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A Characterization of Multipliers of a Lau Algebra Constructed by Semisimple Commutative Banach Algebras

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Abstract. A necessary and sufficient condition for a Lau type binary operation defined by two mappings to be an algebra-operation is given in terms of multipliers. Also a characterization of multipliers of a Lau algebra constructed by semisimple commutative Banach algebras is given in terms of multipliers of original Banach algebras.

1. Introduction

In 2007, Sangani Monfared introduced a product $\times_\theta$ on the Cartesian product $A \times B$ of two Banach algebras A and B, which is of the form

$$(a, b) \times_\theta (c, d) = (ac + \theta(d)a + \theta(b)c, bd),$$

where θ is a multiplicative linear functional on B. He investigated the Banach algebra $(A \times B, \times_\theta)$ in [4]. This type of product was first introduced by A. Lau [3] for a special class of Banach algebras in 1983. After Lau, a product $\times_\theta$ is called a *θ-Lau product* and the algebra $(A \times B, \times_\theta)$, abbreviated to $A \times_\theta B$, is called a *θ-Lau Banach algebra*. Several mathematicians have studied τ-Lau Banach algebras $A \times_\tau B$ defined by a norm-decreasing homomorphism τ from B into A instead of θ. We will note that the unitization and the direct product are special cases of a Lau product. In this paper, we first give a necessary and sufficient condition for a Lau type binary operation defined by two mappings to be an algebra-operation in terms of multipliers. Secondly, we give a characterization of multipliers of the Lau algebra constructed by semisimple commutative Banach algebras in terms of multipliers of original Banach algebras. This extends a characterization obtained by P. A. Dabhi [1].

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2. Lau type binary operations

Let A and B be algebras. Then the Cartesian product $A \times B$ becomes a linear space with pointwise operations. Let $\mathcal{F}_0(A)$ be the set of all mappings ρ from A into itself such that $\rho(0) = 0$. Then $\mathcal{F}_0(A)$ becomes a linear space with pointwise operations:

$$(\rho + \sigma)(a) = \rho(a) + \sigma(a) \quad \text{and} \quad (\lambda\rho)(a) = \lambda\rho(a)$$

for $a \in A$, $\rho, \sigma \in \mathcal{F}_0(A)$ and $\lambda \in \mathbb{C}$. Also a mapping ρ in $\mathcal{F}_0(A)$ is called a *left* (resp. *right*) *multiplier* of A if $\rho(xy) = \rho(x)y$ (resp. $\rho(xy) = x\rho(y)$) holds for all $x, y \in A$. Also an ordered pair (τ, σ) of mappings in $\mathcal{F}_0(A)$ is called a *double multiplier* if $x\tau(y) = \sigma(x)y$ holds for all $x, y \in A$. In particular, the algebra of all linear mappings from A into itself is denoted by $\mathcal{L}(A)$.

For two mappings $S\colon d \mapsto S_d$ and $T\colon b \mapsto T_b$ from B into $\mathcal{F}_0(A)$, we define

$$(a, b) \times_{S,T} (c, d) = (ac + S_d a + T_b c, bd)$$

for each $(a, b), (c, d) \in A \times B$. Then $\times_{S,T}$ is a binary operation on $A \times B$.

Theorem 2.1. *Let S and T be as above. Then $\times_{S,T}$ is an algebra-operation on $A \times B$ if and only if the following conditions hold:*

(i) *S (resp. T) is an anti-homomorphism (resp. a homomorphism) from B into $\mathcal{L}(A)$.*

(ii) *S_b (resp. T_b) is a right (resp. left) multiplier of A for all $b \in B$.*

(iii) *$S_b T_d = T_d S_b$ holds for all $b, d \in B$.*

(iv) *(T_b, S_b) is a double multiplier of A for all $b \in B$.*

Proof. Suppose that $\times_{S,T}$ is an algebra-operation on $A \times B$. Since

$$\begin{aligned}(e, f) \times_{S,T} ((a, b) + (c, d)) &= (e, f) \times_{S,T} (a + c, b + d) \\ &= (e(a + c) + S_{b+d} e + T_f (a + c), f(b + d))\end{aligned}$$

and since

$$\begin{aligned}&(e, f) \times_{S,T} (a, b) + (e, f) \times_{S,T} (c, d) \\ &= (ea + S_b e + T_f a, fb) + (ec + S_d e + T_f c, fd) \\ &= (e(a + c) + (S_b + S_d)e + T_f a + T_f c, f(b + d)),\end{aligned}$$

it follows that

$$S_{b+d} e + T_f (a + c) = (S_b + S_d)e + T_f a + T_f c$$

for all $(a,b),(c,d),(e,f) \in A \times B$. Putting $e = 0$ in the above equation, we see that T_f is additive for each $f \in B$. Also putting $a = c = 0$ in the same equation, we see that S is additive. Similarly, considering the equation

$$((a,b)+(c,d)) \times_{S,T} (e,f) = (a,b) \times_{S,T} (e,f) + (c,d) \times_{S,T} (e,f),$$

we see that T and S_f $(f \in B)$ are additive. Also since

$$\lambda((a,b) \times_{S,T} (c,d)) = (\lambda ac + \lambda S_d a + \lambda T_b c, \lambda bd)$$

and

$$(a,b) \times_{S,T} (\lambda(c,d)) = (a,b) \times_{S,T} (\lambda c, \lambda d) = (\lambda ac + S_{\lambda d} a + T_b(\lambda c), \lambda bd),$$

it follows that

$$\lambda S_d a + \lambda T_b c = S_{\lambda d} a + T_b(\lambda c)$$

for all $(a,b),(c,d) \in A \times B$ and $\lambda \in \mathbb{C}$. Putting $a = 0$ in the above equation, we see that T_b is homogeneous for each $b \in B$. Also putting $c = 0$ in the same equation, we see that S is homogeneous. Similarly, considering the equation

$$\lambda((a,b) \times_{S,T} (c,d)) = (\lambda(a,b)) \times_{S,T} (c,d),$$

we see that T and S_d $(d \in B)$ are homogeneous. Consequently, we obtain that both S and T are linear mappings from B into $\mathcal{L}(A)$.

