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Cross Theorems for Separately $(\cdot, W)$-meromorphic Functions

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Abstract. It is shown that Rothstein's theorem holds for (F, W)-meromorphic functions with F is a sequentially complete locally convex space. We also prove that a meromorphic function on a Riemann domain D over a separable Banach E with values in a sequentially complete locally convex space can be extended meromorphically to the envelope of holomorphy $\widehat{D}$ of D. Using these results, in the remaining parts, we give a version of Kazarian's theorem for the class of separately $(\cdot, W)$-meromorphic functions with values in a sequentially complete locally convex space and generalize cross theorem with pluripolar singularities of Jarnicki and Pflug for separately $(\cdot, W)$-meromorphic functions with values in a Fréchet space.

1. Introduction

The classical Hartogs theorem states that every separately holomorphic function on products of domains in complex Euclidian spaces is holomorphic. This theorem has been a source of inspiration for numerous research works in Complex Analysis for many years. The well known Hartogs theorem on holomorphicity of separately holomorphic functions was extended to the special subsets in $\mathbb{C}^{m+n}$ by several authors, in particular by Siciak [28], Nguyen Thanh Van and Zeriahi [16], Shiffman [26]. It is easy to see that their results are true for vector-valued case. However, in the meromorphic case the situation is different, and more difficult even for scalar functions. Rothstein [22] proved the Hartogs theorem for scalar meromorphic functions. Later, Kazarian [13] and Shiffman [26] extended the Rothstein theorem to the special subsets in $\mathbb{C}^{m+n}$ for the meromorphic case. We known that, there are $\mathbb{C}^N$-valued weakly meromorphic functions which are not meromorphic, and hence Rothstein's theorem for vector-valued meromorphic functions not be deduced from the results for the scalar case.

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The problem of determining the envelope of holomorphy of separately holomorphic functions defined on some cross sets was studied by several authors. The theorems obtained in this type are often called cross theorems. In 1970, by using the relative extremal functions, Siciak [27] established the envelope of holomorphy for separately holomorphic functions for the case where the cross set has a special shape, a product of domains in $\mathbb{C}$. Later, Jarnicki and Pflug generated for the cross theorem with singularities [11]. With the help of generalization of Rothstein's theorem and extension theorem with pluripolar singularities, they considered the cross theorem for meromorphic functions [10]. Recently, Quang and Dai [18] generated Siciak's result to the class of separately $(\cdot, W)$-holomorphic functions.

The paper continues and generalizes the investigations from [18] and [19]. It is also a continuation of an our earlier work [20] in which, using some results of Grosse-Erdmann [6] and modifying the argument of Siu [29], some generalizations of the Levi extension theorem to the class of separately $(\cdot, W)$-meromorphic functions have been introduced.

In the paper we would like to generalize a result of Kazarian [13] on the crosses (Theorems 5.1, 5.3) and cross theorem with pluripolar singularities of Jarnicki and Pflug [10] (Theorem 6.4) to the above class of functions with values in either a sequentially complete locally convex space or a Fréchet space. The proofs are respectively presented in Sections 5 and 6.

One of the most important tools used in the proofs of Kazarian's theorem [13] and cross theorem with pluripolar singularities of Jarnicki and Pflug [10] is Rothstein's theorem. So, before that, the problem to be considered in Section 3 is Rothstein's theorem for $(\cdot, W)$-meromorphic functions with values in a locally complete space (Theorem 3.1). The proof is based on a generalization of the Levi extension theorem (Theorem 3.2) and modification the arguments of Kazarian in [13] and of Jarnicki-Pflug in [10].

Another tool that we will use in the proofs is concerning to domains of existence for Fréchet-valued meromorphic functions. Section 4 is devoted to the discussion this problem for Fréchet-valued meromorphic functions (Theorem 4.1). We prove that a meromorphic function on a Riemann domain D over a locally convex space E with values in a Fréchet space F can be extended meromorphically to the envelope of holomorphy $\widehat{D}$ of D when E is a Banach space with a Schauder basis (Theorem 4.4).

We also review in Section 2 some elements of locally convex spaces and pluripotential theory pertaining to our work.

2. Preliminaries

2.1. General notations

We shall use standard notations of the theory of locally convex spaces as presented in the book of Schaefer [23]. A locally convex space always is a complex vector space with a locally convex Hausdorff topology.

For a Fréchet space E we always assume that its locally convex structure is generated by an increasing system $\{\|\cdot\|_k\}$ of semi-norms. Then we denote by E_k the completion of the canonically normed space $E/\ker\|\cdot\|_k$ and $\omega_k\colon E \to E_k$ denotes the canonical map and U_k denotes the set $\{x \in E : \|x\|_k < 1\}$. Sometimes it is convenient to assume that $\{U_k\}_{k\in\mathbb{N}}$ is a neighbourhood basis of zero (shortly $\mathcal{U}(E)$).

If B is an absolutely convex subset of E we define a norm $\|\cdot\|_B^*$ on E^*, the strongly dual space of E with values in $[0, +\infty]$ by

$$\|u\|_B^* = \sup\{|u(x)| : x \in B\}.$$

Obviously $\|\cdot\|_B^*$ is the gauge functional of B°. Instead of $\|\cdot\|_{U_k}^*$ we write $\|\cdot\|_k^*$. By E_B we denote the linear hull of B which becomes a normed space in a canonical way if B is bounded. The space E is called *locally complete* if every such space E_B is Banach.

For a locally convex space F, a subset $W \subset F'$ is called *separating* if $u(x) = 0$ for each $u \in W$ implies $x = 0$. Clearly, this is equivalent to the span of W being weak*-dense (or dense in the co-topology).

We say that $W \subset F'$ *determines boundedness* if every subset $B \subset F$ is bounded whenever $u(B)$ is bounded in $\mathbb{C}$ for all $u \in W$. This holds if and only if every $\sigma(F, \operatorname{span} W)$-bounded set is F-bounded. Obviously, the linear span of such sets is $\sigma(F', F)$-dense. Clearly, if $W \subset F'$ determines boundedness in F', then W is separating.

2.2. Pluripolar sets and pluriregular sets (L-regular sets)

Let X be a complex space and Ω be a set in X. By $\mathrm{PSH}(\Omega)$ we denote the set of all plurisubharmonic (psh) functions on Ω.

Definition 2.1. A subset K of X is called *pluripolar* if for every $z \in K$ there exist a neighbourhood U of z and $\varphi \in \mathrm{PSH}(U)$, $\varphi \neq -\infty$ such that

$$K \cap U \subset \left\{z' \in U : \varphi(z') = -\infty\right\}.$$

It is well known that a subset X of $\mathbb{C}^p$ is pluripolar if and only if there exists $\varphi \in \mathrm{PSH}(\mathbb{C}^p)$, $\varphi \not\equiv -\infty$ such that

$$X \subset \{z \in \mathbb{C}^p : \varphi(z) = -\infty\}.$$

Let $K \subset \Omega$ with Ω is an open set in X. Put

$$\mathscr{U}(K,\Omega) = \{u \in \mathrm{PSH}(\Omega) : u|_K \leq 0; u|_\Omega \leq 1\}.$$

We denote $u_{K,\Omega}$ the relatively extremal function of the couple (K,Ω) defined by

$$h_{K,\Omega}(z) = \sup\{u(z) : u \in \mathscr{U}(K,\Omega)\}.$$

Let $\omega(\cdot, K, \Omega) := h^*_{K,\Omega}$ be the upper-semicontinuous regularization of $h_{K,\Omega}$:

$$\omega(z, K, \Omega) = \limsup_{\Omega \ni z' \to z} h_{K,\Omega}(z'), \quad z \in \Omega.$$

Definition 2.2. The point $a \in \Omega$ is called the *locally pluriregular* (or *locally L-regular*) point of K if $a \in \overline{K} \cap \Omega$ and $\omega(a, K \cap U, U) = 0$ for all neighbourhood U of a in Ω. Moreover, K is said to be *locally pluriregular* (or *locally L-regular*) if it is locally pluriregular at every point $a \in K$.

Denote K^* the set of all pluriregular points of K (in Ω). If K is non-pluripolar, then K^* is non-pluripolar and $K \setminus K^*$ is pluripolar.

Definition 2.3. The set K is said to be *pluriregular* (or *L-regular*) if $\omega(\cdot, K \cap U, U) = 0$ on K for all neighbourhood U of K.

2.3. The cross

Definition 2.4 (N-fold cross)**.** Let $N \in \mathbb{N}$, $N \geq 2$, and let $\varnothing \neq A_j \subset D_j \subset \mathbb{C}^{k_j}$, $k_j \in \mathbb{N}$, where D_j is a domain, $j = 1, \ldots, N$. The set

$$\begin{aligned} X &:= \mathbb{X}(A_1, \ldots, A_N; D_1, \ldots, D_N) \\ &:= \bigcup_{j=1}^{N} A_1 \times \cdots \times A_{j-1} \times D_j \times A_{j+1} \times \cdots \times A_N \\ &\subset \mathbb{C}^{k_1 + \cdots + k_N} \end{aligned}$$

is called the *N-fold cross* associated to the N pairs (A_j, D_j).

Definition 2.5. Let $N \in \mathbb{N}$, $N \geq 2$, and let $\varnothing \neq A_j \subset D_j \subset \mathbb{C}^{k_j}$, $k_j \in \mathbb{N}$, where D_j is a domain, $j = 1, \ldots, N$ and let $X := \mathbb{X}(A_1, \ldots, A_N; D_1, \ldots, D_N)$. Put

$$\widehat{X} := \left\{ (z_1, \ldots, z_N) \in D_1 \times \cdots \times D_N : \sum_{j=1}^{N} \omega(z_j, A_j, D_j) < 1 \right\}.$$

Definition 2.6 (Generalized N-fold cross)**.** Let $N \in \mathbb{N}$, $N \geq 2$, and let $\varnothing \neq A_j \subset D_j \subset \mathbb{C}^{k_j}$, $k_j \in \mathbb{N}$, where D_j is a domain. Moreover, let $S_j \subset A_j' \times A_j''$, $j = 1, \ldots, N$, where $A_j' = A_1 \times \cdots \times A_{j-1}$ and $A_j'' = A_{j+1} \times \cdots \times A_N$. The set

$$\begin{aligned} T &:= \mathbb{T}(A_1, \ldots, A_N; D_1, \ldots, D_N; S_1, \ldots, S_N) \\ &:= \bigcup_{j=1}^{N} \left\{ (z', z_j, z'') \in A_j' \times D_j \times A_j' : (z', z'') \notin S_j \right\} \end{aligned}$$

is called the *generalized N-fold cross* associated to the N triples (A_j, D_j, S_j).

Here, $z' = (z_1, \ldots, z_{j-1}) \in A_j'$ and $z'' = (z_{j+1}, \ldots, z_N) \in A_j''$.

Remark 2.7. (a) It is clear that $T \subset X$.

(b) $X := \mathbb{X}(A_1, \ldots, A_N; D_1, \ldots, D_N) = \mathbb{T}(A_1, \ldots, A_N; D_1, \ldots, D_N; \varnothing, \ldots, \varnothing)$.

(c) If $N = 2$, then $\mathbb{T}(A_1, A_2; D_1, D_2; S_1, S_2) = \mathbb{X}(A_1 \setminus S_2, A_2 \setminus S_1; D_1, D_2)$. Consequently, any generalized 2-fold cross is a 2-fold cross.

