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Regular Wavelets, Heat Semigroup and Application to the Magneto-hydrodynamic Equations with Data in Critical Triebel-Lizorkin Type Oscillation Spaces

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Abstract. In this paper, based on a Poisson type extension of Triebel-Lizorkin type oscillation spaces $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$, we establish a bilinear estimate on some new tent spaces $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$ associated with $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$. As an application, we get the well-posedness and regularity of the fractional magneto-hydrodynamic equations and quasi-geostrophic equations with initial data in the critical $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$.

1. Introduction

Triebel-Lizorkin spaces $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$ were introduced as generalizations of many classical function spaces such as Lebesgue spaces and Sobolev spaces. We refer the reader to Triebel [48, 49] for an overview of $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$. In the research of harmonic analysis and partial differential equations, Triebel-Lizorkin spaces have been used extensively and attracted the attention of many mathematicians. See Meyer-Yang [40], Chae [4, 5], Kozono-Shimada [28], Chen-Miao-Zhang [8] and the reference therein.

In this paper, we apply wavelets and semigroup to study a class of Triebel-Lizorkin type oscillation spaces $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ defined as

$$\sup_{Q} |Q|^{\gamma_2/n-1/p} \inf_{P_{Q,f}\in S^{\gamma_1,\gamma_2}_{p,q,f}} \|\varphi_Q(f-P_{Q,f})\|_{\dot{F}^{\gamma_1,q}_{p}} < +\infty,$$

where the supremum is taken over all cubes Q and $S^{\gamma_1,\gamma_2}_{p,q,f}$ denotes the set of all polynomials satisfying (2.2) below. See Definition 2.3. The spaces $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ cover many classical function spaces such as Sobolev spaces $W^{p,\gamma_1}(\mathbb{R}^n)$, Triebel-Lizorkin spaces $\dot{F}^{\gamma_1}_{p,q}(\mathbb{R}^n)$, Morrey spaces $L^{p,\gamma_2}(\mathbb{R}^n)$, bounded mean oscillation spaces $\mathrm{BMO}(\mathbb{R}^n)$, Q type spaces $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ and so on. We give the following space structure table to clarify the relation between these spaces and $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$:

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$1<p<\infty, q=2, \gamma_1\in\mathbb{R}, \gamma_2=n/p$	$\dot{F}^{\gamma_1,n/p}_{p,2}(\mathbb{R}^n)=W^{p,\gamma_1}(\mathbb{R}^n)$
$1<p,q<\infty, \gamma_1\in\mathbb{R}, \gamma_2=n/p$	$\dot{F}^{\gamma_1,n/p}_{p,q}(\mathbb{R}^n)=\dot{F}^{\gamma_1}_{p,q}(\mathbb{R}^n)$
$1<p<\infty, q=2, \gamma_1=0, \gamma_2=n/p$	$\dot{F}^{0,\gamma_2}_{p,2}(\mathbb{R}^n)=L^{p,\gamma_2}(\mathbb{R}^n)$
$p=q=2, \gamma_1=0, \gamma_2=-n/2,$	$\dot{F}^{0,-n/2}_{2,2}(\mathbb{R}^n)=\mathrm{BMO}(\mathbb{R}^n)$
$p=q=2, \gamma_1=\alpha-\beta+1, \gamma_2=\alpha+\beta-1$	$\dot{F}^{\alpha-\beta+1,\alpha+\beta-1}_{2,2}(\mathbb{R}^n)=Q^{\beta}_{\alpha}(\mathbb{R}^n)$

In Theorem 2.5, we give a wavelet characterization of $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$. As a consequence, $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ coincide with $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$ introduced by Yang-Yuan [62]. Theorem 2.5 and Lemma 2.7 imply that Calderón-Zygmund operators are bounded on $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$. Moreover, our wavelet characterization is independent of the choice of wavelet basis. See Theorem 2.8 and Corollary 2.9.

It is well-known that for any $f\in\mathrm{BMO}(\mathbb{R}^n)$, the Poisson integral $P_t(f)$ gives a harmonic extension of f. Obviously, the Poisson kernel can be replaced with the fractional heat semigroup $\left\{e^{-t(-\Delta)^{\beta}}\right\}$. In a sense, the function spaces on $\mathbb{R}^n$ can be characterized via the semigroups and the tent spaces on $\mathbb{R}^{n+1}_{+}$, see Coifman-Meyer-Stein [9] for example. In the last decade, such an idea has been applied to the study of the well-posedness of Naiver-Stokes equations. In 2001, Koch-Tataru [26] obtained a semigroup characterizations of $\mathrm{BMO}(\mathbb{R}^n)$. In 2007, by Hausdorff capacity, Xiao [59] gave a semigroup characterization of $Q_{\alpha}(\mathbb{R}^n)$. Li-Zhai [33] developed the idea of [26,59] and obtained a semigroup characterization of $Q^{\beta}_{\alpha}(\mathbb{R}^n)$. For further information, we refer the reader to Cannone [2,3], Dafni-Xiao [14], Lemarié-Rieusset [29], Li-Xiao-Yang [30], Lin-Yang [35] and Miao-Yuan-Zhang [42].

In Section 2.3, we introduce tent type spaces $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$ as follows:

$$\begin{aligned}\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'} &= \mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}\cap\mathbb{F}^{\gamma_1,\gamma_2,II}_{p,q}\cap\mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}\cap\mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}\\ &=: X_1\cap X_2\cap X_3\cap X_4.\end{aligned}$$

Via fractional heat semigroup, we can establish a relation between $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ and $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. See Theorem 2.17 for the details.

Actually, Theorem 2.17 is not a simple generalization of the results in [26,33,59]. In the above structure table, $\mathrm{BMO}(\mathbb{R}^n)$, $Q_{\alpha}(\mathbb{R}^n)$ and $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ are all $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ spaces with $p=q=2$. Technically, the Plancherel identity of Fourier transform plays an important role in the semigroup characterizations of these function spaces. For the cases $p\neq 2$, this method is not valid. To overcome this difficulty, we apply a new method. Let $\left\{\Phi^{\varepsilon}_{j,k}\right\}_{(\varepsilon,j,k)\in\Lambda_n}$ be a wavelet basis and

$$f=\sum_{(\varepsilon,j,k)\in\Lambda_n} a^{\varepsilon}_{j,k}\Phi^{\varepsilon}_{j,k}\in\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n).$$

For any cube Q, based on the relation between j and the radius of Q, we decompose the function

$$F(x,t) := e^{t(-\Delta)^\beta} f(x)$$

into several parts such that every part belongs to some X_i. Such a decomposition provides a clear view of the local structures of $F(x,t)$ related to the spatial variable and the frequency. So the desired semigroup characterization of $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ can be obtained easily.

