-r/m+\varepsilon}}{d_1(d_1\cdots d_{m-2})^{(m-1)/2}} \times \sum_{|n|\ge n_0(\boldsymbol{d})} \frac{|A_f(n,d_{m-2},\ldots,d_1)|}{|n|^{1/2+r/m-\varepsilon}}.$$

The sum over n in $\mathfrak{R}_2(X)$ is

$$\ll n_0(\boldsymbol{d})^{1/2-r/m+\varepsilon} h(\boldsymbol{d})^{1/2} = \left(\frac{q^m}{X}\right)^{1/2-r/m+\varepsilon} h(\boldsymbol{d})^{r/m-\varepsilon}$$

which gives

$$\mathfrak{R}_2(X) \ll q^{(m+1)/2}\sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}} \frac{\sqrt{h(\boldsymbol{d})}}{d_1(d_1\cdots d_{m-2})^{(m-1)/2}} \ll q^{(m+1)/2+\varepsilon}.$$

To estimate $\mathfrak{R}_1(X)$, we distinguish two cases according to $|\lambda| X \leq 1/2$ or not. Suppose $|\lambda| X \leq 1/2$. Then $n_0(\boldsymbol{d}) \geq n_1(\boldsymbol{d})$, hence $\mathfrak{R}_1(X)$ disappears and we get $T(X) \ll q^{(m+1)/2+\varepsilon}$. This together with (3.19) proves

$$\sum_{n\neq 0} A_f(1,\ldots,1,n)e(\alpha n)\phi\left(\frac{n}{X}\right) \ll q^{(m+1)/2+\varepsilon}.$$

Suppose $|\lambda| X > 1/2$. Then $I_\pm(x) \ll (xX)^{-1/(2m)}$, by the second derivative test. Thus $\Psi_1(x) \ll (xX)^{1/2}$ and

$$\mathfrak{R}_1(X) \ll (qX)^{1/2} \sum_{d_1|q}\sum_{d_2|q_1}\cdots\sum_{d_{m-2}|q_{m-3}} \frac{\sqrt{h(\boldsymbol{d})}}{d_1(d_1\cdots d_{m-2})^{(m-1)/2}} \times \sum_{n_0(\boldsymbol{d})\leq|n|\leq n_1(\boldsymbol{d})} \frac{|A_f(n,d_{m-2},\ldots,d_1)|}{\sqrt{|n|}}.$$

The last sum is $\ll (n_1(\boldsymbol{d})h(\boldsymbol{d}))^{1/2} = ((2|\lambda| qX)^m X^{-1})^{1/2}$ which gives

$$\mathfrak{R}_1(X) \ll q^{1/2+\varepsilon}(|\lambda| qX)^{m/2}.$$

Hence

$$T(X) \ll q^{1/2+\varepsilon}(|\lambda| qX)^{m/2} + q^{(m+1)/2+\varepsilon} \ll q^{1/2+\varepsilon}(|\lambda| qX)^{m/2}.$$

This together with (3.19) shows that

$$\sum_{n\neq 0} A_f(1,\ldots,1,n)e(\alpha n)\phi\left(\frac{n}{X}\right) \ll q^{(m+1)/2+\varepsilon}\left((|\lambda| X)^{m/2}+1\right). \qquad \square$$

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