Definition 2.8. Let $N \in \mathbb{N}$, $N \geq 2$, and let $\varnothing \neq D_j \subset \mathbb{C}^{k_j}$ is a domain, $k_j \in \mathbb{N}$, $j = 1, \ldots, N$. Let U be an open subset of $D_1 \times \cdots \times D_N$ and let $M \subset U$ be a relatively closed set. For $(a_1, \ldots, a_N) \in (D_1 \times \cdots \times D_N) \cap U$ and $j = 1, \ldots, N$ we define the *fiber* of M and the fiber of U over $(a_j', \cdot, a_j'')$ as

$$\begin{aligned} M_{(a_j', \cdot, a_j'')} &:= \left\{ z_j \in D_j : (a_j', z_j, a_j'') \in M \right\}, \\ U_{(a_j', \cdot, a_j'')} &:= \left\{ z_j \in D_j : (a_j', z_j, a_j'') \in U \right\}. \end{aligned}$$

Remark 2.9. Let $N \in \mathbb{N}$, $N \geq 2$, and let $\varnothing \neq D_j \subset \mathbb{C}^{k_j}$ is a domain, $k_j \in \mathbb{N}$, $j = 1, \ldots, N$. Let U be an open subset of $D_1 \times \cdots \times D_N$ and let $M \subset U$ be a relatively closed set. Then $M_{(a_j', z_j, a_j'')}$ is closed in $U_{(a_j', \cdot, a_j'')}$ for all $(a_1, \ldots, a_N) \in (D_1 \times \cdots \times D_N) \cap U$ and $j = 1, \ldots, N$.

2.4. Holomorphic functions and meromorphic functions

Definition 2.10. Let E and F be locally convex spaces and let $D \subset E$ be open, $D \neq \varnothing$. A function $f\colon D \to F$ is called to be *holomorphic* if f is continuous and $u \circ f$ is Gâteaux holomorphic for every $u \in F'$.

By $H(D, F)$ we denote the vector space of all holomorphic functions on D with values in F. This space equipped with the compact-open topology. Instead of $H(D, \mathbb{C})$ we write $H(D)$. For details concerning holomorphic functions on locally convex spaces we refer to the book of Dineen [4].

We denote by $H^\infty(D, F)$ the subspace of all bounded functions in $H(D, F)$. Instead of $H^\infty(D, \mathbb{C})$ we write $H^\infty(D)$.

Now assume that K is a compact subset of E and let $H(K)$ denote the space of germs of holomorphic functions on K. This space is equipped with the inductive limit topology

$$H(K) = \liminf_{U \searrow K} H^{\infty}(U)$$

where U ranges over all neighbourhoods of K in E.

Definition 2.11. Let $\varnothing \neq \mathscr{F} \subset H(X \setminus M, F)$. We say that a point $a \in M$ is *non-singular* with respect to $\mathscr{F}$ ($a \in M_{ns,\mathscr{F}}$) if there exists an open neighbourhood U of a such that for each $f \in \mathscr{F}$ there exists a function $\widetilde{f} \in H(U, F)$ with $\widetilde{f} = f$ on $U \setminus M$. If $a \in M_{s,\mathscr{F}} := M \setminus M_{ns,\mathscr{F}}$ then we say that a is *singular with respect to* $\mathscr{F}$.

If $M_{ns,\mathscr{F}} = \varnothing$, i.e., $M_{s,\mathscr{F}} = M$, then we say that M is *singular with respect to* $\mathscr{F}$. If $\mathscr{F} = H(X \setminus M, F)$, then we simply say that M is *singular* and we skip the index $\mathscr{F}$.

Definition 2.12. Let E and F be locally convex spaces and D_0 be a dense open subset of an open set D in E. We say that a function $f \colon D_0 \to F$ is *meromorphic* on D if for every $z \in D$ there exist a neighbourhood U_z of z in E and holomorphic functions $h_{U_z} \colon U_z \to F$, $\sigma_{U_z} \colon U_z \to \mathbb{C}$ such that

$$f\big|_{U_z \cap D_0} = \frac{h_{U_z}}{\sigma_{U_z}}\bigg|_{U_z \cap D_0}.$$

The functions h_{U_z} and σ_{U_z} are called the *local numerator* and *local denominator* of f at z respectively. We say $\frac{h_{U_z}}{\sigma_{U_z}}$ is the *local representation* of f at z. In [14], Khue proved that, if D is a Stein manifold and F is a sequentially complete locally convex space, this representation of f is *global*, that means $f = \frac{h}{\sigma}$ with $h \in H(D, F)$ and $\sigma \in H(D)$.

We denote

$$M(D, F) = \{f \text{ is meromorphic on } D\} \quad \text{and} \quad M(D) = M(D, \mathbb{C}).$$

By

$$P(f) = \left\{ z \in D : \text{there exists a local representation } \frac{h_{U_z}}{\sigma_{U_z}} \text{ such that } h_{U_z}(z) \neq 0,\ \sigma_{U_z}(z) = 0 \right\}$$

and

$$I(f) = \left\{ z \in D : \text{for every local representation } \frac{h_{U_z}}{\sigma_{U_z}} \text{ we have } h_{U_z}(z) = 0,\ \sigma_{U_z}(z) = 0 \right\}$$

we denote the *pole set* and the *indeterminacy set* of f respectively. It is shown in [14] that they are analytic sets in D with

$$\operatorname{codim} P(f) \geq 1 \quad \text{and} \quad \operatorname{codim} I(f) \geq 2.$$

Definition 2.13 (Separately holomorphic on a generalized N-fold cross)**.** Let $N \in \mathbb{N}$, $N \geq 2$, and let $\varnothing \neq A_j \subset D_j \subset \mathbb{C}^{k_j}$, $k_j \in \mathbb{N}$, where D_j is a domain, let $S_j \subset A'_j \times A''_j$, $j = 1, \dots, N$, and let $T := \mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; S_1, \dots, S_N)$ be the generalized N-fold cross.

We say that a function $f\colon T \to F$ is *separately holomorphic* if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus S_j$ the function $f(a'_j, \cdot, a''_j)$ is holomorphic in D_j. Put

$$H_s(T, F) = \{f \text{ is separately holomorphic on } T\}.$$

Definition 2.14 (Separately holomorphic on $X \setminus M$)**.** Let $N \in \mathbb{N}$, $N \geq 2$, and let $\varnothing \neq A_j \subset D_j \subset \mathbb{C}^{k_j}$, $k_j \in \mathbb{N}$, where D_j is a domain, and let $X := \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N)$. Moreover, let U be an open neighbourhood of X and let $M \subset U$ be such that for all $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $j \in \{1, \dots, N\}$ the fiber $M_{(a'_j, \cdot, a''_j)}$ is relatively closed in D_j.

We say that a function $f\colon X \setminus M \to F$ is *separately holomorphic* ($f \in H_s(X \setminus M, F)$) if for any $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $j \in \{1, \dots, N\}$ the function $f(a'_j, \cdot, a''_j)$ is holomorphic in the open set $D_j \setminus M_{(a'_j, \cdot, a''_j)}$. Put

$$H_s(X \setminus M, F) = \{f \text{ is separately holomorphic on } X \setminus M\}.$$

Definition 2.15 (Separately meromorphic on $T \setminus M$)**.** Let $N \in \mathbb{N}$, $N \geq 2$, and let $\varnothing \neq A_j \subset D_j \subset \mathbb{C}^{k_j}$, $k_j \in \mathbb{N}$, where D_j is a domain, let $S_j \subset A'_j \times A''_j$, $j = 1, \dots, N$, and let $T := \mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; S_1, \dots, S_N)$ be the generalized N-fold cross. Let $M \subset T$, $S \subset T \setminus M$ be relatively closed.

We say that a function $f\colon (T \setminus M) \setminus S \to F$ is *separately meromorphic* if for any $(a_1, \dots, a_N) \in (A_1 \times \dots \times A_N) \setminus S_j$ with $(M \cup S)_{(a'_j, \cdot, a''_j)} \neq D_j$ and $j \in \{1, \dots, N\}$ there exists a function $\widetilde{f}(a'_j, \cdot, a''_j) \in M(D_j \setminus M_{(a'_j, \cdot, a''_j)}, F)$ such that $\widetilde{f}(a'_j, \cdot, a''_j) = f(a'_j, \cdot, a''_j)$ on $D_j \setminus (M \cup S)_{(a'_j, \cdot, a''_j)}$.

Observe that $f \in H_s(T \setminus (M \cup S), F)$. Put

$$M_s(T \setminus M, F) = \{f \text{ is separately meromorphic on } T \setminus M\}.$$

2.5. $(\cdot, W)$-holomorphic functions and $(\cdot, W)$-meromorphic functions

Let E, F be locally convex spaces, D an open subset of E, W a subset of F'.

Definition 2.16. A function $f\colon D \to F$ is called

- (F, W)*-holomorphic* if $u \circ f \in H(D)$ for all $u \in W$;
- (F, W)*-holomorphic, bounded* if $u \circ f \in H^\infty(D)$ for all $u \in W$.

We denote

$$H^W(D,F) = \{f : f \text{ is } (F,W)\text{-holomorphic on } D\}\,;$$
$$H^{W,\infty}(D,F) = \{f : f \text{ is } (F,W)\text{-holomorphic, bounded on } D\}\,.$$

Definition 2.17. Let $T = \mathbb{T}(A_1,\dots,A_N; D_1,\dots,D_N; S_1,\dots,S_N)$ be a generalized N-fold cross. Let $M \subset T$ be relatively closed.

We say that a function $f\colon (T\setminus M) \to F$ is *separately* (F,W)*-holomorphic* on $T\setminus M$ if for any $j \in \{1,\dots,N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j)$ the functions $(u\circ f)(a'_j, \cdot, a''_j)$ are holomorphic on $D_j \setminus M_{(a'_j,\cdot,a''_j)}$ for every $u \in W$. We denote

$$H^W_s(T\setminus M,F) = \{f : f \text{ is separately } (F,W)\text{-holomorphic on } T\setminus M\}\,;$$
$$H^{W,\infty}_s(T\setminus M,F) = H^W_s(T\setminus M,F) \cap H^{W,\infty}(T\setminus M,F);$$
$$H^{W,\infty,c}_s(T\setminus M,F) = H^{W,\infty}_s(T\setminus M,F) \cap C(T\setminus M,F).$$

Definition 2.18. A function $f\colon D \to F$ is called (F,W)*-meromorphic* if $u\circ f \in M(D)$ for all $u \in W$. If $u\circ f$ has a meromorphic extension $\widehat{u\circ f} \in M(G)$ with $G \supset D$ for all $u \in W$ we say that f has an (F,W)*-meromorphic extension* to G. Put

$$M^W(D,F) = \{f : f \text{ is } (F,W)\text{-meromorphic on } D\}\,.$$

Definition 2.19. Let $T = \mathbb{T}(A_1,\dots,A_N; D_1,\dots,D_N; S_1,\dots,S_N)$ be a generalized N-fold cross. Let $M \subset T$, $S \subset T\setminus M$ be relatively closed.

We say that a function $f\colon (T\setminus M)\setminus S \to F$ is *separately* (F,W)*-meromorphic* on $T\setminus M$ if for any $j \in \{1,\dots,N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j)\setminus S_j$ with $(M\cup S)_{(a'_j,\cdot,a''_j)} \neq D_j$, there exist functions $(\widetilde{u\circ f})(a'_j, \cdot, a''_j) \in M(D_j \setminus M_{(a'_j,\cdot,a''_j)})$ such that

$$(\widetilde{u\circ f})(a'_j, \cdot, a''_j) = (u\circ f)(a'_j, \cdot, a''_j)$$

on $D_j \setminus (M\cup S)_{(a'_j,\cdot,a''_j)}$ for every $u \in W$. We denote

$$M^W_s(T\setminus M,F) = \{f : f \text{ is separately } (F,W)\text{-meromorphic on } T\setminus M\}\,.$$

3. Rothstein's theorem for $(\cdot,W)$-meromorphic functions

The aim of this section is to consider the Rothstein theorem for class of (F,W)-meromorphic functions where F is a locally complete space.