As an application, with initial data in critical $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$, we consider the well-posedness of the fractional magneto-hydrodynamic equations $(\mathrm{FMHD})_\beta$:

$$\begin{cases} \partial_t u + (-\triangle)^\beta u + u\cdot\nabla u + \nabla p - b\cdot\nabla b = 0; \\ \partial_t b + (-\triangle)^\beta b + u\cdot\nabla b - b\cdot\nabla u = 0; \\ \nabla\cdot u = \nabla\cdot b = 0; \\ u|_{t=0} = u_0,\ b|_{t=0} = b_0 \end{cases} \tag{1.1}$$

with $\beta > 1/2$ and $n \ge 3$.

If $\beta = 1$, equations (1.1) reduce to the classical incompressible magneto-hydrodynamic (MHD) equations:

$$\begin{cases} \partial_t u + (-\triangle) u + u\cdot\nabla u + \nabla p - b\cdot\nabla b = 0; \\ \partial_t b + (-\triangle) b + u\cdot\nabla b - b\cdot\nabla u = 0; \\ u|_{t=0} = u_0,\ b|_{t=0} = b_0, \end{cases} \tag{1.2}$$

with $\nabla\cdot u = \nabla\cdot b = 0$. Further, if the magnetic field $b(t,x)$ identically equals to zero, equations (1.2) reduce to the incompressible fractional Navier-Stokes equations:

$$\begin{cases} \partial_t u + (-\triangle)^\beta u + u\cdot\nabla u + \nabla p - u\cdot\nabla u = 0; \\ \nabla\cdot u = 0; \\ u|_{t=0} = u_0. \end{cases} \tag{1.3}$$

As a basic model in the fluid mechanics, the well-posedness of equations (1.3) with $\beta = 1$ in classical function spaces has been studied by many mathematicians. For example, with initial data in Lebesgue space L^n and Sobolev spaces $\dot{H}^{n/2-1}(\mathbb{R}^n)$, the well-posed results are obtained by Kato [24] and Fujita-Kato [25], respectively. Cannone [2, 3] and Planchon [46] proved the global well-posedness for small initial data in the Besov spaces $\dot{B}^{n/p-1}_{p,\infty}(\mathbb{R}^n)$, $n < p < \infty$. In 2001, for $\beta = 1$, Koch-Tataru [26] proved the well-posedness of (1.3) with data in $\mathrm{BMO}^{-1}(\mathbb{R}^n)$. For further progress on equations (1.3), see Deng-Yao [15], Germain-Pavlović-Staffilani [18], Giga-Miyakawa [19], Giga-Sawada [20], Lemarié-Rieusset [29], Li-Zhai [33], Li-Xiao-Yang [30], Lions [36], Miao-Yuan-Zhang [42], Miura-Sawada [43], Taylor [47], Wang-Xiao [50], Wu [54, 57], Xiao [59, 60] and Zhai [66] and the reference therein.

For equations (1.1), because of the coupling effect between the velocity $u(x,t)$ and the magneto field $b(x,t)$, the situation is more complicated. Kozono [27] proved the existence of a classical solution for the two-dimensional MHD system in a bounded domain. Alexseev [1] established the existence and uniqueness results for the ideal MHD system in Sobolev space $W^k(\mathbb{R}^n)$. For $\beta=1$, Miao-Yuan-Zhang [41] obtained the well-posedness of equations (1.1) with data in $\mathrm{BMO}^{-1}(\mathbb{R}^n)$. For more information on the MHD equations, we refer the reader to Alexseev [1], Wu [52], Wu [53, 58] and the reference therein.

Leaving more notations and terminologies later, let us outline roughly our approach of the main result. With data in Besov spaces or Triebel-Lizorkin spaces, one usual method to derive the well-posedness of (1.3) is the "energy inequality" based on some commutator estimates and Littlewood-Paley theory. See Wu [54]. We apply a new method. The key step is a decomposition of the bilinear operator

$$B_\ell(u,v)(t,x)=\int_0^t e^{-(t-s)(-\Delta)^\beta}\frac{\partial}{\partial x_\ell}(uv)\,ds,\quad \ell=1,2,\ldots,n$$

based on multi-resolution analysis. We decompose $B_\ell(u,v)$ as follows.

$$B_\ell(u,v)(t,x):=\int_0^t\sum_{j'\in\mathbb{Z}}\sum_{i=1}^4 I^{i,\ell}_{j'}(s,t,x)\,ds,$$

where $I^{i,\ell}_{j'}$ are composed of the wavelet coefficients of u and v. See (3.2) for details. By this decomposition, Lemma 2.11 enables us to convert the bilinear estimate of $B_\ell(u,v)$ into various efficient computations involved in the wavelets coefficients of u and v. Finally we obtain the desired bilinear estimate on $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}\times\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. See Sections 3.2–3.3. The well-posedness of equations (1.1) can be derived from such bilinear estimates together with a usual procedure. Our method can be also applied to the well-posedness of quasi-geostrophic equations (4.4). See Section 4.

We point out that when we consider the well-posedness of equation (1.1) with data in $\dot{F}^{\gamma_1,\gamma_2}_{p,q}$, the tent type space $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$ has some distinct advantages. On one hand, for $F(x,t)\in\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$, the four parts of $\|F\|_{\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}}$ have different meanings:

- the norms $\|F\|_{X_1}$ and $\|F\|_{X_2}$ denote the Bloch-parts of $F(x,t)$,
- the norms $\|F\|_{X_3}$ and $\|F\|_{X_4}$ denote the L^p-parts of $F(x,t)$.

Furthermore, the index m represents the regularities for the variable x. Compared with the results in [26,33,59], if m becomes bigger, the elements in $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$ have higher regularities.