Now for $(a,b),(c,d),(e,f) \in A \times B$, we have

$$\begin{aligned}
&((a,b) \times_{S,T} (c,d)) \times_{S,T} (e,f)\\
&= (ac + S_d a + T_b c, bd) \times_{S,T} (e,f)\\
&= (ace + (S_d a)e + (T_b c)e + S_f(ac + S_d a + T_b c) + T_{bd} e, bdf)\\
&= (ace + (S_d a)e + (T_b c)e + S_f(ac) + S_f(S_d a) + S_f(T_b c) + T_{bd} e, bdf)
\end{aligned}$$

and

$$\begin{aligned}
&(a,b) \times_{S,T} ((c,d) \times_{S,T} (e,f))\\
&= (a,b) \times_{S,T} (ce + S_f c + T_d e, df)\\
&= (ace + a(S_f c) + a(T_d e) + S_{df} a + T_b(ce + S_f c + T_d e), bdf)\\
&= (ace + a(S_f c) + a(T_d e) + S_{df} a + T_b(ce) + T_b(S_f c) + T_b(T_d e), bdf).
\end{aligned}$$

Therefore $\times_{S,T}$ is associative if and only if

$$\begin{aligned}
&(S_d a)e + (T_b c)e + S_f(ac) + S_f(S_d a) + S_f(T_b c) + T_{bd} e\\
&= a(S_f c) + a(T_d e) + S_{df} a + T_b(ce) + T_b(S_f c) + T_b(T_d e)
\end{aligned} \tag{2.1}$$

holds for all $(a,b),(c,d),(e,f) \in A \times B$. Putting $e = 0$ in (2.1), we have

$$S_f(ac) + S_f(S_d a) + S_f(T_b c) = a(S_f c) + S_{df} a + T_b(S_f c) \tag{2.2}$$

for all $a, c \in A$ and $b, d, f \in B$. Putting $c = 0$ in (2.2), we have

$$S_f(S_d a) = S_{df} a \tag{2.3}$$

for all $a \in A$ and $d, f \in B$. Putting $a = 0$ in (2.2), we have

$$S_f(T_b c) = T_b(S_f c) \tag{2.4}$$

for all $c \in A$ and $b, f \in B$. By (2.2), (2.3) and (2.4), we have

$$S_f(ac) = a(S_f c) \tag{2.5}$$

for all $a, c \in A$ and $f \in B$. By (2.3), (2.4) and (2.5), the equation (2.1) becomes

$$(S_d a)e + (T_b c)e + T_{bd} e = a(T_d e) + T_b(ce) + T_b(T_d e) \tag{2.6}$$

for all $a, c, e \in A$ and $b, d \in B$. Putting $a = c = 0$ in (2.6), we have

$$T_{bd} e = T_b(T_d e) \tag{2.7}$$

for all $e \in A$ and $b, d \in B$. By (2.7) and (2.6), we have

$$(S_d a)e + (T_b c)e = a(T_d e) + T_b(ce) \tag{2.8}$$

for all $a, c, e \in A$ and $b, d \in B$. Putting $c = 0$ in (2.8), we have

$$(S_d a)e = a(T_d e) \tag{2.9}$$

for all $a, e \in A$ and $d \in B$. By (2.8) and (2.9), we have

$$(T_b c)e = T_b(ce) \tag{2.10}$$

for all $c, e \in A$ and $b \in B$. Therefore (2.1) implies (2.3), (2.4), (2.5), (2.7), (2.9) and (2.10). Conversely, we can easily see that (2.3), (2.4), (2.5), (2.7), (2.9) and (2.10) imply (2.1).

Also we have the following equivalences:

$$\begin{aligned}
(2.3) \quad &\Longleftrightarrow \quad S_{df} = S_f S_d \ (d, f \in B) \\
&\Longleftrightarrow \quad S \text{ is an anti-homomorphism from } B \text{ into } \mathcal{L}(A). \\
(2.4) \quad &\Longleftrightarrow \quad S_f T_b = T_b S_f \ (b, f \in B). \\
(2.5) \quad &\Longleftrightarrow \quad \text{each } S_b \text{ is a right multiplier of } A \ (b \in B). \\
(2.7) \quad &\Longleftrightarrow \quad T_{bd} = T_b T_d \ (b, d \in B) \\
&\Longleftrightarrow \quad T \text{ is an algebra-homomorphism from } B \text{ to } \mathcal{L}(A). \\
(2.9) \quad &\Longleftrightarrow \quad \text{each } (T_b, S_b) \text{ is a double multiplier of } A \ (b \in B). \\
(2.10) \quad &\Longleftrightarrow \quad \text{each } T_b \text{ is a left multiplier of } A \ (b \in B).
\end{aligned}$$

Therefore we obtain the desired conditions (i)–(iv).

Conversely, suppose that (i)–(iv) hold. By a similar argument, we can easily see that $\times_{S,T}$ is an algebra-operation on $A \times B$. □

If $\times_{S,T}$ is an algebra-operation on $A \times B$, then we call $(A \times B, \times_{S,T})$ a *Lau algebra* defined by S and T, and denote $(A \times B, \times_{S,T})$ by $A \times_{S,T} B$.