Throughout this section we shall adopt the following notations:

$$\Delta_r(s) := \{t \in \mathbb{C} : |t-s| < r\}\,; \quad \Delta_r = \Delta_r(0); \quad \Delta = \Delta_1;$$
$$\Delta^N_r(s) := \Delta_r(s_1)\times\cdots\times\Delta_r(s_N), \quad \text{for } s = (s_1,\dots,s_N) \in \mathbb{C}^N.$$

The following is the main result of this section.

Theorem 3.1. (cf. [10, 22]) *Let $f \in M(\Delta^p \times \Delta^q, F)$ where $\Delta^p \subset \mathbb{C}^p$, $\Delta^q \subset \mathbb{C}^q$ are the unit polydiscs and F is a sequentially complete space. Assume that $A \subset \Delta^p$ is a locally pluriregular set such that for any $a \in A$ we have $(P(f))_{(a,\cdot)} \neq \Delta^q$. Let $G \subset \mathbb{C}^q$ be a domain such that $\Delta^q \subset G$. Assume that for every $a \in A$ the function f_a has an (F, W)-meromorphic extension to G where W is a subspace of F' that determining boundedness. Then there exist an open neighbourhood Ω of $(\Delta^p \times \Delta^q) \cup (A \times G)$ and a function $\widetilde{f} \in M(\Omega, F)$ such that $\widetilde{f} = f$ on $\Delta^p \times \Delta^q$.*

The first is a generalization of the Levi extension theorem for $(\cdot, W)$-meromorphic functions.

Let F be a locally convex space, $\Omega \subset \mathbb{C}$ a domain and $f\colon \Omega \to F$ be a meromorphic function. Then for every point $t \in \Omega$ there exists some non-negative integer N so that $(\lambda - t)^N f(\lambda)$ has a holomorphic extension to t. We write $o_t(f)$ for the least such number N, possibly 0. If $o_t(f) > 0$ then t is a pole of f.

Theorem 3.2. *Let F be a sequentially complete locally convex space, W a separating subset of F' and $f \in M(D \times (\Delta_r \setminus \overline{\Delta}), F)$, where $r > 1$ and D is an open set in $\mathbb{C}^n$. Suppose D_* is a dense set in D such that for each $z \in D_*$ there exist a set $P_z \subset \Delta_r$ without limit points in Δ_r and a locally bounded function $f_z^- \colon \Delta_r \setminus P_z \to F$ satisfying*

(i) *$f_z^- = f_z$ on $\Delta_r \setminus \overline{\Delta}$;*

(ii) *f_z^- has an (F, W)-meromorphic extension to Δ_r with $P(\widehat{u \circ f_z^-}) \subset P_z$ for all $u \in W$ and for all $t \in P_z$*

$$\max_{u \in W} o_t(\widehat{u \circ f_z^-}) < \infty.$$

Then f extends meromorphically to $D \times \Delta_r$.

Proof. (a) From the hypotheses (i) and (ii), by Grosse-Erdmann [6, Theorem 4] for each $z \in D$ the function f_z is extended to a meromorphic function $\widehat{f_z}$ on Δ_r.

(b) Now we consider the case where F is a Fréchet space. As in [29] we may assume that f is holomorphic on $D \times (\Delta_r \setminus \overline{\Delta})$. We have a Laurent series expansion

$$f(z, \lambda) = \sum_{k=-\infty}^{+\infty} C_k(z) \lambda^k, \quad (1 < |\lambda| < r)$$

where $C_k(z)$ are holomorphic on D. Let

$$D_p = \left\{ z \in D : \widehat{f_z} \text{ has at most } p \text{ poles} \right\},$$

where the number of the pole points is counted with their multiplicities. Since $D = \bigcup_p D_p$ there exists some p such that D_p is dense in D. Let $\mathscr{K}$ denote the quotient field of

$H(F'_{\mathrm{bor}} \times D)$, where F'_{bor} is the space F' equipped with the bornological topology associated to the strongly topological dual F' of F. Note that F'_{bor} is a bornological (DF)-space. Consider the subspace $\mathscr{V}$ of $\mathscr{K}^{p+1}$ generated by $\{V_k\}_{k\geq 1}$ with

$$V_k = (\widehat{C}_{-k}, \widehat{C}_{-k-1}, \dots, \widehat{C}_{-k-p}) \quad (k \geq 1),$$

where $\widehat{C}_{-k}(u, z) = u(C_{-k}(z))$ for $u \in F'_{\mathrm{bor}}$ and $z \in D$. Observe that $\widehat{C}_k \in \mathscr{K}$ for $k > -\infty$, because $\widehat{C}_k$ is Gâteaux holomorphic and bounded on sets of the form $B \times K$ where B is bounded in F'_{bor} and K is compact in K, and hence, $\widehat{C}_k$ is holomorphic [15]. We can check that $\dim \mathscr{V} \leq p$, that means

$$H = \det \begin{pmatrix} \widehat{C}_{-k_1} & \cdots & \widehat{C}_{-k_1-p} \\ \cdots & \cdots & \cdots \\ \widehat{C}_{-k_{1+p}} & \cdots & \widehat{C}_{-k_{1+p}-p} \end{pmatrix} \equiv 0 \quad \text{on } F' \times D$$

for $1 \leq k_1 < \cdots < k_{1+p}$. For each $z \in D_p$, from the meromorphicity of $\widehat{f}_z$, there exist $a_0, \dots, a_p \in F$ such that

$$\left(\sum_{m=0}^{p} a_m \lambda^m\right) \left(\sum_{k=-\infty}^{+\infty} C_k(z) \lambda^k\right)$$

is holomorphic on Δ_r. This means

$$a_0 C_{-k}(z) + \cdots + a_p C_{-k-p}(z) = 0 \quad \text{for } k \geq 1.$$

Hence $H(z) = 0$ for $z \in D_p$. Since D_p is dense in D we have $H \equiv 0$ on $F' \times D$. Choose a basis $V_{k_1}, \dots, V_{k_q}$ of $\mathscr{V}$, where $q \leq p$. Then for each $k \geq 1$ there exist holomorphic functions $\alpha^{(k)}, \alpha_1^{(k)}, \dots, \alpha_q^{(k)}$ with $\alpha^{(k)} \not\equiv 0$ such that

$$\alpha^{(k)} V_k = \sum_{i=1}^{q} \alpha_i^{(k)} V_{k_i}. \tag{3.1}$$

Since $q \leq p$, there exist holomorphic functions $\beta_0, \beta_1, \dots, \beta_p$ on D with values in F such that

$$\sum_{m=0}^{p} \widehat{\beta}_m \widehat{C}_{-k_i-m} \equiv 0 \quad \text{on } F'_{\mathrm{bor}} \times D \text{ for } 1 \leq i \leq q \tag{3.2}$$

where $\widehat{\beta}_m \colon F'_{\mathrm{bor}} \times D \to \mathbb{C}$ are given by

$$\widehat{\beta}_m(u, z) = u(\beta_m(z)), \quad \forall\, u \in F'_{\mathrm{bor}},\ 0 \leq m \leq p.$$

From (3.1) and (3.2) it follows that

$$\sum_{m=0}^{p} \widehat{\beta}_m \widehat{C}_{-k-m} \equiv 0 \quad \text{on } F' \times D \text{ for } k \geq 1.$$

Hence,

$$\widehat{g} := \left(\sum_{m=0}^{p} \widehat{\beta}_m(u,z)\lambda^m \right) \left(\sum_{k=-\infty}^{+\infty} \widehat{C}_k(u,z)\lambda^k \right)$$

is holomorphic on $F'_{\text{bor}} \times D \times \Delta_r$. Then the meromorphic function

$$\widehat{h} := \left(\sum_{m=0}^{p} \widehat{\beta}_m(u,z)\lambda^m \right)^{-1} \widehat{g}$$

is the extension of $\widehat{f}$, where $\widehat{f}\colon F'_{\text{bor}} \times D \times (\Delta_r \setminus \overline{\Delta}) \to \mathbb{C}$ is given by $\widehat{f}(u,z,\lambda) = u(f(z,\lambda))$. Now, for each continuous semi-norm α on F we consider $f_\alpha = \omega_\alpha f$, where F_α is the Banach space associated to α and $\omega_\alpha\colon F \to F_\alpha$ is the canonical map. By [8], f_α is extended to a meromorphic function g_α on $D \times \Delta_r$. Put

$$P = \bigcup_{\alpha} P(g_\alpha).$$

We have $P(\widehat{h}) \cap (F'_\alpha \times D \times \Delta_r) = P(\widehat{g}_\alpha) = F'_\alpha \times P(g_\alpha)$ for every α. Hence

$$P(\widehat{h}) = \bigcup_{\alpha} (F'_\alpha \times P(g_\alpha)) = F' \times P.$$

The equality yields that P is a hypersurface in $D \times \Delta_r$. Let $(z_0, \lambda_0) \in R(P)$, the regular locus of P. Then there exist a neighbourhood U of (z_0, λ_0) in $D \times \Delta_r$ and a holomorphic function σ on U such that $\sigma|_{P \cap U} = 0$ and $\sigma'(z,\lambda) \neq 0$ for $(z,\lambda) \in U$. Hence $\sigma^p \widehat{h}$ is holomorphic on $F'_{\text{bor}} \times U$ for some $p \geq 0$. On the other hand, since $\widehat{h}\big|_{F' \times [(D \times \Delta_r) \setminus P]}$ is a holomorphic extension of $\widehat{f}$ which is continuous linear in $u \in F'_{\text{bor}}$ for $(z,\lambda) \in D \times (\Delta_r \setminus \Delta)$, we infer that $\sigma^p \widehat{h}$ so is for $(z,\lambda) \in D \times \Delta_r$. Thus, $\sigma^p \widehat{h}$ induced a holomorphic function $\sigma^p h\colon U \to [F'_{\text{bor}}]'$. This yields that $\widehat{h}$ is induced by a meromorphic function h on $D \times \Delta_r$ with values in $[F'_{\text{bor}}]'$, since $\operatorname{codim} S(P) \geq 2$, where $S(P)$ is a singular locus of P. Moreover, by uniqueness and since $h = f$ on $[D \times (\Delta_r \setminus \Delta)] \setminus P$ it follows that h can be considered as a meromorphic function on $D \times \Delta_r$ with values in F. Hence f is extended meromorphically to $D \times \Delta_r$.