On the other hand, Lin-Yang [35] and Li-Yang [31] used another tent type spaces to study the well-posedness of equations (1.3) and (1.1) with data in the critical Besov-Q spaces $\dot{B}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$, respectively. In [35] and [31], the tent type spaces consist of two parts:

the L^∞-part and L^p-part. Hence the index β is restricted to the scope $(1/2, 1]$. In this paper, Bloch parts $\|F\|_{X_1}$ and $\|F\|_{X_2}$ are substituted for the L^∞-part. This change will enable us to obtain the well-posedness of equations (1.3) and (1.1) with $\beta > 1/2$.

Remark 1.1. Interestingly, Federbush [16] employed the divergence-free wavelets to study the classical Navier-Stokes equations, while the wavelets used in this paper are classical Meyer wavelets. In addition, when constructing a contraction mapping, Federbush's method was based on the estimates of "long wavelength residues". Nevertheless, our wavelet approach is based on Lemmas 3.2–3.7 and the Cauchy-Schwarz inequality.

Remark 1.2. In recent years, some special cases of $\dot{F}^{\gamma_1,\gamma_2}_{2,2}$ have been applied to the study well-posedness of equations (1.3) by several authors. We refer the reader to Wang-Xiao [50, 51], Xiao [60] for further information.

The rest of this paper is organized as follows. In Section 2, we state some preliminary knowledge, notations and terminologies. In Section 3, we introduce a class of tent spaces $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. For $f \in \dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$, we prove its Poisson type extension belongs to $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. In Sections 3.2–3.3, we establish a bilinear estimate on $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'} \times \mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. In Section 4, we prove the well-posedness of equations (1.1) and (4.4) with data in $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ being small.

2. Notations and preliminaries

In this paper, the symbols $\mathbb{Z}$ and $\mathbb{N}$ denote the sets of all integers and natural numbers, respectively. For $n \in \mathbb{N}$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, with Euclidean norm denoted by $|x|$ and the Lebesgue measure denoted by dx. $\mathbb{R}^{n+1}_+$ is the upper half-space $\{(t,x) \in \mathbb{R}^{n+1}_+ : t > 0, x \in \mathbb{R}^n\}$ with Lebesgue measure $dtdx$. $B(x,r)$ denotes a ball in $\mathbb{R}^n$ with center x, radius r and the volume $|B|$. Denote by Q a cube in $\mathbb{R}^n$ with the sides parallel to the coordinate axes. The volume and side length of Q are denoted by $|Q|$ and $l(Q)$, respectively.

For convenience, the positive constants C may change from one line to another and usually depend on the dimension n, α, β and other fixed parameters. The Schwartz class of rapidly decreasing functions and its dual will be denoted by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$, respectively. For $f \in \mathscr{S}(\mathbb{R}^n)$, $\widehat{f}$ means the Fourier transform of f.

2.1. Wavelets

In this paper, we use real valued tensor product orthogonal wavelets Φ^ϵ, which will be Daubechies wavelets or Meyer wavelets. Daubechies wavelets are used only to get a wavelet characterization of $\dot{F}^{\gamma_1,\gamma_2}_{p,q}$ and Meyer wavelets will be used throughout this paper. If Φ^ϵ is a Daubechies wavelet, we assume that there exists an integer m_0 which is greater than some constant depending on the index of $\dot{F}^{\gamma_1,\gamma_2}_{p,q}$ such that

(1) for any $\epsilon \in \{0,1\}^n$, $\Phi^\epsilon \in C_0^{m_0}([-2^M, 2^M]^n)$;

(2) for any $\epsilon \in E_n$, Φ^ϵ has the vanishing moments up to the order $m_0 - 1$.

We present some preliminaries on Meyer wavelets. Let $\Psi^0 \in C_0^\infty([-4\pi/3, 4\pi/3])$ be an even function satisfying $\Psi^0(\xi) \in [0,1]$ and $\Psi^0(\xi) = 1$, if $|\xi| \leq 2\pi/3$. Let $\Omega(\xi) = [(\Psi^0(\xi/2)^2 - (\Psi^0(\xi))^2]^{1/2}$. Then $\Omega \in C_0^\infty([-8\pi/3, 8\pi/3])$ is an even function satisfying

$$\begin{cases} \Omega(\xi) = 0, & \text{if } |\xi| \leq 2\pi/3; \\ \Omega^2(\xi) + \Omega^2(2\xi) = \Omega^2(\xi) + \Omega^2(2\pi - \xi) = 1, & \text{if } \xi \in [2\pi/3, 4\pi/3]. \end{cases}$$

Let $\Psi^1(\xi) = \Omega(\xi) e^{-i\xi/2}$. We define

$$\begin{cases} \widehat{\Phi^\epsilon} = \prod_{i=1}^n \Psi^{\epsilon_i}(\xi_i), & \epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n; \\ \Phi^\epsilon_{j,k}(x) = 2^{nj/2} \Phi^\epsilon(2^j x - k), & j \in \mathbb{Z},\ k \in \mathbb{Z}^n. \end{cases}$$

In this paper, we denote $E_n := \{0,1\}^n \setminus \{0\}$, $F_n := \{(\epsilon,k) : \epsilon \in E_n, k \in \mathbb{Z}^n\}$ and $\Lambda_n = \{(\epsilon, j, k), \epsilon \in E_n, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$. For further information about wavelets, we refer the reader to Meyer [38] and Meyer-Coifman [39]. The following result is well-known.

Lemma 2.1. $\left\{\Phi^\epsilon_{j,k}\right\}_{(\epsilon,j,k)\in\Lambda_n}$ *is an orthogonal basis in* $L^2(\mathbb{R}^n)$.