Now let $M(A)$ be the set of all double multipliers of A. Then it becomes an algebra with natural operations:

$$\begin{aligned}(T_1, S_1) + (T_2, S_2) &= (T_1 + T_2, S_1 + S_2),\\ \lambda(T_1, S_1) &= (\lambda T_1, \lambda S_1),\\ (T_1, S_1)(T_2, S_2) &= (T_1T_2, S_2S_1).\end{aligned}$$

Also we denote by $M_l(A)$ and $M_r(A)$ the algebra of all left multipliers of A and the algebra of all right multipliers of A, respectively. If a left annihilator of A is only zero or if a right annihilator of A is only zero, then A is said to be *without order*.

Lemma 2.2. *Assume that A is without order. If $(T, S), (T', S') \in M(A)$, $T, T' \in M_l(A)$ and $S, S' \in M_r(A)$, then $TS' = S'T$.*

Proof. First assume that a left annihilator of A is only zero. Since

$$\begin{aligned}(TS')(x)y &= T(S'x)y = T((S'x)y) = T(xT'y)\\ &= (Tx)(T'y) = S'(Tx)y = (S'T)(x)y\end{aligned}$$

for all $x, y \in A$, the assumption implies that $TS' = S'T$. Assume next that a right annihilator is only zero. Since

$$\begin{aligned}y(TS')x &= y(T(S'x)) = (Sy)(S'x) = S'((Sy)x)\\ &= S'(yTx) = yS'(Tx) = y(S'T)x\end{aligned}$$

for all $x, y \in A$, the assumption implies that $TS' = S'T$. □

A semisimple Banach algebra is, of course, without order. It is known that if A is a semisimple Banach algebra and $(T, S) \in M(A)$, then:

(v) T is a left multiplier of A and S is a right multiplier of A.

(vi) T and S are bounded linear operators on A.

From Theorem 2.1, Lemma 2.2 and the above facts, we obtain the following.

Corollary 2.3. *Assume that A is a semisimple Banach algebra. Then $\times_{S,T}$ is an algebra-operation on $A \times B$ if and only if the mapping $b \mapsto (T_b, S_b)$ is a homomorphism from B into $M(A)$.*

Assume that A is a semisimple commutative Banach algebra. If $(T, S) \in M(A)$, then $T = S$. Indeed, since

$$z(Tx)y = zyTx = z(Sy)x = x(Sy)z = xyTz = yxTz = y(Sx)z = z(Sx)y$$

for all $x, y, z \in A$, it follows from the semisimplicity of A that $T = S$ as required. As a consequence, $M(A)$ becomes the usual multiplier algebra of A. Therefore for any two mappings $S, T\colon B \to \mathcal{F}_0(A)$, $\times_{S,T}$ is an algebra-operation on $A \times B$ if and only if $S = T$ and T is a homomorphism from B into $M(A)$. In this case, we write $\times_T$ for $\times_{T,T}$ and $A \times_T B$ for $A \times_{T,T} B$. We can easily see that if B is commutative, then $A \times_T B$ is also commutative.

3. A characterization of multipliers of Lau algebras

In this section, we focus on the semisimple commutative Banach algebras. Let A and B be semisimple commutative Banach algebras. By Φ_A and Φ_B, we denote the Gelfand spaces of A and B, respectively. Let $M(A)$ be the multiplier algebra of A with Gelfand space $\Phi_{M(A)}$. Put $L_a(x) = ax$ for each $a, x \in A$. Then L_a is a multiplier of A. We sometimes identify L_a with a. Then A is an ideal of $M(A)$. Let T be a norm-decreasing homomorphism from B into $M(A)$. Then the Lau algebra $A \times_T B$ becomes a commutative Banach algebra with the l^1-norm:

$$\|(a, b)\| = \|a\| + \|b\| \quad ((a, b) \in A \times B).$$

For any $\varphi \in A^*$, the dual space of A, and for any $\psi \in B^*$, the dual space of B, we put

$$(\varphi, \psi)(a, b) = \varphi(a) + \psi(b) \quad ((a, b) \in A \times B).$$

Then (φ, ψ) is a continuous linear functional on $A \times_T B$ with the norm $\max\{\|\varphi\|, \|\psi\|\}$. Let $\varphi \in \Phi_A$. Choose $e_\varphi \in A$ with $\varphi(e_\varphi) = 1$ and put

$$\widetilde{\varphi}(S) = \varphi(Se_\varphi)$$

for all $S \in M(A)$. Here $\widetilde{\varphi}$ does not depend on a choice of e_φ. Indeed, if $a \in A$ with $\varphi(a) = 1$, then

$$\varphi(Sa) = \varphi(e_\varphi Sa) = \varphi((Se_\varphi)a) = \varphi(Se_\varphi)\varphi(a) = \varphi(Se_\varphi).$$

We have the following.

Lemma 3.1. *Let $\varphi \in \Phi_A$. Then $\widetilde{\varphi} \in \Phi_{M(A)}$ and $(\varphi, \widetilde{\varphi} \circ T) \in \Phi_{A \times_T B}$.*

Proof. (i) Observe that $\widetilde{\varphi}$ is a nonzero continuous linear functional on $M(A)$. If $S_1, S_2 \in M(A)$, then

$$\begin{aligned}\widetilde{\varphi}(S_1 S_2) &= \varphi(S_1(S_2(e_\varphi))) = \varphi(e_\varphi S_1(S_2(e_\varphi))) \\ &= \varphi(S_1(e_\varphi) S_2(e_\varphi)) = \varphi(S_1 e_\varphi)\varphi(S_2 e_\varphi) = \widetilde{\varphi}(S_1)\widetilde{\varphi}(S_2)\end{aligned}$$

for all $S_1, S_2 \in M(A)$, and hence $\widetilde{\varphi} \in \Phi_{M(A)}$.