(c) General case. Assume that F is a sequentially complete locally convex space. As in notations of (b), put

$$P = \bigcup_{\alpha} P(g_\alpha).$$

First we show that P is a hypersurface in $D \times \Delta_r$. Let D' be a relatively compact open subset of D and $0 < r' < r$. For each continuous semi-norm α of F, let n_α denote the number of irreducible branches of $P(g_\alpha) \cap (D \times \Delta_{r'})$. To prove the analyticity of $P \cap (D \times \Delta_{r'})$ we have to check $\sup_\alpha n_\alpha < \infty$. Otherwise there exists a sequence $\{\alpha_k\}$ of continuous semi-norms on F such that $\alpha_k \nearrow +\infty$. Consider the space $E = \lim\operatorname{proj}_k F_{\alpha_k}$

and the canonical map $\omega\colon F \to E$. By (b), $\omega\circ f$ has a meromorphic extension $h\colon D\times\Delta_r \to E$. Since $P(g_{\alpha_k}) \subset P(h)$ for $k \geq 1$, and hence, the number of irreducible branches of $P(h) \cap (D' \times \Delta_r)$ is equal to $+\infty$. This is impossible. Since D' and r' are arbitrary, P is a hypersurface. Since $u \circ f$ has a holomorphic extension to $(D \times \Delta_r) \setminus P$ for every $u \in F'$, it follows that f has also a holomorphic extension $\widehat{f}$ to $(D \times \Delta_r) \setminus P$. It remains to show that $\widehat{f}$ in fact is meromorphic on $(D \times \Delta_r) \setminus S(P)$ and hence, on $D \times \Delta_r$. Given $w_0 \in (D \times \Delta_r) \setminus S(P)$. After changing of coordinates we can write $\widehat{f}$ in the form

$$\widehat{f}(z,\lambda) = \sum_{k=-\infty}^{+\infty} c_k(z)\lambda^k$$

in the neighbourhood $\Delta^n \times \Delta$ of w_0, where $c_k(z)$ are holomorphic on Δ^n. If $\widehat{f}$ is not meromorphic at w_0, then there exists a sequence $k_j \searrow -\infty$ such that $c_{k_j} \neq 0$ on Δ^n. Choose for each $j \geq 1$, $u_j \in F'$ such that $u_j \circ c_{k_j} \neq 0$. Consider $g = \eta \circ f$, where $\eta\colon F \to \mathbb{C}^{\mathbb{N}}$ is the map induced by $\{u_j\}$, that means $\eta(y) = \{u_j(y)\}_{j\geq 1}$. By (b), g has a meromorphic extension to $\Delta^n \times \Delta$. This yields that there exists j_0 such that

$$u_j \circ c_{k_j} \equiv 0 \quad \text{for } j < j_0.$$

It is impossible. □

We shall prove the following proposition.

Proposition 3.3. *Let $D_1 \subset \mathbb{C}^{n_1}$, $D_2 \subset \mathbb{C}^{n_2}$ be domains and $A_1 \subset D_1$, $A_2 \subset D_2$ be locally pluriregular subsets. Let F be a locally complete space and $W \subset F'$ be a subspace determining boundedness. Assume that $f\colon X \to F$ is a function such that*

(i) *for every $a_1 \in A_1$ the function $f_{a_1} \in H^{W,\infty}(D_2, F)$;*

(ii) *for every $a_2 \in A_2$ the function $f^{a_2} \in H^{W,\infty}(D_1, F)$.*

Then there exists exactly one $\widehat{f} \in H(\widehat{X}, F)$ with $\widehat{f} = f$ on X.

For the proof of Proposition 3.3 we need the following lemma.

Lemma 3.4. *Let $D_1 \subset \mathbb{C}^{n_1}$, $D_2 \subset \mathbb{C}^{n_2}$ be domains and $A_1 \subset D_1$, $A_2 \subset D_2$ be locally pluriregular subsets. Then for any $f \in H_s(X)$ there exists exactly one $\widehat{f} \in H(\widehat{X})$ with $\widehat{f} = f$ on X.*

Proof. Since A_1 and A_2 are locally pluriregular we have $X \subset \widehat{X}$. Observe that $A_1 \times A_2 \subset X$ is locally pluriregular, hence, is non-pluripolar. By [1, Theorem 3.4.2] there exists $\widehat{f} \in H(\widehat{X})$ with $\widehat{f} = f$ on X. Suppose there is a $\widehat{g} \in H(\widehat{X})$ with $\widehat{g} = f$ on X. Then by the non-pluripolarity of $A_1 \times A_2$ we get $\widehat{g} = \widehat{f}$ on $\widehat{X}$. Thus $\widehat{f}$ is uniquely determined. □

Now we can prove Proposition 3.3 as follows.

Since A_1, A_2 are locally pluriregular, they are non-pluripolar, hence, are of uniqueness and $X \subset \widehat{X}$. Then, by [5], for each $z \in A_1$ and $w \in A_2$ we have

$$f_z \in H^\infty(D_2, F) \quad \text{and} \quad f^w \in H^\infty(D_1, F).$$

Now, let $\varphi \in F'$ be an arbitrary continuous linear form on F. Consider the separately holomorphic function

$$\varphi \circ f\colon X \to \mathbb{C}.$$

By Lemma 3.4, there exists exactly one $\widehat{\varphi \circ f} \in H(\widehat{X})$ with $\widehat{\varphi \circ f} = \varphi \circ f$ on X. By the identity principle we can define the mapping

$$T\colon F'_{\text{bor}} \to H(\widehat{X}),$$

given by

$$T(\varphi)(z) = \widehat{\varphi \circ f}(z), \quad z \in \widehat{X},\ \varphi \in F'_{\text{bor}},$$

where F'_{bor} is F' equipped with the bornological topology associated with the strong topology β. By the uniqueness of extensions $\widehat{\varphi \circ f}$ and by using the identity principle, it follows that T is linear and has the closed graph. Hence, in view of the closed graph theorem of Grothendieck [7] we derive that T is continuous.

Now we can define the map $\widehat{f}\colon \widehat{X} \to [F'_{\text{bor}}]'_\beta$ by the formula

$$\widehat{f}(z)(\varphi) = T(\varphi)(z), \quad z \in \widehat{X},\ \varphi \in F'_{\text{bor}}.$$

For each $\varphi \in F'_{\text{bor}}$ we have

$$\widehat{f}(z)(\varphi) = T(\varphi)(z) = (\widehat{\varphi \circ f})(z), \quad z \in \widehat{X}$$

and hence, we deduce that $\widehat{f}\colon \widehat{X} \to [F'_{\text{bor}}]'_\beta$ is holomorphic. Since $(\varphi \circ \widehat{f})(z) = \widehat{f}(z)(\varphi) = (\widehat{\varphi \circ f})(z) = (\varphi \circ f)(z)$ for all $z \in X$ and for all $\varphi \in F'$, we have $\widehat{f} = f$ on the non-pluripolar set X. However, F is a closed subspace of $[F'_{\text{bor}}]'_\beta$, by the identity principle it follows that $\widehat{f}(\widehat{X}) \subset F$ and hence $\widehat{f}\colon \widehat{X} \to F$ is holomorphic.

It is obvious that $\widehat{u \circ f}$ is a holomorphic extension of $u \circ (f\big|_{A_1 \times A_2})$ for all $u \in W$. From the uniqueness of $A_1 \times A_2$ (because $A_1 \times A_2$ is non-pluripolar), by [5] $\widehat{f}$ is uniquely determined. The proposition is proved.

Since the equality $\omega(z, A_1, D_1) = 0$ for every locally pluriregular point z of A_1 from Proposition 3.3 we obtain

Corollary 3.5. *Let X be a 2-fold cross and $f\colon X \to F$ a function as in Proposition 3.3. Assume that $z_* \in \overline{A}_1$ is a locally pluriregular point of A_1. Then f can be extended holomorphically to a neighbourhood $\{z_*\} \times D'_2$ for all $D'_2 \subset D_2$.*

Lemma 3.6. *Let $f\colon \mathbb{C}^p \times \mathbb{C}^q \to F$ be a meromorphic function on a neighbourhood of $D \times \{0\}$ in $\mathbb{C}^{p+q}$. Then $-\log(R_f)_*$ is plurisubharmonic on D where $R_f(z)$ is the meromorphic radius of f_z at $0 \in \mathbb{C}^q$ defined by*

$$R_f(z) = \sup\{r > 0 : f_z \textit{ has a meromorphic extension on } \Delta_r^q\}$$

and $(R_f)_*$ is the lower semicontinuous regularization of R_f.

The lemma is proved similarly as of Proposition 1.4 and Remark 1.5 in [29] by using Theorem 3.2.

Lemma 3.7. *Let $D \subset \mathbb{C}^m$, $G_0 \subset \mathbb{C}^n$ be domains. Let $f\colon D \times G_0 \to F$ be a meromorphic function and let G be an open subset containing G_0, where F is a sequentially complete space. Assume that for almost $z \in D$ the function f_z has an (F, W)-meromorphic extension to G where W is a subspace of F' that determining boundedness. Then f is extended meromorphically to $D \times G$.*

Proof. The following argument is a small modification of [26].

Let $\mathbb{CP}(F)$ denote the projective space associated to F and $\pi\colon F \setminus \{0\} \to \mathbb{CP}(F)$ the canonical map. By the uniqueness, we may assume that F is separable.

Let us consider the following two cases.

Case 1: $n = 1$.

Because for almost $z \in D$ the function $u \circ f_z$ is extended meromorphically to G for all $u \in W$, by [6, Theorem 4], for almost $z \in D$ the function f_z is extended to a meromorphic function $\widehat{f_z}$ on G.

Let G_1 be an arbitrary open set with $G_1 \subset G_0$ and $D_1 = D \setminus \pi_D(I(f) \cap (D \times \overline{G}_1))$, where $\pi_D\colon D \times G \to D$ is the canonical projection. Then g_z is holomorphic on G_1 for $z \in D_1$, where $g = \pi \circ f$.

By Corollary 3.5 and [25, Theorem 1], for every open set G_2 with $G_1 \subset G_2 \subset G$ there exist an open subset $D_2 \subset D$ of full measure and a holomorphic function $g_2\colon D_2 \times G_2 \to \mathbb{CP}(F)$ such that

$$g_2\big|_{D_2 \times G_1} = g\big|_{D_2 \times G_1}.$$

Let $f_2\colon D_2 \times G_2 \to F$ be the meromorphic function induced by g_2. Then

$$f_2\big|_{D_2 \times G_1} = f\big|_{D_2 \times G_1}.$$

By [24], the envelope of holomorphy of $(D \times G_0) \cup (D_2 \times G_2)$ contains $D \times G_2$. Then, f_2 and hence f has a meromorphic extension to $D \times G_2$. Since G_2 is arbitrary, f is extended to $D \times G$.

Case 2: $n > 1$.

We can easily reduce the lemma to the case $G_0 = \Delta^n$, $G = \Delta^n_R$ with $R > 1$. Suppose that the lemma has been verified for $G_0 = \Delta^{n-1}$, $G = \Delta^{n-1}_R$. Assume that for all $u \in W$ the function $u \circ f_z$ has a meromorphic extension on Δ^n_R, where $z \in D \setminus A$ such that $\mathrm{mes}(A) = 0$. For each $r \in (0, R)$ put

$$B = \{(z, \xi) \in D \times \Delta : \{z\} \times \Delta_r \times \{\xi\} \subset I(f)\},$$
$$C = (A \times \Delta) \cup B.$$

Then $\mathrm{mes}(C) = 0$ and $u \circ f_{z,\xi}$ has a meromorphic extension to Δ^{n-1}_R for $(z, \xi) \in (D \times \Delta) \setminus C$ for all $u \in W$. By the inductive assumption (with D is replaced by $D \times \Delta_r$), it follows that f can be extended meromorphically to $D \times \Delta^{n-1}_R \times \Delta$.

Similarly, let

$$B' = \left\{(z, w) \in D \times \Delta^{n-1}_R : (z, w) \times \Delta_r \subset I(f)\right\},$$
$$C' = (A \times \Delta^{n-1}_R) \cup B'.$$

Then we also have $\mathrm{mes}(C') = 0$ and $u \circ f_{z,w}$ has a meromorphic extension to Δ_R for $(z, w) \notin C'$ for all $u \in W$. Hence, by Case 1, f can be extended meromorphically to $D \times \Delta^n_R$. □

We are now in a position to prove the Rothstein theorem.