For a function f, denote by $f^\epsilon_{j,k} = \left\langle f, \Phi^\epsilon_{j,k} \right\rangle$ the wavelet coefficients of f, where $\epsilon \in \{0,1\}^n$, $k \in \mathbb{Z}^n$. Let

$$P_j f(x) = \sum_{k\in\mathbb{Z}^n} f^0_{j,k} \Phi^0_{j,k}(x) \quad \text{and} \quad Q_j f(x) = \sum_{(\epsilon,k)\in F_n} f^\epsilon_{j,k} \Phi^\epsilon_{j,k}(x).$$

By Lemma 2.1, we can see that P_j and Q_j are projection operators on $L^2(\mathbb{R}^n)$. In fact, for any two functions u and v, we have

$$\begin{aligned} uv = &\sum_{j\in\mathbb{Z}} P_{j-3} u Q_j v + \sum_{j\in\mathbb{Z}} Q_j u Q_j v + \sum_{0<j-j'\leq 3} Q_j u Q_{j'} v \\ &+ \sum_{0<j'-j\leq 3} Q_j u Q_{j'} v + \sum_{j\in\mathbb{Z}} Q_j u P_{j-3} v. \end{aligned} \tag{2.1}$$

2.2. Triebel-Lizorkin type oscillation spaces and wavelet characterization

As a generalization of Triebel-Lizorkin spaces and Morrey spaces, Triebel-Lizorkin type oscillation spaces have been introduced by Yang [61] via wavelets. We refer the reader to Yuan-Sickel-Yang [65] for further information on Triebel-Lizorkin type oscillation spaces and their generalizations. To make the mathematicians on PDE to understand such spaces better, we give here a direct definition by using classical Triebel-Lizorkin spaces.

Suppose that ϕ is a function on $\mathbb{R}^n$ such that

$$\operatorname{supp}\widehat{\phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 1\} \quad \text{and} \quad \widehat{\phi}(\xi) = 1 \text{ for } \{\xi \in \mathbb{R}^n : |\xi| \le 1/2\}.$$

For $v \in \mathbb{Z}$, define

$$\phi_v(x) = 2^{n(v+1)}\phi(2^{v+1}x) - 2^{nv}\phi(2^v x)$$

as the Littlewood-Paley functions. See Frazier-Jawerth-Weiss [17] and Triebel [48, 49] for more information on the Littlewood-Paley theory. The Triebel-Lizorkin spaces are defined as follows.

Definition 2.2. Let $-\infty < r < \infty$, $1 \le p < \infty$ and $1 \le q \le \infty$. A function $f \in \mathscr{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ belongs to $\dot{F}^r_{p,q}(\mathbb{R}^n)$ if

$$\|f\|_{\dot{F}^r_{p,q}} = \left\| \left(\sum_{v\in\mathbb{Z}} 2^{qvr} \left|\phi_v * f(x)\right|^q \right)^{1/q} \right\|_p < \infty.$$

Take $\varphi \in C_0^\infty(B(0,n))$ such that $\varphi(x) = 1$ for $x \in B(0,\sqrt{n})$. Denote by $Q(x_0, r)$ a cube centered at x_0 and with side length r. For simplicity, sometimes, we write $Q(x_0, r)$ as Q or $Q(r)$. Let $\varphi_Q(x) = \varphi\left(\frac{x-x_Q}{r}\right)$. For $0 < p, q \le \infty$, $\gamma_1, \gamma_2 \in \mathbb{R}$, let $m_0 = m^{\gamma_1,\gamma_2}_{p,q}$ be sufficiently big positive real number. For any function f, let $S^{\gamma_1,\gamma_2}_{p,q,f}$ be the set of polynomials $P_{Q,f}$ such that for any $|\alpha| \le m_0$,

$$\int x^\alpha \varphi_Q(x)(f(x) - P(x))\, dx = 0. \tag{2.2}$$

The Triebel-Lizorkin type oscillation spaces are defined as follows.

Definition 2.3. Let $0 < p, q \le \infty$, $\gamma_1, \gamma_2 \in \mathbb{R}$. We say that f belongs to Triebel-Lizorkin type oscillation spaces $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ provided

$$\sup_Q |Q|^{\gamma_2/n-1/p} \inf_{P_{Q,f}\in S^{\gamma_1,\gamma_2}_{p,q,f}} \|\varphi_Q(f - P_{Q,f})\|_{\dot{F}^{\gamma_1,q}_p} < +\infty, \tag{2.3}$$

where the supremum is taken over all the cubes with center x_Q and length r in $\mathbb{R}^n$.

It is easy to verify the following results.

Proposition 2.4. *Let $0 < p, q \le \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$.*

(i) *The definition of $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ is independent of the choice of Littlewood-Paley function ϕ.*

(ii) *For $1 \le p, q \le \infty$, $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ are Banach spaces.*

Applying Hausdorff capacity and Littlewood-Paley theory, Yang-Yuan [62] introduced a new class of function spaces $\dot{F}^{s,\tau}_{p,q}(\mathbb{R}^n)$, $p \in (0,\infty)$, which generalize many classical function spaces. For more information, we refer to Liang et al. [34], Yang-Yuan [63, 64] and Yuan-Sickel-Yang [65].

We call Φ^ϵ a regular Daubechies wavelet, if there exist two integers m_0 and M such that

$$\begin{cases} \text{for } \epsilon \in \{0,1\}^n, \ \Phi^\epsilon \in C_0^m([-2^M, 2^M]^n); \\ \text{for } \epsilon \in E_n \text{ and } |\alpha| \le m_0, \ \int x^\alpha \Phi^\epsilon(x)\, dx = 0. \end{cases} \tag{2.4}$$

Let $0 < p, q \le \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. In Theorem 2.5 below, we take an integer $m^{\gamma_1,\gamma_2}_{p,q}$ large enough such that the constant m_0 in (2.4) satisfies $m_0 \ge m^{\gamma_1,\gamma_2}_{p,q}$. For any cube Q, let $\Lambda^Q_n = \{(\epsilon, j, k) \in \Lambda_n, Q_{j,k} \subset Q\}$. We have the following wavelet characterization of $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$. We omit the proof and refer the reader to Yuan-Sickel-Yang [65] and Li-Yang-Zheng [32].

Theorem 2.5. *Let $0 < p, q \le \infty$, $\gamma_1, \gamma_2 \in \mathbb{R}$. $f = \sum_{(\epsilon,j,k)\in\Lambda_n} a^\epsilon_{j,k} \Phi^\epsilon_{j,k} \in \dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ if and only if*

$$\sup_Q |Q|^{\gamma_2/n - 1/p} \left\| \left(\sum_{(\epsilon,j,k)\in\Lambda^n_Q} 2^{qj(\gamma_1+n/2)} \left|a^\epsilon_{j,k}\right|^q \chi(2^j \cdot -k) \right)^{1/q} \right\|_{L^p} < +\infty, \tag{2.5}$$

where the supremum is taken over all the dyadic cubes Q in $\mathbb{R}^n$.