(ii) By (i), we have $\widetilde{\varphi} \circ T \in B^*$ and hence $(\varphi, \widetilde{\varphi} \circ T)$ is a nonzero continuous linear functional on $A \times_T B$. We next show that $(\varphi, \widetilde{\varphi} \circ T)$ is multiplicative. To do this, let $(a, b), (c, d) \in A \times_T B$. Then

$$\begin{aligned}(\varphi, \widetilde{\varphi} \circ T)((a, b) \times_T (c, d)) &= (\varphi, \widetilde{\varphi} \circ T)(ac + T_d a + T_b c, bd) \\ &= \varphi(ac) + \varphi(T_d a) + \varphi(T_b c) + (\widetilde{\varphi} \circ T)(bd) \\ &= \varphi(a)\varphi(c) + \varphi(T_d a) + \varphi(T_b c) + \widetilde{\varphi}(T_{bd}) \\ &= \varphi(a)\varphi(c) + \varphi(T_d a) + \varphi(T_b c) + \widetilde{\varphi}(T_b)\widetilde{\varphi}(T_d)\end{aligned}$$

and

$$\begin{aligned}(\varphi, \widetilde{\varphi} \circ T)(a, b)(\varphi, \widetilde{\varphi} \circ T)(c, d) &= (\varphi(a) + \widetilde{\varphi}(T_b))(\varphi(c) + \widetilde{\varphi}(T_d)) \\ &= \varphi(a)\varphi(c) + \varphi(a)\widetilde{\varphi}(T_d) + \varphi(c)\widetilde{\varphi}(T_b) + \widetilde{\varphi}(T_b)\widetilde{\varphi}(T_d) \\ &= \varphi(a)\varphi(c) + \varphi(a)\varphi(T_d e_\varphi) + \varphi(c)\varphi(T_b e_\varphi) + \widetilde{\varphi}(T_b)\widetilde{\varphi}(T_d) \\ &= \varphi(a)\varphi(c) + \varphi(e_\varphi T_d a) + \varphi(e_\varphi T_b c) + \widetilde{\varphi}(T_b)\widetilde{\varphi}(T_d) \\ &= \varphi(a)\varphi(c) + \varphi(T_d a) + \varphi(T_b c) + \widetilde{\varphi}(T_b)\widetilde{\varphi}(T_d).\end{aligned}$$

Therefore

$$(\varphi, \widetilde{\varphi} \circ T)((a, b) \times_T (c, d)) = (\varphi, \widetilde{\varphi} \circ T)(a, b)(\varphi, \widetilde{\varphi} \circ T)(c, d)$$

holds. Consequently, $(\varphi, \widetilde{\varphi} \circ T) \in \Phi_{A \times_T B}$. □

By the above lemma, we have $\{(\varphi, \widetilde{\varphi} \circ T) : \varphi \in \Phi_A\} \subset \Phi_{A \times_T B}$. Also observe that if $\psi \in \Phi_B$, then $(0, \psi) \in \Phi_{A \times_T B}$. Then we have $\{(0, \psi) : \psi \in \Phi_B\} \subset \Phi_{A \times_T B}$. Put

$$E = \{(\varphi, \widetilde{\varphi} \circ T) : \varphi \in \Phi_A\} \quad \text{and} \quad F = \{(0, \psi) : \psi \in \Phi_B\}.$$

Then we have the following.

Lemma 3.2. *The set E (resp. F) is open (resp. closed) in $\Phi_{A \times_T B}$ and $\Phi_{A \times_T B} = E \cup F$ (disjoint union).*

Proof. Take $f \in \Phi_{A\times_T B}$ arbitrarily. Assume that $f|_{A\times\{0\}} \neq 0$. Put

$$\varphi(a) = f(a,0)$$

for each $a \in A$. Then $\varphi \in \Phi_A$. Moreover we have

$$\begin{aligned}(\varphi, \widetilde{\varphi}\circ T)(a,b) &= \varphi(a) + \widetilde{\varphi}(T_b) = f(a,0) + \varphi(T_b e_\varphi)\\ &= f(a,0) + f(T_b e_\varphi, 0) = f(a,0) + f((e_\varphi,0)\times_T (0,b))\\ &= f(a,0) + f(e_\varphi,0)f(0,b) = f(a,0) + \varphi(e_\varphi)f(0,b) = f(a,b)\end{aligned}$$

for all $(a,b) \in A\times_T B$. In other words, $(\varphi, \widetilde{\varphi}\circ T) = f$.

Next assume that $f|_{A\times\{0\}} = 0$. Put

$$\psi(b) = f(0,b)$$

for each $b \in B$. Then ψ is a multiplicative linear functional on B. Since

$$\psi(b) = f(0,b) = f(a,0) + f(0,b) = f(a,b)$$

for all $(a,b) \in A\times_T B$, it follows that $\psi \in \Phi_B$ and $f = (0,\psi)$. These observations imply $\Phi_{A\times_T B} = E \cup F$. It is evident that $E \cap F = \varnothing$. Also it is easy to see that F is closed in $\Phi_{A\times_T B}$, and hence E is open. $\square$

Lemma 3.3. *The mapping $\varphi \mapsto (\varphi, \widetilde{\varphi}\circ T)$ (resp. $\psi \to (0,\psi)$) is a homeomorphism from Φ_A (resp. Φ_B) onto E (resp. F).*