Proof of Theorem 3.1. (i) *The case where* $G = \Delta^q_R$.

For each $z \in D$, let $R_f(z)$ denote the meromorphic radius of f_z at $0 \in \mathbb{C}^q$. Obviously, $R_f \geq 1$ on Δ^p and $R_f \geq R$ on A. Using Lemma 3.7 with $D = \Delta^p$ we can easily conclude that f extends meromorphically to the Hartogs domain

$$\Omega := \{(z, w) \in \Delta^p \times \mathbb{C}^q : |w| < (R_f)_*(z)\}.$$

By Lemma 3.6 and $R_f \geq R$ on A, using the local pluriregularity of A we conclude that $(R_f)_* \geq R$ on A. Thus $A \times \Delta^q_R \subset \Omega$, and therefore Ω is the required neighbourhood.

(ii) *The case where* G *is arbitrary.*

Fix $a \in A$. Let G_0 denote the set of all $b \in G$ such that there exist $r_b > 0$ and ${}^b f \in M(\Delta^{p+q}_{r_b}(a, b), F)$ with $\Delta^{p+q}_{r_b}(a, b) \subset \Delta^p \times G$ such that for all $\alpha \in A \cap \Delta^p_{r_b}(a)$ the function $u \circ {}^b f_\alpha$ has a meromorphic extension $(u \circ {}^b f_\alpha)\tilde{}$ on $\Delta^q_{r_b}(b)$ for all $u \in W$.

Obviously G_0 is open, $G_0 \neq \varnothing$ ($\Delta^q \subset G_0$). Using (i) one can prove that G_0 is closed in G. Thus $G_0 = G$.

Moreover, one can prove that if $\Delta^q_{r_{b_1}}(b_1) \cap \Delta^q_{r_{b_2}}(b_2) \neq \varnothing$ then ${}^{b_1} f = {}^{b_2} f$ on $\Delta^{p+q}_{r_{b_1}}((a, b_1)) \cap \Delta^{p+q}_{r_{b_2}}((a, b_2))$. This gives a meromorphic extension of f to an open neighbourhood of $a \times G$. Since a was arbitrary, we get the required neighbourhood Ω. The proof of the Rothstein theorem is complete. □

4. Domains of existence for Fréchet-valued meromorphic functions

Let D be a Riemann domain over a Banach space E and $f\colon D \to F$ be a holomorphic function on D with values in a Banach space F. By D^h_f we denote the domain of existence of f over E. Hirschowitz [9] proved that if E is separable the set

$$Z^h_f := \left\{x^* \in F' : D^h_f = D^h_{x^*f}\right\}$$

is not a first category set. Moreover, he also showed that in the general case this result is not true.

In the first part of this section we study the above problem for meromorphic functions. Then the obtained result is used to extend Fréchet-valued meromorphic functions.

Now, as in [2], by the sheaf theory method we can construct a Riemann domain D^m_f over E (which is called the domain of existence of f) such that it is the largest domain on which f can be extended to a meromorphic function $\widehat{f}$.

We shall prove the following theorem.

Theorem 4.1. *Let D be a Riemann domain over a separable Banach space E and f be a meromorphic function on D with values in a Banach space F. Then the set*

$$Z^m_f := \left\{x^* \in F' : D^m_f = D^m_{x^*f}\right\}$$

is dense in F'.

For the proof of Theorem 4.1 we need the following lemmas.

Lemma 4.2. *Let D be an open set in a Banach space E and S an analytic set in D with* $\operatorname{codim} S \geq 2$. *Then every meromorphic function f on $D \setminus S$ with values in a Banach space F can be extended meromorphically to D.*

Proof. Given $f\colon D \setminus S \to F$ a meromorphic function.

(i) First consider the case where $F = \mathbb{C}$. Let $f_0\colon (D \setminus S) \setminus I(f) \to \mathbb{CP}^1$ be the holomorphic function defined by f, where $\mathbb{CP}^1$ is the complex projective space. Then f is defined uniquely by the analytic set $\Gamma(f)$ in $(D \setminus S) \times \mathbb{CP}^1$ given by

$$\Gamma(f) := \operatorname{Cl}_{(D\setminus S)\times\mathbb{CP}^1}\left\{(z, f_0(z)) \in (D \setminus S) \times \mathbb{CP}^1\right\}.$$

By the Remmert-Stein-Ramis theorem [21], $\operatorname{Cl}_{D\times\mathbb{CP}^1}\Gamma(f)$ is an analytic set in $D \times \mathbb{CP}^1$ defining a meromorphic extension of f to D.

(ii) Now assume that F is an arbitrary Banach space. By (i) for every $x^* \in F'$, the meromorphic function x^*f on $D \setminus S$ can be extended to a meromorphic function $\widehat{x^*f}$ on D. On the other hand, by the Remmert-Stein-Ramis theorem [21], $V = \operatorname{Cl}[P(f)]$ is an analytic set in D. We can assume that $V \neq \varnothing$ and hence $\operatorname{codim} V = 1$.

Fix $z_0 \in S$. Take a neighbourhood W of z_0 in D such that

$$W \cap V = Z(\sigma)$$

where $Z(\sigma)$ is the zero-set of a holomorphic function σ on W.

For each pair (k, n) we put

$$A_{(k,n)} := \left\{ x^* \in F' : \sigma^n \widehat{x^* f} \in H(z_0 + U_k) \right\}$$

where $U_k = \left\{ x \in E : \|z\| < \frac{1}{k} \right\}$.

By the factoriality of the local ring $\mathcal{O}_{E,z_0}$ of germs of holomorphic functions at z_0 [17], we can find holomorphic functions g and β on $z_0 + U_k$ for some k such that

$$\widehat{x^* f}\Big|_{z_0+U_k} = \frac{g}{\beta}$$

and g_y, the germ of g at y, is prime with respect to β_y for all $y \in z_0 + U_k$. This implies that

$$P(\widehat{x^* f}) \cap (z_0 + U_k) = Z(\beta) \subseteq V \cap (z_0 + U_k) = Z(\sigma) \cap (z_0 + U_k).$$

Hence there exist i, n and $\gamma \in H(z_0 + U_i)$ such that

$$\sigma^n\big|_{z_0+U_i} = \gamma\beta\big|_{z_0+U_i}.$$

We prove that the sets $A_{(k,n)}$ are closed. Let $\left\{x_p^*\right\} \subset A_{(k,n)}$ and $x_p^* \to x^*$ in F'. Then $\left\{\sigma^n \widehat{x_p^* f}\right\} \subset H(z_0 + U_k)$ and this sequence converges to $\sigma^n \widehat{x^* f}$ in $H((z_0 + U_k) \setminus V)$. Since $\operatorname{codim} V = 1$, by the maximum principle, it follows that $\left\{\sigma^n \widehat{x_p^* f}\right\}$ converges to $\sigma^n \widehat{x^* f}$ in $H(z_0 + U_k)$. This means that $x^* \in A_{(k,n)}$ and hence $A_{(k,n)}$ is closed in F'.

Using the Baire theorem to $F' = \bigcup_{(k,n)} A_{(k,n)}$ we can find (k, n) such that $\sigma^n \widehat{x^* f}$ is holomorphic on $z_0 + U_k$ for all $x^* \in F'$. Thus $\sigma^n f$ is holomorphic at z_0 and hence f is meromorphic at z_0. The lemma is proved. □

Lemma 4.3. *Let D be an open set in a Banach space E and S an analytic set in D with* $\operatorname{codim} S = 1$. *Assume that G is an open subset of D such that every irreducible branch of S meets G. Then every meromorphic function f on $(D \setminus S) \cup G$ with values in a Banach space F can be extended meromorphically to D.*

Proof. It suffices to show that f can be meromorphically extended to every $z \in \partial G \cap S$. Given $z_0 \in \partial G \cap S$. By Lemma 4.2 we can assume that $z_0 \in R(S)$, the regular locus of S, and $z_0 = 0 \in E$. Take a neighbourhood U of z_0 of the form $\Delta \times V$, where Δ is the open unit disc in $\mathbb{C}$ and V is connected neighbourhood of zero in a Banach space such that $R(S) \cap U = 0 \times V$. Consider the Laurent expansion of f in $\Delta \times V$ at $(0, 0)$

$$f(t, z) = \sum_{k=-\infty}^{+\infty} a_k(z) t^k$$

where a_k are holomorphic functions on V.

Since f is meromorphic on G and $z_0 \in \partial G$, we can find a non-empty open subset W of V such that f is meromorphic on $\Delta \times W$. This yields that there exists k_0 such that

$$a_k(z) = 0 \quad \text{for every } z \in W \text{ and every } k < k_0.$$

By the connectedness of V we have

$$a_k = 0 \quad \text{for every } k < k_0.$$

Thus f is meromorphic at z_0. □

Now we can prove Theorem 4.1 as follows.

Proof of Theorem 4.1. Given $f\colon D \to F$ a meromorphic function, where D is a Riemann domain over separable Banach space E and F is a Banach space. Put

$$D^0 = D \setminus P(f) \quad \text{and} \quad f_0 = f\big|_{D^0}.$$

By the Hirschowitz's result in [9] it suffices to consider the case where there exists $x_0^* \in Z^h(f_0) \setminus Z^m(f)$. Since

$$(D^0)^h_{f_0} = (D^0)^h_{x_0^* f} \quad \text{and} \quad D^m_f \setminus (D^0)^h_{f_0} = P(\widehat{f}),$$

by the local biholomorphicity of the canonical map $D^m_f \to D^m_{x_0^* f}$, it follows that D^m_f is contained in $D^m_{x_0^* f}$ as an open set. We first have the following relations

$$D^m_{x_0^* f} \setminus D^m_f = P(\widehat{x_0^* f}) \setminus P(\widehat{f}) \quad \text{and} \quad P(\widehat{f}) \subseteq P(\widehat{x_0^* f}).$$

Indeed, let $z \in D^m_{x_0^* f} \setminus D^m f$, but $z \notin P(\widehat{x_0^* f}) \setminus P(\widehat{f})$. Then

$$z \in (D^0)^h_{x_0^* f_0} = (D^0)^h_{f_0} \subseteq D^m_f$$

which is impossible.

The inverse inclusion is trivial. The second relation follows from that

$$(D^0)^h_{f_0} = (D^0)_{x_0^* f_0}.$$

Let $z_0 \in \partial R(P(\widehat{f}))$, the boundary of $R(P(\widehat{f}))$ in $R(P(\widehat{x_0^* f}))$. We may assume that $z_0 = 0 \in E$. Take a neighbourhood U of z_0 of the form $\Delta \times V$, where Δ is the open unit disc in $\mathbb{C}$ and V is a connected neighbourhood of zero in a Banach space. As in Lemma 4.3, consider the Laurent expansion of $\widehat{f}$ in $\Delta \times V$ at $(0, 0)$

$$\widehat{f}(t, z) = \sum_{k=-\infty}^{+\infty} a_k(z) t^k$$

where a_k are holomorphic functions on V.

Since $z_0 \in \partial R(P(f))$ we can find a non-empty open subset W of V such that $\widehat{f}$ is meromorphic on $\Delta \times W$. As in Lemma 4.3 we infer that for some k_0 we have

$$a_k = 0 \quad \text{for every } k < k_0.$$

Hence $\widehat{f}$ is meromorphic at z_0.