Remark 2.6. If $\gamma_2 = n/p$, $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n) = \dot{F}^{\gamma_1,q}_{p}(\mathbb{R}^n)$. Moreover, for $\gamma_2 > n/p$, if $f \in \dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$, $f^\epsilon_{j,k} = 0$. Hence f is only a polynomial.

Now we state some preliminaries about Calderón-Zygmund operators and refer to Meyer [38] and Meyer-Coifman [39] for further information. For $x \ne y$, let $K(x,y)$ be a smooth function such that there exists a sufficiently large $N_0 \le m_0$ satisfying that

$$\left|\partial^\alpha_x \partial^\beta_y K(x,y)\right| \le C\,|x-y|^{-(n+|\alpha|+|\beta|)}, \quad \forall\ |\alpha| + |\beta| \le N_0. \tag{2.6}$$

A linear operator T is said to be a Calderon-Zygmund operator if

(i) T is continuous from $C^1(\mathbb{R}^n)$ to $(C^1(\mathbb{R}^n))'$;

(ii) $Tf(x) = \int K(x,y) f(y)\, dy$, where $K(x,y)$ satisfies (2.6);

(iii) $Tx^\alpha = T^* x^\alpha = 0$ for $\alpha \in \mathbb{N}^n$ with $|\alpha| \le N_0$.

We denote $T \in \mathrm{CZO}(N_0)$. By Schwartz kernel theorem, $K(\cdot,\cdot)$ is a distribution in $S'(\mathbb{R}^{2n})$. For (ϵ, j, k), $(\epsilon', j', k') \in \Lambda_n$, let

$$a^{\epsilon,\epsilon'}_{j,k,j',k'} = \left\langle K(\cdot,\cdot), \Phi^\epsilon_{j,k} \Phi^{\epsilon'}_{j',k'} \right\rangle.$$

If T is a Calderon-Zygmund operator, then its kernel $K(\cdot,\cdot)$ and the related coefficients satisfy the following relations.

Lemma 2.7. [39, Section 8.3, Proposition 1]

(i) *If* $T \in \mathrm{CZO}(N_0)$*, then for* $(\epsilon, j, k), (\epsilon', j', k') \in \Lambda_n$*,* $\left\{a^{\epsilon,\epsilon'}_{j,k,j',k'}\right\}$ *satisfy the following condition:*

$$\left|a^{\epsilon,\epsilon'}_{j,k,j',k'}\right| \le C 2^{-|j-j'|(n/2+N_0)} \left(\frac{2^{-j}+2^{-j'}}{2^{-j}+2^{-j'}+|k2^{-j}-k'2^{-j'}|}\right)^{n+N_0}. \tag{2.7}$$

(ii) *If* $a^{\epsilon,\epsilon'}_{j,k,j',k'}$ *satisfies the above condition* (2.7)*, then*

$$K(x,y) = \sum_{(\epsilon,j,k)\in\Lambda_n} \sum_{(\epsilon',j',k')\in\Lambda_n} a^{\epsilon,\epsilon'}_{j,k,j',k'} \Phi^{\epsilon}_{j,k}(x) \Phi^{\epsilon'}_{j',k'}(y)$$

in the sense of distributions and for any small positive real number δ*,* $T \in \mathrm{CZO}(N_0 - \delta)$*.*

Following the ideas of Meyer-Yang [40], we can prove the following result. We omit the proof and refer the reader to Yuan-Sickel-Yang [65].

Theorem 2.8. *Let* $1 < p, q < \infty$ *and* $\gamma_1, \gamma_2 \in \mathbb{R}$*. The Calderón-Zygmund operators are bounded on* $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$*.*

Let $\left\{\Phi^{i,\epsilon}_{j,k}\right\}_{(\epsilon,j,k)\in\Lambda_n}$, $i = 1, 2$, be two regular orthogonal wavelets basis. For (ϵ, j, k), $(\epsilon', j', k') \in \Lambda_n$, denote $a^{\epsilon,\epsilon'}_{j,k,j',k'} = \left\langle \Phi^{1,\epsilon}_{j,k}, \Phi^{2,\epsilon'}_{j',k'}\right\rangle$. We know that $\left\{a^{\epsilon,\epsilon'}_{j,k,j',k'}\right\}_{(\epsilon,j,k),(\epsilon',j',k')\in\Lambda_n}$ satisfies the condition (2.7). The following result is a corollary of Theorem 2.8. We refer to Yuan-Sickel-Yang [65] for a proof.

Corollary 2.9. *Theorem* 2.5 *is also true for Meyer wavelets.*

For $A > 0$ and a sequence $f := \{f_j\}$, we define the vector-valued maximal function $M_A(f)$ as

$$M_A(f)(x) = \left(\sum_j M(|f_j|^A)(x)\right)^{1/A}.$$

For $\left\{a^{\epsilon}_{j,k}\right\}_{(\epsilon,j,k)\in\Lambda_n}$, we set $f_{j'} = \sum_{(\epsilon,j,k)\in\Lambda_n} 2^{j(s+n/2)} \left|a^{\epsilon}_{j,k}\right| \chi(2^j x - k)$. Let

$$g^k_{j,j'} = \begin{cases} \displaystyle\sum_{\epsilon',k'} \frac{2^{j'(s+n/2)}\left|a^{\epsilon'}_{j',k'}\right|}{(1+|k'-2^{j'-j}k|)^{n+\gamma}}, & j \ge j',\ k \in \mathbb{Z}^n; \\ \displaystyle\sum_{\epsilon',k'} \frac{2^{j'(s+n/2)}\left|a^{\epsilon'}_{j',k'}\right|}{(1+|k-2^{j-j'}k'|)^{n+\gamma}}, & j < j',\ k \in \mathbb{Z}^n. \end{cases}$$

Yang [61] obtained the following result.

Lemma 2.10. [61, Chapter 5, Lemma 3.2] *For any* $\gamma > n/A+1$ *and* $x \in Q_{j,k}$*, we have*

$$g^k_{j,j'} = \begin{cases} CM_A(f_{j'})(x), & \text{if } j \geq j'; \\ C2^{n(j'-j)/A}M_A(f_{j'})(x), & \text{if } j < j'. \end{cases}$$

2.3. Semigroup extension of $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$

In this section, we will introduce some tent spaces associated with $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$, which establishes a clear relationship among time t, frequency ξ and position x by wavelet methods. We would like to remind the reader that, for wavelets $\left\{\Phi^{\epsilon}_{j,k} : (\epsilon,j,k) \in \Lambda_n\right\}$, 2^j represents the range of frequency ξ and $2^{-j}k$ represents the range of position x in some sense. In the sequel, we only use the classical tensorial Meyer wavelets.