Proof. It is clear that the mapping $\varphi \mapsto (\varphi, \widetilde{\varphi}\circ T)$ is a bijection from Φ_A onto E. Also this mapping is continuous. To see this, let $\{\varphi_\lambda\}$ be a net in Φ_A which converges to $\varphi \in \Phi_A$. Take $(a,b) \in A\times_T B$ arbitrarily. Then $\lim_\lambda \varphi_\lambda(e_\varphi) = \varphi(e_\varphi) = 1$. Also we have

$$\begin{aligned}\lim_\lambda \varphi_\lambda(T_b(e_{\varphi_\lambda}))\varphi_\lambda(e_\varphi) &= \lim_\lambda \varphi_\lambda(T_b(e_{\varphi_\lambda})e_\varphi) = \lim_\lambda \varphi_\lambda(T_b(e_\varphi)e_{\varphi_\lambda})\\ &= \lim_\lambda \varphi_\lambda(T_b e_\varphi) = \varphi(T_b e_\varphi) = \widetilde{\varphi}(T_b),\end{aligned}$$

and hence $\lim_\lambda \varphi_\lambda(T_b(e_{\varphi_\lambda})) = \widetilde{\varphi}(T_b)$. Therefore

$$\begin{aligned}\lim_\lambda(\varphi_\lambda, \widetilde{\varphi}_\lambda\circ T)(a,b) &= \lim_\lambda \varphi_\lambda(a) + \lim_\lambda \widetilde{\varphi}_\lambda(T_b) = \varphi(a) + \lim_\lambda \varphi_\lambda(T_b(e_{\varphi_\lambda}))\\ &= \varphi(a) + \widetilde{\varphi}(T_b) = (\varphi, \widetilde{\varphi}\circ T)(a,b)\end{aligned}$$

holds for all $(a,b) \in A\times_T B$. In other words, $\lim_\lambda(\varphi_\lambda, \widetilde{\varphi}_\lambda\circ T) = (\varphi, \widetilde{\varphi}\circ T)$. It is evident that the inverse mapping is continuous.

Moreover, it will be obvious that the mapping $\psi \mapsto (0,\psi)$ is a homeomorphism from Φ_B onto F. $\square$

Hereafter, according to the above lemma, we may identify Φ_A and Φ_B with E and F, respectively. Moreover, we may identify $A \times \{0\}$ and $\{0\} \times B$ with A and B, respectively. Thus Lemma 3.2 is restated as follows.

Lemma 3.4. $\Phi_{A\times_T B} = \Phi_A \cup \Phi_B$ *(disjoint union).*

The above disjoint union implies that the commutative Banach algebra $A \times_T B$ is semisimple. Also note that Φ_A is an open subset of $\Phi_{A\times_T B}$ and Φ_B is a closed subset of $\Phi_{A\times_T B}$.

Now if S is a bounded linear mapping from $A \times_T B$ into itself, then there exist a unique pair of bounded linear mappings $S_1\colon A \times_T B \to A$ and $S_2\colon A \times_T B \to B$ such that $S(a, b) = (S_1(a, b), S_2(a, b))$ for all $(a, b) \in A \times_T B$. We will express this by

$$S = (S_1, S_2).$$

The next theorem describes the multipliers of $A \times_T B$ completely.

Theorem 3.5. *Let S be a bounded linear mapping from $A \times_T B$ into itself with $S = (S_1, S_2)$. Then $S \in M(A \times_T B)$ if and only if S_1 and S_2 satisfy the following conditions:*

(i) $S_1|_A \in M(A)$.

(ii) $S_2|_B \in M(B)$.

(iii) $S_2|_A = 0$.

(iv) $(S_1 b)a = T_b(S_1 a) - T_{S_2 b}(a)$ *for all $a \in A$ and $b \in B$.*

Proof. First assume $S \in M(A \times_T B)$. Let $(a, b), (c, d) \in A \times_T B$. Then

$$\begin{aligned}(a, b) \times_T (S_1(c, d), S_2(c, d)) &= (a, b) \times_T S(c, d)\\ &= (S(a, b)) \times_T (c, d)\\ &= (S_1(a, b), S_2(a, b)) \times_T (c, d).\end{aligned}$$

Therefore it follows that

$$aS_1(c, d) + T_{S_2(c,d)}(a) + T_b(S_1(c, d)) = S_1(a, b)c + T_d(S_1(a, b)) + T_{S_2(a,b)}(c)$$

and

$$bS_2(c, d) = S_2(a, b)d.$$

Taking $b = d = 0$, we have

$$aS_1 c + T_{S_2 c}(a) = (S_1 a)c + T_{S_2 a}(c). \tag{3.1}$$

Taking $a = c = 0$, we get

$$T_b(S_1 d) = T_d(S_1 b) \quad \text{and} \quad bS_2 d = (S_2 b)d. \tag{3.2}$$

Taking $a = d = 0$, we get

$$T_b(S_1 c) = (S_1 b)c + T_{S_2 b}(c) \quad \text{and} \quad bS_2 c = 0. \tag{3.3}$$

By the second equation of (3.3) and the semisimplicity of B, we obtain that $S_2 c = 0$ for all $c \in A$, i.e., $S_2|_A = 0$. Then $T_{S_2 c} = T_{S_2 a} = 0$ for all $a, c \in A$. So we have from (3.1) that $aS_1 c = (S_1 a)c$ for all $a, c \in A$, i.e., $S_1|_A \in M(A)$. Also note that the second equation of (3.2) implies that $S_2|_B \in M(B)$. By the first equation of (3.3), we have

$$(S_1 b)c = T_b(S_1 c) - T_{S_2 b}(c)$$

holds for all $c \in A$ and $b \in B$. Consequently, S_1 and S_2 satisfy the conditions (i)–(iv).