Thus we can write

$$P(\widehat{x_0^* f}) \setminus \mathrm{Cl}_{D^m_{x_0^* f}}(P(\widehat{f})) = \left(\bigcup_{j \geq 1} X_j \right) \cup S(P(\widehat{x_0^* f}))$$

where X_j are irreducible branches and $S(P(\widehat{x_0^* f}))$ is the singular locus of $P(\widehat{x_0^* f})$. Since $\operatorname{codim} P(\widehat{x_0^* f}) \geq 2$, by Lemma 4.2, we have

$$P(\widehat{x_0^* f}) \setminus \mathrm{Cl}_{D^m_{x_0^* f}}(P(\widehat{f})) = \bigcup_{j \geq 1} X_j.$$

For each $j \geq 1$ take $z_j \in R(X_j)$ and write in a neighbourhood U_j of z_j of the form $\Delta \times V_j$ as in Lemma 4.3 the Laurent expansion of $\widehat{f}$ at $(0,0)$

$$f(t,z) = \sum_{k=-\infty}^{+\infty} a_k^j(z) t^k.$$

We may assume that

$$a_k^j \neq 0 \quad \text{for every } k < 0 \text{ and } j \geq 1.$$

Put

$$A_j = \left\{ x^* \in F' : x^*(a_k^j) \neq 0 \text{ for every } k < 0 \right\},$$
$$G_j^k = \left\{ x^* \in F' : x^*(a_k^j) \neq 0 \right\}.$$

Obviously, G_j^k is open for every $k < 0$ and $j \geq 1$. Moreover, G_j^k is dense in F'. Indeed, in the converse case we can find $x_1^* \in F'$ and $\varepsilon > 0$ such that

$$(x_1^* - x^*)(a_k^j) = 0 \quad \text{for every } x^* \in F' \text{ with } \|x^*\| < \varepsilon.$$

Then $a_k^j = 0$ which is impossible. By the Baire theorem

$$T = \bigcap_{j \geq 1} A_j = \bigcap_{\substack{k<0 \\ j \geq 1}} G_j^k$$

is dense in F'.

For each $x^* \in T$ consider the continuous linear map $\Theta_{x^*}\colon F \to \mathbb{C}^2$ given by (x_0^*, x^*). Then the set

$$Z_{x^*} = \left\{y^* \in \mathbb{C}^2 : D^m_{y^*\Theta_{x^*}f} = D^m_{\Theta_{x^*}f}\right\}$$

is dense in $\mathbb{C}^2$. Then the set

$$Z := \{y^*\Theta_{x^*} : y^* \in Z_{x^*}, x^* \in T\}$$

is dense in F'. Indeed, given $x^* \in F'$ and $\varepsilon > 0$. Take $x_1^* \in T$ such that $\|x^* - x_1^*\| < \varepsilon$. Since $Z_{x_1^*}$ is dense in $\mathbb{C}^2$, there exists $y = (a,b) \in Z_{x_1^*}$ such that

$$|a| + |b-1| < \frac{\varepsilon}{3(\|x_0^*\| + \|x^*\| + 1)}$$

with $|b| < 1$. Then

$$\begin{aligned}\left\|y^*\Theta_{x_1^*} - x^*\right\| &= \|ax_0^* + bx_1^* - x^*\| \\ &\le |a|\,\|x_0^*\| + |b|\,\|x_1^* - x^*\| + |b-1|\,\|x_0^*\| < \varepsilon.\end{aligned}$$

Since $D^m_{\Theta_{x^*}f} \subseteq D^m_{x_0^*f}$, by Lemma 4.3, $\Theta_{x^*}f$ is not meromorphic at every point belonging to $\bigcup_{j\ge1} X_j$, we infer that

$$D^m_{y^*\Theta_{x^*}f} = D^m_{\Theta_{x^*}f} = D^m_f$$

for all $y^*\Theta_{x^*} \in Z$. Theorem 4.1 is proved. □

Now, using Theorem 4.1 we prove the following result.

Theorem 4.4. *Let $f\colon D \to F$ be a meromorphic function, where D is a Riemann domain over a Banach space E with a Schauder basis and F is a sequentially complete space. Then f can be extended meromorphically to $\widehat{D}$, the envelope of holomorphy of D.*

Proof. As in the proof of Theorem 3.2, it suffices to consider the space F is Fréchet.

Given $f\colon D \to F$ a meromorphic function as in Theorem 4.4. Cover D by a sequence of open sets U_j such that for every $j \ge 1$ there exist bounded holomorphic functions h_j and σ_j on U_j for which $f\big|_{U_j} = \frac{h_j}{\sigma_j}$. Since F is a Fréchet space there exists a sequence $\varepsilon_j \searrow 0$ such that

$$B = \overline{\operatorname{conv}} \bigcup_{j\ge1} \varepsilon_j h_j(U_j)$$

is bounded. It is easy to see that in the fact f is a meromorphic function on D with values in the canonical Banach space F_B generated by B. From Theorem 4.1 and from the pseudoconvexity of the domain of existence of a scalar meromorphic function [2] it follows that D^m_f is pseudoconvex. Since E has a Schauder basis, D^m_f is a domain of holomorphy [21]. This implies that f can be meromorphically extended to $\widehat{D}$. □

5. A generalization of the Kazarian theorem

In [13] Kazarian proved that if f is a separately meromorphic function on a 2-fold cross $X := \mathbb{X}(E, F; D, G) = (D \times F) \cup (E \times G)$ where $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ are domains and $E \subset D$, $F \subset G$ are pluriregular compact sets then there exists a meromorphic function $\widetilde{f}$ on

$$\widehat{X} = \{(z, w) \in D \times G : \omega(z, E, D) + \omega(w, F, G) < 1\}$$

such that $\widetilde{f} = f$ on X.

The following theorem is a generalization of this result for the class of separately $(\cdot, W)$-meromorphic functions.

Theorem 5.1. *Let $D_1 \subset \mathbb{C}^{n_1}$, $D_2 \subset \mathbb{C}^{n_2}$ be domains and $A_1 \subset D_1$, $A_2 \subset D_2$ be locally pluriregular compact subsets. Let F be a sequentially complete locally convex space, W be a separating subset of F' and $f\colon X = \mathbb{X}(A_1, A_2; D_1, D_2) \to F$ be a function such that*

(i) *for every $a_1 \in A_1$, $f_{a_1} \in M^W(D_2, F)$;*

(ii) *for every $a_2 \in A_2$, $f^{a_2} \in M^W(D_1, F)$*

where $f_{a_1}(w) := f(a_1, w)$ for all $w \in D_2$ and $f^{a_2}(z) := f(z, a_2)$ for all $z \in D_1$. Then f is extended meromorphically to

$$\widehat{X} = \{(z_1, z_2) \in D_1 \times D_2 : \omega(z_1, A_1, D_1) + \omega(z_2, A_2, D_2) < 1\}\,.$$

Some lemmas in the previous section and the following proposition will be used to prove the theorem.

Proposition 5.2. *Let $f\colon X \to F$ be as in the Theorem* 5.1. *Then there exist open subsets $D_1^0 \subset D_1$, $D_2^0 \subset D_2$ and non-pluripolar compact sets $A_1^0 \subset A_1 \cap D_1^0$, $A_2^0 \subset A_2 \cap D_2^0$ such that*

$$f\big|_{X^0}\colon X^0 := (D_1^0 \times A_2^0) \cup (A_1^0 \times D_2^0) \to F$$

is separately holomorphic.

Proof. Recall that if S is a pluripolar closed set in $D_1 \times D_2$ then the sets

$$S_1 := \{z \in A_1 : \{z\} \times D_2 \subset S\}$$
$$S_2 := \{w \in A_2 : D_1 \times \{w\} \subset S\}$$

are pluripolar. Since $A_1 \backslash S_1$ and $A_2 \backslash S_2$ are non-pluripolar, by Bedford and Taylor [3] there exist locally pluriregular points $z_0 \in A_1 \setminus S_1$ and $w_0 \in A_2 \setminus S_2$ for A_1 and A_2 respectively. Choose $a \in D_1$ and $b \in D_2$ such that $(a, w_0) \notin S$ and $(z_0, b) \notin S$. Let $r > 0$ be sufficiently small such that

$$Z^0 := (B(a,r) \times (\overline{B}(w_0,r) \cap A_2)) \cup ((\overline{B}(z_0,r) \cap A_1) \times B(b,r)) \subset X \setminus S.$$

Note that $A_1^0 := \overline{B}(z_0,r) \cap A_1$ and $A_2^0 := \overline{B}(w_0,r) \cap A_2$ are non-pluripolar.

It remains to show that f is separately holomorphic on Z^0. Given $z_1 \in A_1^0$. By the hypothesis, $u \circ f_{z_1}$ has a meromorphic extension $\widehat{u \circ f}_{z_1}$ on $D_2^0 := B(b,r)$ such that

$$\widehat{u \circ f}_{z_1}\Big|_{D_2^0 \setminus I(\widehat{u\circ f}_{z_1})} = u \circ f_{z_1}\Big|_{D_2^0 \setminus I(\widehat{u\circ f}_{z_1})} \quad \text{for all } u \in W.$$

Thus $\widehat{u \circ f}_{z_1}\Big|_{D_2^0 \setminus I(\widehat{u\circ f}_{z_1})}$ and hence, $\widehat{u \circ f}_{z_1}$ is holomorphic on D_2^0, because

$$\operatorname{codim} I(\widehat{u \circ f}_{z_1}) \geq 2 \quad \text{and} \quad \{z_1\} \times (D_2^0 \setminus I(\widehat{u \circ f}_{z_1})) \subset X \setminus S.$$

Since A_2^0 is non-pluripolar, by [5, Theorem 2.2], the function $\widehat{f}_{z_1}$ is holomorphic on D_2^0.

Similarly, $\widehat{f}^{w_1}$ is holomorphic on $D_1^0 = B(a,r)$ for $w_1 \in A_2^0$. □

Proof of Theorem 5.1. From Proposition 5.2 there exists $X^0 \subset X$ such that $f\big|_{X^0}$ is separately holomorphic. In the case where A_1^0 and A_2^0 are locally pluriregular (then $A_1^0 = (A_1^0)^*$ and $A_2^0 = (A_2^0)^*$), by Quang and Dai [19, Theorem 4.5], the function f extends holomorphically to the neighbourhood $\widehat{X^0} = \{(z_1,z_2) \in D_1 \times D_2 : \omega(z_1, A_1^0, D_1^0) + \omega(z_2, A_2^0, D_2^0) < 1\}$ of X^0.

By Corollary 3.5, the function f extends holomorphically to an open neighbourhood $U_* \times G_2^0$ of $\{z_*\} \times A_2^0$ for all $z_* \in H_{A_1^0} = \{z \in A_1^0 : z \text{ is a locally pluripolar point of } A_1^0\}$, where $U_* \subset D_1^0$ and $A_2^0 \subset G_2^0 \subset D_2^0$. Let $E_1 \subset A_1 \cap U_*$ be an arbitrary compact set. Then by the hypothesis, for each $z_* \in A_1 \setminus S_1$ the function $u \circ f_{z_*}$ is meromorphic on D_2 for all $u \in W$. Because f is holomorphic on $U_* \times G_2^0$ by Lemma 3.7, f has a meromorphic extension $\widehat{f}$ to a neighbourhood of $H_{E_1} \times D_2$. That means for each relatively compact domain $D_2' \subset D_2$ with $A_2 \subset D_2'$ there exists a neighbourhood U_{E_1} of H_{E_1} such that $\widehat{f}$ is meromorphic on $U_{E_1} \times D_2'$.