In the sequel, we use $N > 0$ to denote a fixed real number which is large enough. For fixed $\beta > 0$, we may choose a radial function $\phi \in \mathscr{S}(\mathbb{R}^n)$ (cf. [17, Lemma 1.1], [38, Chap. 3, §2]) such that there exists $C_\beta > 0$ satisfying

(i) $\displaystyle\int_{\mathbb{R}^n} x^\gamma \phi(x)\,dx = 0$ for all $\gamma \in \mathbb{N}^n$;

(ii) $\displaystyle\int_0^\infty \left(\widehat{\phi}(t^{1/(2\beta)}\xi)\right)^2 \frac{dt}{t} = 1$ for all $\xi \neq 0$;

(iii) $\displaystyle\int_0^\infty \widehat{\phi}(t^{1/(2\beta)})e^{-t}\,\frac{dt}{t} = \frac{1}{C_\beta}$.

Define $\phi^\beta_t(x) = t^{-n/(2\beta)}\phi(t^{-1/(2\beta)}x)$. Then $\widehat{\phi^\beta_t}(\xi) = \widehat{\phi}(t^{1/(2\beta)}\xi)$, and hence

$$f(t,x) := e^{-t(-\Delta)^\beta}f(x) = K^\beta_t * f(x).$$

Applying the inverse Fourier transform, we can get

$$\widehat{f}(\xi) = C_\beta \int_0^\infty \widehat{\phi}(t^{1/(2\beta)})e^{-t}\,\frac{dt}{t}\,\widehat{f}(\xi) = C_\beta \int_0^\infty \widehat{\phi}(t^{1/(2\beta)}\,|\xi|)e^{-t|\xi|^{2\beta}}\widehat{f}(\xi)\,\frac{dt}{t}$$

which implies that

$$f(x) = C_\beta \int_0^\infty \int_{\mathbb{R}^n} f(t,x-y)\phi^\beta_t(y)\,dy\frac{dt}{t} := \pi_\phi f(\cdot,x). \tag{2.8}$$

For $(\epsilon,j,k) \in \Lambda_n$, let $a^\epsilon_{j,k}(t) = \left\langle f(t,\cdot),\Phi^\epsilon_{j,k}\right\rangle$ and $a^\epsilon_{j,k} = \left\langle f,\Phi^\epsilon_{j,k}\right\rangle$. Then

$$f = \sum_{(\epsilon,j,k)\in\Lambda_n} a^\epsilon_{j,k}\Phi^\epsilon_{j,k} \quad \text{and} \quad f(t,\cdot) = \sum_{(\epsilon,j,k)\in\Lambda_n} a^\epsilon_{j,k}(t)\Phi^\epsilon_{j,k}.$$

We first represent $a^\epsilon_{j,k}(t)$ by $a^{\epsilon'}_{j',k'}$. If $f(t,x) = K^\beta_t * f(x)$, then

$$\begin{aligned} a^\epsilon_{j,k}(t) &= \sum_{\epsilon',|j-j'|\leq 3,k'} a^{\epsilon'}_{j',k'}\left\langle K^\beta_t\Phi^{\epsilon'}_{j',k'},\Phi^\epsilon_{j,k}\right\rangle \\ &= \sum_{\epsilon',|j-j'|\leq 3,k'} a^{\epsilon'}_{j',k'}\int e^{-t2^{2j\beta}|\xi|^{2\beta}}\widehat{\Phi}^{\epsilon'}(2^{j-j'}\xi)\widehat{\Phi}^\epsilon(\xi)e^{-i(2^{j-j'}k'-k)\xi}\,d\xi. \end{aligned}$$

By a simple application of integration by parts, we could control $a^{\epsilon}_{j,k}(t)$ by $\left\{a^{\epsilon'}_{j',k'}\right\}$ as follows.

Lemma 2.11. *There exists a fixed small constant $\widetilde{c} > 0$ depending only on β and on the size of support for the Fourier transformation of Meyer wavelets such that*

(i) *For $t2^{2\beta j} \geq 1$,*

$$\left|a^{\epsilon}_{j,k}(t)\right| \leq Ce^{-\widetilde{c}t2^{2j\beta}} \sum_{\epsilon',|j-j'|\leq 3,k'} \left|a^{\epsilon'}_{j',k'}\right| \left(1 + \left|2^{j-j'}k' - k)\right|\right)^{-N};$$

(ii) *For $0 \leq t2^{2\beta j} \leq 1$,*

$$\left|a^{\epsilon}_{j,k}(t)\right| \leq C \sum_{|j-j'|\leq 3} \sum_{\epsilon',k'} \left|a^{\epsilon'}_{j',k'}\right| \left(1 + \left|2^{j-j'}k' - k\right|\right)^{-N}.$$

Furthermore, if f is a function obtained by (2.8), then we could express $a^{\epsilon}_{j,k}$ by $\left\{a^{\epsilon'}_{j',k'}(t)\right\}$ as follows:

$$a^{\epsilon}_{j,k} = \int_{\mathbb{R}^{n+1}_{+}} \sum_{(\epsilon',j',k')\in\Lambda_n} a^{\epsilon'}_{j',k'}(t)(\phi^{\beta}_t * \Phi^{\epsilon'}_{j',k'}(x))\Phi^{\epsilon}_{j,k}(x)\, dx\frac{dt}{t}.$$

Similarly, we apply integration by parts to obtain the following estimate.