Conversely, assume that S_1 and S_2 satisfy the conditions (i)–(iv). Let $a, c \in A$ and $b, d \in B$. We observe

$$T_d(S_1 a) + T_b(S_1(c, d)) = T_b(S_1 c) + T_d(S_1(a, b)). \tag{3.4}$$

In fact, let x be any element of A. Then we have

$$\begin{aligned} x[T_d(S_1 a) + T_b(S_1(c, d))] &= T_d(xS_1 a) + T_b(xS_1(c, d)) \\ &= T_d(xS_1 a) + T_b[xS_1 c + (S_1 d)x] \\ &= T_d(xS_1 a) + T_b[xS_1 c + T_d(S_1 x) - T_{S_2 d}(x)] && \text{(by (iv))} \\ &= T_d(xS_1 a) + T_b(xS_1 c) + (T_b T_d)(S_1 x) - (T_b T_{S_2 d})(x) \\ &= T_d(xS_1 a) + xT_b(S_1 c) + T_{bd}(S_1 x) - T_{S_2(bd)}(x) && \text{(by (ii))} \end{aligned}$$

and

$$\begin{aligned} x[T_b(S_1 c) + T_d(S_1(a, b))] &= xT_b(S_1 c) + T_d(xS_1(a, b)) \\ &= xT_b(S_1 c) + T_d(xS_1 a + xS_1 b) \\ &= xT_b(S_1 c) + T_d[(xS_1 a + T_b(S_1 x) - T_{S_2 b}(x)] && \text{(by (iv))} \\ &= xT_b(S_1 c) + T_d(xS_1 a) + (T_d T_b)(S_1 x) - (T_d T_{S_2 b})(x) \\ &= xT_b(S_1 c) + T_d(xS_1 a) + T_{bd}(S_1 x) - T_{S_2(bd)}(x) && \text{(by (ii)).} \end{aligned}$$

Consequently we have

$$x[T_d(S_1 a) + T_b(S_1(c, d))] = x[T_b(S_1 c) + T_d(S_1(a, b))]$$

for all $x \in A$. Since A is semisimple, we obtain the equality (3.4) as required.

Now take $(a,b),(c,d) \in A \times_T B$ arbitrarily. Then we have

$$
\begin{aligned}
&(a,b) \times_T S(c,d) \\
&= (a,b) \times_T (S_1(c,d), S_2(c,d)) \\
&= (aS_1(c,d) + T_{S_2(c,d)}(a) + T_b(S_1(c,d)), bS_2(c,d)) \\
&= (aS_1(c,d) + T_{S_2 d}(a) + T_b(S_1(c,d)), bS_2 d) && \text{(by (iii))} \\
&= (aS_1 c + aS_1 d + T_{S_2 d}(a) + T_b(S_1(c,d)), bS_2 d) \\
&= (aS_1 c + T_d(S_1 a) - T_{S_2 d}(a) + T_{S_2 d}(a) + T_b(S_1(c,d)), bS_2 d) && \text{(by (iv))} \\
&= ((S_1 a)c + T_d(S_1 a) + T_b(S_1(c,d)), dS_2 b) && \text{(by (i) and (ii))}
\end{aligned}
$$

and

$$
\begin{aligned}
&(S(a,b)) \times_T (c,d) \\
&= (S_1(a,b), S_2(a,b)) \times_T (c,d) \\
&= (S_1(a,b)c + T_d(S_1(a,b)) + T_{S_2(a,b)}(c), S_2(a,b)d) \\
&= ((S_1 a)c + (S_1 b)c + T_d(S_1(a,b)) + T_{S_2 b}(c), (S_2 b)d) && \text{(by (iii))} \\
&= ((S_1 a)c + T_b(S_1 c) - T_{S_2 b}(c) + T_d(S_1(a,b)) + T_{S_2 b}(c), dS_2 b) && \text{(by (iv))} \\
&= ((S_1 a)c + T_b(S_1 c) + T_d(S_1(a,b)), dS_2 b).
\end{aligned}
$$

Therefore it follows from (3.4) that

$$(a,b) \times_T S(c,d) = (S(a,b)) \times_T (c,d).$$

Consequently, we have $S \in M(A \times_T B)$. □

If $\{T_b : b \in B\} \subseteq A$, then the above theorem is just [1, Theorem 1] obtained by P. A. Dabhi.

For each $T \in M(A)$, there exists a unique bounded continuous function $\widehat{T}$ on Φ_A such that $\widehat{Ta}(\varphi) = \widehat{T}(\varphi)\widehat{a}(\varphi)$ for all $a \in A$ and $\varphi \in \Phi_A$ (see [2]). Put

$$\widehat{M}(A) = \left\{ \widehat{T} : T \in M(A) \right\}.$$

Definition 3.6. Let $\widehat{U} \in \widehat{M}(A)$ and $\widehat{V} \in \widehat{M}(B)$. We say that the ordered pair $(\widehat{U}, \widehat{V})$ satisfies the condition $(\flat)$ if

$$T_b U - T_{V(b)} \in A \quad (\cong \{L_a : a \in A\} \subseteq M(A))$$

for all $b \in B$.

Given a topological space X, we denote by $C^b(X)$ the set of all bounded continuous complex-valued functions on X. Then we have the following.