Now, we prove that

$$\widehat{f}\Big|_{[(A_1 \setminus S_1) \cap U_{E_1}] \times D_2'} = f\Big|_{[(A_1 \setminus S_1) \cap U_{E_1}] \times D_2'}.$$

Indeed, fix a $w_* \in A_2 \setminus S_2$. The functions $\widehat{f}^{w_*}$ and f^{w_*} are meromorphic on $U_{H_{E_1}} \subset D_1$ and coincide on the non-pluripolar set H_{E_1}. By the unique theorem, $\widehat{f}^{w_*} = f^{w_*}$ on U_{E_1}. Now, we fix $z_* \in (A_1 \cap U_{E_1}) \setminus S_1$. Then, the meromorphic functions $\widehat{f}_{z_*}$ and f_{z_*} on D_2' coincide on the non-pluripolar set $A_2 \setminus S_2$. This implies that $\widehat{f}_{z_*} = f_{z_*}$ on D_2'. Since z_* is arbitrary we have

$$\widehat{f} = f \quad \text{on } [(A_1 \setminus S_1) \cap U_{E_1}] \times D_2'.$$

Thus, for each fix $w_* \in A_2 \setminus S_2$ the function $u \circ \widehat{f}^{w_*}$ is meromorphic on D_1 for all $u \in W$. Again by Lemma 3.7, $\widehat{f}$ has a meromorphic extension $\widetilde{f}$ on a neighbourhood of $D_1 \times H_{A_2}$. That means for each relatively compact domain $D_1' \subset D_1$ with $A_1 \subset D_1'$ there exists a neighbourhood U_{A_2} of H_{A_2} such that $\widetilde{f}$ is meromorphic on $D_1' \times U_{A_2}$.

Now, because $\widetilde{f}$ is meromorphic on a neighbourhood of $D_1 \times H_{A_2}$, for each fix $z_* \in A_1 \setminus S_1$ the function $u \circ \widetilde{f}_{z_*}$ is meromorphic on D_2. Using the same argument as above, we deduce that $\widetilde{f}$ has a meromorphic extension (we use the same denote $\widetilde{f}$) on a neighbourhood of $(D_1 \times H_{A_2}) \cup (H_{A_1} \times D_2)$. But, by locally pluriregularity of A_1, A_2 we have $H_{A_1} = A_1$ and $H_{A_2} = A_2$. Thus, the function f extends meromorphically to a neighbourhood of $(D_1 \times A_2) \cup (A_1 \times D_2)$.

The theorem is proved. □

Theorem 5.3. *Let D be an open in $\mathbb{C}^n$ and A_1, A_2 be non-pluripolar compact sets in D and $\mathbb{C}^m$ respectively. Let F be a sequentially complete locally convex space, W be a separating subset of F' and $f\colon X = \mathbb{X}(A_1, A_2; D, \mathbb{C}^m) \to F$ be a function such that*

(i) *for every $a_1 \in A_1$, $f_{a_1} \in M^W(D, F)$,*

(ii) *for every $a_2 \in A_2$, $f^{a_2} \in M^W(\mathbb{C}^m, F)$.*

Then f can be extended meromorphically to $D \times \mathbb{C}^m$.

Proof. Put

$$H_{A_1} = \{z \in A_1 : A_1 \text{ not thin at } z\} \cup \left\{z \in A_1 : A_1 \cap \overline{B}(z, r) \text{ is not pluripolar}\right\}$$

and

$$H_{A_2} = \{w \in A_2 : A_2 \text{ not thin at } w\} \cup \left\{w \in A_2 : A_2 \cap \overline{B}(w, r) \text{ is not pluripolar}\right\}.$$

Since if A_1 is locally pluriregular at z, then z is not thin for A_1 and since every non-pluripolar set contains a pluriregular point, it follows that $A_1 \setminus H_{A_1}$ is pluripolar. Similarly, $A_2 \setminus H_{A_2}$ is also pluripolar. Then by N. T. Van and Zeriahi [16] we have

$$\widehat{H} = D \times \mathbb{C}^m \quad \text{where} \quad H = (D \times H_{A_2}) \cup (H_{A_1} \times \mathbb{C}^m).$$

By the same argument as in the proof of Theorem 5.1, f can be extended to a meromorphic function on a neighbourhood Z of H. Since $\widehat{Z} \supset \widehat{H} = D \times \mathbb{C}^m$ it follows that f has a meromorphic extension on $D \times \mathbb{C}^m$. □

6. Cross theorems for $(\cdot, W)$-meromorphic functions with pluripolar singularities

In this section we shall consider an extension of the cross theorem for separately meromorphic functions with pluripolar singularities of Jarnicki and Pflug in [10] to the case of separately $(\cdot, W)$-meromorphic functions.

In order to do that we need some results on the extension of separately $(\cdot, W)$-holomorphic functions.

First of all, we generalize the extension theorem for separately holomorphic functions with singularities which was proved in [11] to separately (F, W)-holomorphic functions.

Theorem 6.1. *Let $D_j \subset \mathbb{C}^{k_j}$ be a pseudoconvex domain, let $A_j \subset D_j$ be a locally pluriregular set, $k_j \in \mathbb{N}$, $j = 1, 2$, and let U be an open neighbourhood of 2-fold cross $X := \mathbb{X}(A_1, A_2; D_1, D_2)$. Let $M \subset U$ be a relatively closed subset of U such that*

$$\Sigma_1 := \Sigma_1(A_1, A_2; M) = \left\{z_2 \in A_2 : M_{(\cdot, z_2)} \text{ is not pluripolar}\right\},$$
$$\Sigma_2 := \Sigma_2(A_1, A_2; M) = \left\{z_1 \in A_1 : M_{(z_1, \cdot)} \text{ is not pluripolar}\right\}$$

are pluripolar. Put

$$X' := \mathbb{T}(A_1, A_2; D_1, D_2; \Sigma_1, \Sigma_2) = \mathbb{X}(A_1 \setminus \Sigma_2, A_2 \setminus \Sigma_1; D_1, D_2).$$

Let F be a sequentially complete locally convex space and let $W \subset F'$ be a subspace which determines boundedness in F.

Then there exists a relatively closed pluripolar subset $\widehat{M} \subset \widehat{X}$ such that

(1) *$\widehat{M} \cap X' \subset M$;*

(2) *for every $f \in H_s^{W,\infty}(X' \setminus M, F)$ there is exactly one $\widehat{f} \in H^\infty(\widehat{X} \setminus \widehat{M}, F)$ with $\widehat{f} = f$ on $X' \setminus M$;*

(3) *$\widehat{M}$ is singular with respect to the family $\left\{\widehat{f} : f \in H_s^{W,\infty}(X' \setminus M, F)\right\}$;*

(4) *$\widehat{X} \setminus \widehat{M}$ is a pseudoconvex domain.*

Proof. For each $z \in A_1 \setminus \Sigma_2$ and $w \in A_2 \setminus \Sigma_1$ we write

$$D_1 \setminus M_{(\cdot, w)} = \bigcup_{\alpha \in I} (D_1 \setminus M_{(\cdot, w)})_\alpha \quad \text{and} \quad D_2 \setminus M_{(z, \cdot)} = \bigcup_{\beta \in J} (D_2 \setminus M_{(z, \cdot)})_\beta$$

where $(D_1 \setminus M_{(\cdot, w)})_{\alpha \in I}$ and $(D_2 \setminus M_{(z, \cdot)})_{\beta \in J}$ are connected components of $D_1 \setminus M_{(\cdot, w)}$ and $D_2 \setminus M_{(z, \cdot)}$ respectively.

For each $z_0 \in A_1 \setminus \Sigma_2$ and $w_0 \in A_2 \setminus \Sigma_1$ put

$$f_{z_0}(w) := f(z_0, w) \quad \text{for all } w \in D_2,$$
$$f^{w_0}(z) := f(z, w_0) \quad \text{for all } z \in D_1.$$

By the hypothesis $f \in H_s^{W,\infty}(X' \setminus M, F)$ for each $z \in A_1 \setminus \Sigma_2$ and $w \in A_2 \setminus \Sigma_1$ we have $u \circ f_z \in H^\infty(D_2 \setminus M_{(z,\cdot)})$ and $u \circ f^w \in H^\infty(D_1 \setminus M_{(\cdot,w)})$.

Since A_j is locally pluriregular and Σ_i is pluripolar, the set $A_j \setminus \Sigma_i$ $(i, j = 1, 2, i \neq j)$ is non-pluripolar, hence, is of uniqueness. Then, by [5, Theorem 2.2], for each $z \in A_1 \setminus \Sigma_2$ and $w \in A_2 \setminus \Sigma_1$ we have

$$f_{z,\beta} \in H^\infty((D_2 \setminus M_{(z,\cdot)})_\beta, F) \quad \text{and} \quad f^{w,\alpha} \in H^\infty((D_1 \setminus M_{(\cdot,w)})_\alpha, F)$$

for all $\alpha \in I$ and $\beta \in J$, where $f_{z,\beta} = f_z\big|_{(D_2 \setminus M_{(z,\cdot)})_\beta}$ and $f^{w,\alpha} = f^w\big|_{(D_1 \setminus M_{(\cdot,w)})_\alpha}$. Because $(D_1 \setminus M_{(\cdot,w)})_\alpha$ and $(D_2 \setminus M_{(z,\cdot)})_\beta$ are connected components of $D_1 \setminus M_{(\cdot,w)}$ and $D_2 \setminus M_{(z,\cdot)}$ respectively, the families $\{f_{z,\beta}\}_{\beta \in J}$ and $\{f^{w,\alpha}\}_{\alpha \in I}$ define functions

$$f_z \in H^\infty(D_2 \setminus M_{(z,\cdot)}, F) \quad \text{and} \quad f^w \in H^\infty(D_1 \setminus M_{(\cdot,w)}, F)$$

respectively.

Now, let $\varphi \in F'$ be an arbitrary continuous linear form on F. Consider the separately holomorphic function

$$\varphi \circ f \colon X' \setminus M \to \mathbb{C}.$$

By a result of Jarnicki and Pflug [10, Theorem 1.1] there exists a relatively closed pluripolar subset $\widehat{M} \subset \widehat{X'}$ such that

(1) $\widehat{M} \cap X' \subset M$;

(2') there is exactly one $\widehat{\varphi \circ f} \in H^\infty(\widehat{X'} \setminus \widehat{M})$ with $\widehat{\varphi \circ f} = \varphi \circ f$ on $X' \setminus M$;

(3') $\widehat{M}$ is singular with respect to the family $\left\{\widehat{\varphi \circ f} : \varphi \circ f \in H_s^{W,\infty}(X' \setminus M)\right\}$;

(4) $\widehat{X} \setminus \widehat{M}$ is a pseudoconvex domain.

Note that, since Σ_1, Σ_2 are pluripolar, we have $\widehat{X'} = \widehat{X}$.

By the identity principle we can define the mapping

$$T \colon F'_{\text{bor}} \to H(\widehat{X} \setminus \widehat{M}),$$

given by

$$T(\varphi)(z) = \widehat{\varphi \circ f}(z), \quad z \in \widehat{X} \setminus \widehat{M},\ \varphi \in F'_{\text{bor}},$$

where F'_{bor} is F' equipped with the bornological topology associated with the strong topology β.

By the uniqueness of extensions $\widehat{\varphi \circ f}$ and by using the identity principle, it follows that T is linear and has the closed graph. Hence, in view of the closed graph theorem of Grothendieck [7, Introduction, Theoreme B] we derive that T is continuous.