Lemma 2.12.

$$\left|a^{\epsilon}_{j,k}\right| \leq C \sum_{|j-j'|\leq 3} \int_0^{\infty} \left(\max\left\{t2^{2j'\beta}, t^{-1}2^{-2j'\beta}\right\}\right)^{-N} \sum_{(\epsilon',k')\in F_n} \frac{\left|a^{\epsilon'}_{j',k'}(t)\right|}{(1 + |2^{j-j'}k' - k|)^N} \frac{dt}{t}.$$

We introduce some tent type spaces associated with $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$. For any $a(t,x)$ defined on $\mathbb{R}^{n+1}_{+}$, by wavelet theory there exists a family $\left\{a^{\epsilon}_{j,k}(t)\right\}_{(\epsilon,j,k)\in\Lambda_n}$ such that $a(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} a^{\epsilon}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x)$. Let $\gamma_1, \gamma_2 \in \mathbb{R}$, $1 < p < \infty$, $m, \widetilde{m} \in \mathbb{R}$, $m' > 0$ and $m'' \in \mathbb{Z}_{+}$. For any dyadic cube $Q_{j,k}$, we denote by $\chi_{j,k} = \chi(2^j \cdot -k)$ the characteristic function of $Q_{j,k}$.

The tent spaces associated with $\dot{F}^{\gamma_1,\gamma_2}_{p,q}$ consist of four parts. For the first two parts, we first take $\dot{F}^{r}_{p,q}$-norms for space variable x, then take L^{∞}-norm for variable t. We call them the Bloch type tent spaces. Precisely,

Definition 2.13. Given $\gamma_1, \gamma_2 \in \mathbb{R}$, $1 < p < \infty$ and $1 < q < \infty$. Let

$$\Lambda_{Q_r,1} = \{(\epsilon, j, k) : j \geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}, Q_{j,k} \subset Q_r\}$$

and

$$\Lambda_{Q_r,2} = \{(\epsilon, j, k) : -\log_2 r < j < -(\log_2 t)/(2\beta), Q_{j,k} \subset Q_r\}.$$

(i) We call $f \in \mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}(\mathbb{R}^{n+1}_+)$, if

$$\sup_{t>0} I^{\gamma_1,\gamma_2}_{p,q,m}(t) := \sup_{t,Q_r} |Q_r|^{\gamma_2/n-1/p}\, t^m \left\| \left(\sum_{(\epsilon,j,k)\in\Lambda_{Q_r,1}} 2^{qj(\gamma_1+n/2+2m\beta)} \left|a^\epsilon_{j,k}(t)\right|^q \chi_{j,k}(x) \right)^{1/q} \right\|_p < \infty.$$

(ii) We call $f \in \mathbb{F}^{\gamma_1,\gamma_2,II}_{p,q}(\mathbb{R}^{n+1}_+)$, if for any $t>0$

$$\sup_{t>0} II^{\gamma_1,\gamma_2}_{p,q}(t) := \sup_{t,Q_r} |Q_r|^{\gamma_2/n-1/p} \left\| \left(\sum_{(\epsilon,j,k)\in\Lambda_{Q_r,2}} 2^{qj(\gamma_1+n/2)} \left|a^\epsilon_{j,k}(t)\right|^q \chi_{j,k}(x) \right)^{1/q} \right\|_p < \infty.$$

For the other two parts, we first take L^p-norm for variable t, then take $\dot{F}^r_{p,q}$-norm for variable x. We call them the L^p type tent spaces. Precisely,

Definition 2.14. Given $\gamma_1,\gamma_2\in\mathbb{R}$, $1<p<\infty$ and $1<q<\infty$.

(i) $f \in \mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}(\mathbb{R}^{n+1}_+)$, if

$$III^{\gamma_1,\gamma_2}_{p,q,m} := \sup_{Q_r} |Q_r|^{\gamma_2/n-1/p} \left\| \left(\sum_{(\epsilon,j,k)\in\Lambda^n_Q} 2^{qj(\gamma_1+n/2+2m\beta)} \int_{2^{-2j\beta}}^{r^{2\beta}} \left|a^\epsilon_{j,k}(t)\right|^q t^{qm} \frac{dt}{t} \chi_{j,k}(x) \right) \right\|_p < \infty.$$

(ii) $f \in \mathbb{F}^{\gamma_1,\gamma_2,II}_{p,q}(\mathbb{R}^{n+1}_+)$, if

$$IV^{\gamma_1,\gamma_2}_{p,q,m'} := \sup_{Q_r} |Q_r|^{\gamma_2/n-1/p} \left\| \left(\sum_{(\epsilon,j,k)\in\Lambda^n_Q} 2^{qj(\gamma_1+n/2+2m'\beta)} \int_0^{2^{-2j\beta}} \left|a^\epsilon_{j,k}(t)\right|^q t^{qm'} \frac{dt}{t} \chi_{j,k}(x) \right) \right\|_p < \infty.$$

The tent spaces associated to $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$ are defined as follows.

Definition 2.15. Let $\gamma_1,\gamma_2,m\in\mathbb{R}$, $m'>0$ and $1<p,q<\infty$. The tent spaces $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$ are defined as

$$\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'} = \mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m} \cap \mathbb{F}^{\gamma_1,\gamma_2,II}_{p,q} \cap \mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m} \cap \mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}.$$

The following lemma can be obtained immediately.

Lemma 2.16. *Let $1<p,q<\infty$, $\gamma_1,\gamma_2\in\mathbb{R}$, $m>p$, $m'>0$ and $a(x,t)\in\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$.*

(i) *If $m>0$, then $a(t,x)$ satisfies*

$$\sup_{(\epsilon,j,k)\in\Lambda_n} \left\{ \sup_{t2^{2j\beta}\geq 1} (t2^{2j\beta})^m 2^{nj/2} 2^{j(\gamma_1-\gamma_2)} \left|a^\epsilon_{j,k}(t)\right| + \sup_{0<t2^{2j\beta}\leq 1} 2^{nj/2} 2^{j(\gamma_1-\gamma_2)} \left|a^\epsilon_{j,k}(t)\right| \right\} < \infty.$$

(ii) *If* $-2\beta m < \gamma_1 - \gamma_2 < 0 < \beta$, *then*

$$\sup_{t>0} \sup_{j\in\mathbb{Z},k\in\mathbb{Z}^n} t^{-(\gamma_1-\gamma_2)/(2\beta)} 2^{nj/2} \left|\left\langle a(t,\cdot), \Phi^0_{j,k}\right\rangle\right| < \infty.$$

Proof. (i) We divide the proof into two cases.