Theorem 3.7. *$\widehat{M}(A\times_T B)$ equals the set $\mathcal{S}$ of all $\sigma \in C^b(\Phi_{A\times_T B})$ such that $\sigma|_{\Phi_A} \in \widehat{M}(A)$, $\sigma|_{\Phi_B} \in \widehat{M}(B)$ and the ordered pair $(\sigma|_{\Phi_A}, \sigma|_{\Phi_B})$ satisfies the condition $(\flat)$.*

Proof. Take $S \in M(A \times_T B)$ arbitrarily. Write

$$S = (S_1, S_2),$$

where $S_1\colon A \times_T B \to A$ and $S_2\colon A \times_T B \to B$ are bounded linear mappings. Then S_1 and S_2 must satisfy the conditions (i)–(iv) in Theorem 3.5. Take $\varphi \in \Phi_A$ arbitrarily. By (i) and (iii), we have

$$\begin{aligned}\widehat{Se_\varphi}(\varphi) &= (\varphi, \widetilde{\varphi} \circ T)(S(e_\varphi, 0)) = (\varphi, \widetilde{\varphi} \circ T)(S_1 e_\varphi, S_2 e_\varphi)\\ &= (\varphi, \widetilde{\varphi} \circ T)(S_1 e_\varphi, 0) = \varphi(S_1 e_\varphi) = \widehat{S_1|_A}(\varphi)\widehat{e_\varphi}(\varphi)\\ &= \widehat{S_1|_A}(\varphi).\end{aligned}$$

On the other hand, we have

$$\widehat{Se_\varphi}(\varphi) = \widehat{S}(\varphi, \widetilde{\varphi} \circ T)\widehat{(e_\varphi, 0)}(\varphi, \widetilde{\varphi} \circ T) = \widehat{S}(\varphi, \widetilde{\varphi} \circ T) = \widehat{S}(\varphi).$$

Therefore we have $\widehat{S}(\varphi) = \widehat{S_1|_A}(\varphi)$. In other words, $\widehat{S}|_{\Phi_A} = \widehat{S_1|_A} \in \widehat{M}(A)$. Take $\psi \in \Phi_B$ arbitrarily. By (ii), we have

$$\begin{aligned}\widehat{Se_\psi}(\psi) &= (0, \psi)(S_1(e_\psi), S_2(e_\psi)) = \psi(S_2 e_\psi) = \psi(S_2|_B(e_\psi))\\ &= \widehat{S_2|_B}(\psi)\widehat{e_\psi}(\psi) = \widehat{S_2|_B}(\psi).\end{aligned}$$

On the other hand, we have

$$\widehat{Se_\psi}(\psi) = \widehat{S}(\psi)\widehat{(0, e_\psi)}(0, \psi) = \widehat{S}(\psi)\psi(e_\psi) = \widehat{S}(\psi).$$

Therefore we have $\widehat{S}(\psi) = \widehat{S_2|_B}(\psi)$. In other words, $\widehat{S}|_{\Phi_B} = \widehat{S_2|_B} \in \widehat{M}(B)$. Now put

$$U = S_1|_A \quad \text{and} \quad V = S_2|_B.$$

Since $L_{S_1 b} = T_b(S_1|_A) - T_{S_2 b}$ holds for all $b \in B$ from (iv), it follows that the ordered pair $(\widehat{U}, \widehat{V})$ satisfies the condition $(\flat)$. Then $\widehat{S}$ must be in $\mathcal{S}$. Consequently, $\widehat{M}(A \times_T B) \subseteq \mathcal{S}$.

Conversely, let $\sigma \in \mathcal{S}$. Then $\sigma \in C^b(\Phi_{A\times_T B})$, $\sigma|_{\Phi_A} = \widehat{U} \in \widehat{M}(A)$, $\sigma|_{\Phi_B} = \widehat{V} \in \widehat{M}(B)$ and the pair $(\widehat{U}, \widehat{V})$ satisfies the condition $(\flat)$. So we have

$$\widehat{U(a)}(\varphi) = \sigma(\varphi, \widetilde{\varphi} \circ T)\widehat{a}(\varphi) \quad (\varphi \in \Phi_A, a \in A)$$

and

$$\widehat{V(b)}(\psi) = \sigma(0, \psi)\widehat{b}(\psi) \quad (\psi \in \Phi_B, b \in B).$$

Define $S_1\colon A \times_T B \to A$ and $S_2\colon A \times_T B \to B$ by

$$S_1(a,b) = U(a) + T_bU - T_{V(b)} \quad \text{and} \quad S_2(a,b) = V(b)$$

for each $(a,b) \in A \times_T B$. Then both S_1 and S_2 are bounded linear mappings. Put $S = (S_1, S_2)$. Then we can easily see that S is a bounded linear mapping from $A \times_T B$ into itself and that S_1, S_2 satisfy the conditions (i)–(iv) in Theorem 3.5. Hence we have from Theorem 3.5 that $S \in M(A \times_T B)$. Let $(a,b) \in A \times_T B$, $\varphi \in \Phi_A$ and $\psi \in \Phi_B$. Then

$$\begin{aligned}
\widehat{S(a,b)}(\varphi, \widetilde{\varphi} \circ T) &= (\varphi, \widetilde{\varphi} \circ T)(S_1(a,b), S_2(a,b)) \\
&= \varphi(S_1(a,b)) + \widetilde{\varphi}(T_{S_2(a,b)}) \\
&= \varphi\big((Ua)e_\varphi + (T_bU)e_\varphi - T_{V(b)}e_\varphi\big) + \varphi(T_{S_2(a,b)}e_\varphi) \\
&= \varphi(Ua) + \varphi((T_bU)e_\varphi)) \\
&= \sigma(\varphi, \widetilde{\varphi} \circ T)\widehat{a}(\varphi) + \widehat{T_b}(\varphi)\widehat{U}(\varphi) \\
&= \sigma(\varphi, \widetilde{\varphi} \circ T)\widehat{a}(\varphi) + \widehat{T_b}(\varphi)\sigma(\varphi, \widetilde{\varphi} \circ T) \\
&= \sigma(\varphi, \widetilde{\varphi} \circ T)(\widehat{a}(\varphi) + \widehat{T_b}(\varphi)).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\widehat{S(a,b)}(\varphi, \widetilde{\varphi} \circ T) &= \widehat{S}(\varphi, \widetilde{\varphi} \circ T)\widehat{(a,b)}(\varphi, \widetilde{\varphi} \circ T) \\
&= \widehat{S}(\varphi, \widetilde{\varphi} \circ T)(\varphi(a) + \widetilde{\varphi}(T_b)) \\
&= \widehat{S}(\varphi, \widetilde{\varphi} \circ T)(\widehat{a}(\varphi) + \widehat{T_b}(\varphi)).
\end{aligned}$$