Now we can define the map $\widehat{f}\colon \widehat{X} \setminus \widehat{M} \to [F'_{\mathrm{bor}}]'_\beta$ by the formula

$$\widehat{f}(z)(\varphi) = T(\varphi)(z), \quad z \in \widehat{X} \setminus \widehat{M},\ \varphi \in F'_{\mathrm{bor}}.$$

For each $\varphi \in F'_{\mathrm{bor}}$ we have

$$\widehat{f}(z)(\varphi) = T(\varphi)(z) = (\widehat{\varphi \circ f})(z), \quad z \in \widehat{X} \setminus \widehat{M}$$

and hence, we deduce that $\widehat{f}\colon \widehat{X} \setminus \widehat{M} \to [F'_{\mathrm{bor}}]'_\beta$ is holomorphic.

Since $(\varphi \circ \widehat{f})(z) = \widehat{f}(z)(\varphi) = (\widehat{\varphi \circ f})(z) = (\varphi \circ f)(z)$ for all $z \in X' \setminus M$ and for all $\varphi \in F'$, we have $\widehat{f} = f$ on the non-pluripolar set $X' \setminus M$. Thus, (2) is proved.

However, F is a closed subspace of $[F'_{\mathrm{bor}}]'_\beta$, by the identity principle it follows that $\widehat{f}\colon \widehat{X} \setminus \widehat{M} \to F$ is holomorphic.

Obviously, (3') implies (3). The theorem is proved. □

Theorem 6.2. *Let D_j be a Riemann domain over $\mathbb{C}^{k_j}$, let $A_j \subset D_j$ be a locally pluriregular set, $k_j \in \mathbb{N}$, and let $\Sigma_j \subset \Sigma_j^0 \subset A_i$ be such that Σ_j^0 is pluripolar, $i, j = 1, 2$, $i \neq j$. Consider 2-fold crosses*

$$\begin{aligned} X &:= \mathbb{X}(A_1, A_2; D_1, D_2), \\ T &:= \mathbb{T}(A_1, A_2; D_1, D_2; \Sigma_1, \Sigma_2) = \mathbb{X}(A_1 \setminus \Sigma_2, A_2 \setminus \Sigma_1; D_1, D_2), \\ T^0 &:= \mathbb{T}(A_1, A_2; D_1, D_2; \Sigma_1^0, \Sigma_2^0) = \mathbb{X}(A_1 \setminus \Sigma_2^0, A_2 \setminus \Sigma_1^0; D_1, D_2) \end{aligned}$$

and the center of T^0

$$c(T^0) := T^0 \cap A_1 \times A_2 = (A_1 \times A_2) \setminus (\Sigma_2^0 \times \Sigma_1^0).$$

Let $M \subset T$ be relatively closed satisfying

(a) *for any $a_1 \in A_1 \setminus \Sigma_2$, $a_2 \in A_2 \setminus \Sigma_1$ the fibers $M_{(a_1,\cdot)}$, $M_{(\cdot,a_2)}$ are closed in D_2 and D_1 respectively;*

(b) *for any $a_1 \in A_1 \setminus \Sigma_2^0$, $a_2 \in A_2 \setminus \Sigma_1^0$ the fibers $M_{(a_1,\cdot)}$, $M_{(\cdot,a_2)}$ are pluripolar in D_2 and D_1 respectively.*

Let F be a sequentially complete locally convex space and let $W \subset F'$ be a subspace which determines boundedness in F and let

$$f \in \begin{cases} H_s^{W,\infty}(X \setminus M, F) & \text{if } \Sigma_1 = \Sigma_2 = \varnothing, \\ H_s^{W,\infty,c}(T \setminus M, F) & \text{otherwise.} \end{cases}$$

Then there exist $T' := \mathbb{T}(A_1, A_2; D_1, D_2; \Sigma_1', \Sigma_2')$ with $\Sigma_j^0 \subset \Sigma_j' \subset A_i$, Σ_j' pluripolar, $i, j = 1, 2$ and a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ such that

(1) $\widehat{M} \cap (c(T^0) \cup T') \subset M$;

(2) *there exists an* $\widehat{f} \in H^\infty(\widehat{X} \setminus \widehat{M}, F)$ *with* $\widehat{f} = f$ *on* $(c(T^0) \cup T') \setminus M$;

(3) $\widehat{f}(\widehat{X} \setminus \widehat{M}) \subset f(T \setminus M)$.

Proof. It is sufficient to prove for the case $H_s^{W,\infty,c}(T \setminus M, F)$ because the remain can be proved in a similar way.

Because $T^0 \subset T$ we can consider ${}^0f := f|_{T^0 \setminus M}$. By $f \in H_s^{W,\infty,c}(T \setminus M, F)$, for each $a_i \in A_i \setminus \Sigma_j$ $(i \neq j)$, the functions $u \circ {}^0f_{a_1}$ and $u \circ {}^0f^{a_2}$ admit holomorphic extensions $\widehat{f}_{a_1} \in H^\infty(D_2 \setminus M_{(a_1,\cdot)})$ and $\widehat{f}_{a_2} \in H^\infty(D_1 \setminus M_{(\cdot,a_2)})$ respectively. Since Σ_i^0 is pluripolar and A_j is locally pluriregular $(i, j = 1, 2)$, the sets $A_j \setminus \Sigma_i^0$ are non-pluripolar. Then, by [5, Theorem 2.2] and similar arguments as in the proof of Theorem 6.1 the functions ${}^0f_{a_1}$, ${}^0f^{a_2}$ admit (unique) holomorphic extensions $\widehat{f}_{a_1} \in H^\infty(D_2 \setminus M_{(a_1,\cdot)}, F)$ and $\widehat{f}^{a_2} \in H^\infty(D_1 \setminus M_{(\cdot,a_2)}, F)$ respectively. Thus, by the uniqueness of holomorphic extensions we have

$$\varphi \circ f \in H_s^{W,\infty,c}(T \setminus M), \quad \forall \varphi \in F'.$$

Now, by [12, Theorem 10.2.9] there exist $T' := \mathbb{T}(A_1, A_2; D_1, D_2; \Sigma_1', \Sigma_2')$ with $\Sigma_j^0 \subset \Sigma_j' \subset A_i$, Σ_j' pluripolar, $i, j = 1, 2$, and a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ such that

(1) $\widehat{M} \cap (c(T^0) \cup T') \subset M$;

(2') there exists a $\widehat{\varphi \circ f} \in H^\infty(\widehat{X} \setminus \widehat{M})$ with $\widehat{\varphi \circ f} = \varphi \circ f$ on $(c(T^0) \cup T') \setminus M$ for all $\varphi \in F'$;

(3') $\widehat{\varphi \circ f}(\widehat{X} \setminus \widehat{M}) \subset \varphi \circ f(T \setminus M)$.

By the identity principle we can define the mapping

$$T\colon F'_{\text{bor}} \to H(\widehat{X} \setminus \widehat{M}),$$

given by

$$T(\varphi)(z) = \widehat{\varphi \circ f}(z), \quad z \in \widehat{X} \setminus \widehat{M},\ \varphi \in F'_{\text{bor}},$$

where F'_{bor} is F' equipped with the bornological topology associated with the strong topology β.

From now, the proof of (2) runs as in the last part in the proof of Theorem 6.1. Because F' is separating, from (3') we obtain (3). □

From this theorem, as in [12] (see Theorem 10.2.9 implies Theorem 10.2.12) we get the following.

Theorem 6.3. *Assume that $D_j, A_j, \Sigma_j, \Sigma_j^0, X, T, T_0$ and function f are as in Theorem 6.2 where $M \subset T$ is analytic in the sense that $M = T \cap S$ where $S \subset U$ is an analytic subset of an open neighbourhood $U \subset \widehat{X}$ of T with $\operatorname{codim} S \geq 1$. Then there exist an analytic set $\widehat{M} \subset \widehat{X}$ and an open neighbourhood $U_0 \subset U$ of T such that*

(1) $\widehat{M} \cap U_0 \subset S$;

(2) *there exists an $\widehat{f} \in H^\infty(\widehat{X} \setminus \widehat{M}, F)$ with $\widehat{f} = f$ on $T \setminus M$;*

(3) *$\widehat{M}$ is singular with respect to f;*

(4) *if $U = \widehat{X}$ then $\widehat{M}$ is the union of all one codimensional components of S;*

(5) $\widehat{f}(\widehat{X} \setminus \widehat{M}) \subset f(T \setminus M)$.

Now we prove the cross theorem for separately $(\cdot, W)$-meromorphic functions with pluripolar singularities.

Theorem 6.4. *Let $A_j, D_j, X, M, \widehat{M}$ and F, W be as in Theorem 6.1. Let $S \subset X \setminus M$ be relatively closed and let $f\colon (X \setminus M) \setminus S \to F$ be a separately (F, W)-meromorphic function on $X \setminus M$, (i.e., $f \in M_s^W(X \setminus M, F)$) such that*

$$\Sigma_1(S) := \Sigma_1(A_1, A_2; S) = \left\{z_2 \in A_2 : S_{(\cdot, z_2)} \text{ is not pluripolar}\right\},$$
$$\Sigma_2(S) := \Sigma_2(A_1, A_2; S) = \left\{z_1 \in A_1 : S_{(z_1, \cdot)} \text{ is not pluripolar}\right\}$$

are pluripolar. Put $Q_f = M \cup S$. Then there exists exactly one $\widehat{f} \in M(\widehat{X} \setminus \widehat{M}, F)$ such that

(i) *$\widehat{f} \in H(\widehat{X} \setminus \widehat{Q}_f, F)$, where the set $\widehat{Q}_f$ is constructed via Theorem 6.1 (in the same way as $\widehat{M}$ for M), (note that $\widehat{M} \subset \widehat{Q}_f$);*

(ii) *$\widehat{f} = f$ on $X'_f \setminus Q_f$, where*

$$X'_f := \mathbb{T}(A_1, A_2; D_1, D_2; \Sigma_1(Q_f), \Sigma_2(Q_f)).$$

Proof. Fix a function $f \in M_s^W(X \setminus M, F) \cap H_s^W(X \setminus Q_f, F)$. By Theorem 6.1 there exists exactly one $\widehat{f} \in H(\widehat{X} \setminus \widehat{Q}_f, F)$ with $\widehat{f} = f$ on $X'_f \setminus Q_f$. It remains to prove that $\widehat{f} \in M(\widehat{X} \setminus \widehat{M}, F)$.

By Theorem 6.3 it is sufficient to prove that $\widehat{f} \in M(\Omega \setminus \widehat{M}, F)$ where $\Omega \subset \widehat{X}$ is an open neighbourhood of X'_f. Consequently, by virtue of Theorem 4.4 the function $\widehat{f}$ extends to $\widehat{X} \setminus \widehat{M}$.

Fix $a_1 \in A_1 \setminus \Sigma_2(Q_f)$. Take $a_2 \in D_2 \setminus (Q_f)_{(a_1, \cdot)}$ and let $r > 0$ be such that $\Delta_r(a) \subset \widehat{X} \setminus \widehat{Q}_f$, where $a = (a_1, a_2)$. Take a $D'_2 \subset D_2 \setminus \widehat{M}_{(a_1, \cdot)}$. We may assume that $\Delta_r(a_1) \times D'_2 \subset \widehat{X} \setminus \widehat{M}$ and $\Delta_r(a_2) \subset D'_2$. By the Rothstein theorem (Theorem 3.1) with $A := A_1 \cap \Delta_r(a_1)$ we get an open set $\Omega_a \supset A \times D'_2$ such that $\widehat{f}$ extends meromorphically to Ω_a. The proof of Theorem 6.4 is complete. □

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