Therefore we have

$$\sigma(\varphi, \widetilde{\varphi} \circ T)(\widehat{a}(\varphi) + \widehat{T_b}(\varphi)) = \widehat{S}(\varphi, \widetilde{\varphi} \circ T)(\widehat{a}(\varphi) + \widehat{T_b}(\varphi)).$$

In particular taking $a = e_\varphi$ and $b = 0$ in the above equation, we have

$$\widehat{S}(\varphi) = \widehat{S}(\varphi, \widetilde{\varphi} \circ T) = \sigma(\varphi, \widetilde{\varphi} \circ T) = \sigma(\varphi) \quad (\varphi \in \Phi_A),$$

and hence $\widehat{S}|_{\Phi_A} = \sigma|_{\Phi_A}$. Note that

$$\begin{aligned}
\widehat{S(a,b)}(0,\psi) = (0,\psi)(S_1(a,b), S_2(a,b)) &= \psi(S_2(a,b)) \\
= \psi(Vb) = \widehat{V}(\psi)\widehat{b}(\psi) &= \sigma(\psi)\widehat{b}(\psi).
\end{aligned}$$

On the other hand, we have

$$\widehat{S(a,b)}(0,\psi) = \widehat{S}(0,\psi)\widehat{(a,b)}(0,\psi) = \widehat{S}(\psi)\widehat{b}(\psi).$$

Therefore we have that $\widehat{S}(\psi) = \sigma(\psi)$ for all $\psi \in \Phi_B$, and hence $\widehat{S}|_{\Phi_B} = \sigma|_{\Phi_B}$. Then we have $\sigma = \widehat{S} \in \widehat{M}(A \times_T B)$. Consequently, $\mathcal{S} \subseteq \widehat{M}(A \times_T B)$. Thus we have the desired result. $\square$

Note that if $\{T_b : b \in B\} \subseteq A$, then any ordered pair $(\widehat{U}, \widehat{V})$ with $U \in M(A)$ and $V \in M(B)$ always satisfies the condition ($\flat$). Therefore the next corollary follows from Theorem 3.7 immediately.

Corollary 3.8. *Assume that $\{T_b : b \in B\} \subseteq A$. Then*

$$\widehat{M}(A \times_T B) = \left\{\sigma \in C^b(\Phi_{A \times_T B}) : \sigma|_{\Phi_A} \in \widehat{M}(A), \sigma|_{\Phi_B} \in \widehat{M}(B)\right\}.$$

Let $\theta \in \Phi_B$ and id_A the identity mapping of A. Put

$$T_b = \theta(b)\,\mathrm{id}_A$$

for each $b \in B$. Then T is a norm-decreasing homomorphism from B into $M(A)$. In this case, $\times_T$ is just the θ-Lau product $\times_\theta$ defined in Sangani Monfared [4]. Therefore we have the following.

Corollary 3.9. *$\widehat{M}(A \times_\theta B)$ equals the set of all $\sigma \in C^b(\Phi_{A \times_\theta B})$ such that $\sigma|_{\Phi_A} \in \widehat{M}(A)$, $\sigma|_{\Phi_B} \in \widehat{M}(B)$ and $\sigma|_{\Phi_A} - \sigma(\theta)1 \in \widehat{A}$.*

Proof. Let $b \in B$, $U \in M(A)$ and $V \in M(B)$. Then

$$\begin{aligned} \widehat{T_bU - T_{V(b)}}(\varphi) &= \widehat{T_b}(\varphi)\widehat{U}(\varphi) - \widehat{T_{V(b)}}(\varphi) = \theta(b)\widehat{U}(\varphi) - \theta(V(b)) \\ &= \widehat{U}(\varphi)\widehat{b}(\theta) - \widehat{V}(\theta)\widehat{b}(\theta) = \left(\widehat{U}(\varphi) - \widehat{V}(\theta)\right)\widehat{b}(\theta) \end{aligned}$$

for all $\varphi \in \Phi_A$. Then we have

$$T_bU - T_{V(b)} = \widehat{b}(\theta)\left(U - \widehat{V}(\theta)\,\mathrm{id}_A\right)$$

for all $b \in B$. Then an ordered pair $(\widehat{U}, \widehat{V})$ satisfies the condition ($\flat$) if and only if $U - \widehat{V}(\theta)\,\mathrm{id}_A \in A$ or equivalently, $\widehat{U} - \widehat{V}(\theta)1 \in \widehat{A}$. Therefore the desired result follows from Theorem 3.7. □

The above corollary immediately implies the following.

Corollary 3.10. *Suppose that A is a non-unital commutative C^*-algebra. Then*

$$\widehat{M}(A \times_\theta B) = \left\{\sigma \in C^b(\Phi_{A \times_\theta B}) : \sigma|_{\Phi_B} \in \widehat{M}(B), \lim_{\varphi \to \infty} \sigma|_{\Phi_A}(\varphi) = \sigma(\theta)\right\}.$$

In particular, if B is a commutative C^*-algebra, then

$$\widehat{M}(A \times_\theta B) = \left\{\sigma \in C^b(\Phi_{A \times_\theta B}) : \lim_{\varphi \to \infty} \sigma|_{\Phi_A}(\varphi) = \sigma(\theta)\right\}.$$

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