Case 1: $t2^{2j\beta} \geq 1$. Because $f \in \mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}$, for fixed j and k, we have

$$\begin{aligned}&|Q_r|^{\gamma_1/n-1/p}\, t^m \left(\int 2^{pj(\gamma_1+n/2+2m\beta)} \left|a^\epsilon_{j,k}(t)\right|^p \chi_{j,k}(x)\,dx\right)^{1/p}\\ &\lesssim |Q_r|^{\gamma_2/n-1/p}\, t^m 2^{j(\gamma_1+n/2+2m\beta)} \left|a^\epsilon_{j,k}(t)\right| 2^{-jn/p}.\end{aligned}$$

Take $r = 2^{-j}$. We can get $\left|a^\epsilon_{j,k}(t)\right| \lesssim 2^{j(\gamma_2-\gamma_1-n/2)}(t2^{2j\beta})^{-m}$.

Case 2: $0 < t2^{2j\beta} < 1$. Because $f \in \mathbb{F}^{\gamma_1,\gamma_2,II}_{p,q}$, we can obtain that

$$|Q_r|^{\gamma_2/n-1/p} \left\| \left(\sum_{-\log_2 r<j<-(\log_2 t)/(2\beta)} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2)} \left|a^\epsilon_{j,k}(t)\right|^q \chi_{j,k}(x)\right)^{1/q} \right\|_p \lesssim 1.$$

This implies that for fixed ϵ and k, $\left|a^\epsilon_{j,k}(t)\right| \lesssim 2^{-j(\gamma_1-\gamma_2)}2^{-jn/2}$.

Now we prove (ii). Because $\left\{\Phi^\epsilon_{j,k}\right\}$ are regular wavelets,

$$\left\langle a(t,\cdot), \Phi^0_{j,k}\right\rangle = \left\langle \sum_{j'<j} a^{\epsilon'}_{j',k'}(t)\Phi^{\epsilon'}_{j',k'}, \Phi^0_{j,k}\right\rangle.$$

For $0 < t2^{2j\beta} \leq 1$, a direct computation implies the desired result. For $t2^{2j\beta} > 1$, it suffices to divide into two subcases of $j' \leq -(\log_2 t)/(2\beta)$ and $-(\log_2 t)/(2\beta) < j' < j$. □

In the rest of this paper, for any dyadic cube Q_r with side length r, we denote $\widetilde{Q}_r$ the dyadic cube which contains Q_r with side length 2^7r. For all $w \in \mathbb{Z}^n$, write $\widetilde{Q}^w_r := 2^7rw+\widetilde{Q}_r$. Define

$$S^{w,j'}_{Q_r} = \left\{(\epsilon',k') : (\epsilon',j',k') \in \Lambda_n \text{ and } Q_{j',k'} \subset \widetilde{Q}^w_r\right\}.$$

Li-Yang-Zheng [32] established the following semigroup extension for $\dot{F}^{\gamma_1,\gamma_2}_{p,q}(\mathbb{R}^n)$.

Theorem 2.17. [32, Theorem 4.1] *Suppose that* $\gamma_1,\gamma_2 \in \mathbb{R}$, $1 < p < m < \infty$, $\gamma_1 - \gamma_2 < 0 < \beta$, $m' > 0$ *and* $\tau + (\gamma_1-\gamma_2)/(2\beta) > 0$.

(i) *If* $f \in \dot{F}^{\gamma_1,\gamma_2}_{p,q}$, *then* $f * K^\beta_t \in \mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$;

(ii) *The operator* π_ϕ *is a bounded and surjective operator from* $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$ *to* $\dot{F}^{\gamma_1,\gamma_2}_{p,q}$.

For the Riesz transforms R_ℓ, $\ell = 1, 2, \ldots, n$, let

$$a^{\epsilon,\epsilon',\ell}_{j,k,j',k'} = \left\langle \Phi^{\epsilon}_{j,k}, R_\ell \Phi^{\epsilon'}_{j',k'} \right\rangle, \quad (\epsilon', j', k') \in \Lambda_n.$$

If $|j - j'| \geq 2$, then $a^{\epsilon,\epsilon',\ell}_{j,k,j',k'} = 0$. By a similar method, we can obtain the boundedness of Riesz transforms on $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$.

Theorem 2.18. *Given $1 < p, q < \infty$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $m > p$ and $m' > 0$. The Riesz transforms $R_1, R_2, \ldots, R_n$ are bounded on $\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$.*

Proof.

$$R_\ell g(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} g^{\epsilon}_{j,k}(t) R_\ell \Phi^{\epsilon}_{j,k}(x) := \sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon}_{j,k}(t) \Phi^{\epsilon}_{j,k}(x),$$

where

$$b^{\epsilon}_{j,k}(t) = \left\langle R_\ell g(t,\cdot), \Phi^{\epsilon}_{j,k} \right\rangle = \sum_{|j-j'|\leq 1} \sum_{\epsilon',k'} a^{\epsilon,\epsilon'}_{j,k,j',k'} g^{\epsilon'}_{j',k'}(t).$$

By Lemma 2.7, we have

$$\left| a^{\epsilon,\epsilon'}_{j,k,j',k'} \right| \lesssim 2^{-|j-j'|(n/2+N_0)} \left(\frac{2^{-j} + 2^{-j'}}{2^{-j} + 2^{-j'} + |2^{-j}k - 2^{-j'}k'|} \right)^{n+N_0}.$$

The rest of the proof is similar to that of Theorem 2.17. We omit the details and refer the reader to [32, Theorem 4.2]. □

3. Bilinear estimate on tent spaces associated with $\dot{F}^{\gamma_1,\gamma_2}_{p,q}$

3.1. Decomposition of bilinear operator

In Lemmas 3.2–3.7 below, we assume that the index $(\beta, p, q, \gamma_1, \gamma_2, m, m')$ satisfies the following conditions:

$$(3.1) \qquad \begin{cases} 1 < p < \infty,\ 1 < q < 2; \\ \beta > \frac{1}{2},\ \gamma_1 = \gamma_2 - 2\beta + 1; \\ m > \max\left\{p, \frac{n}{2\beta}\right\}; \\ 0 < m' < \min\left\{1, \frac{p}{2\beta}\right\}; \\ \frac{2\beta-2}{q} + \frac{n}{p}(1 - \frac{1}{q}) < \gamma_2 \leq \frac{n}{p} \end{cases} \quad \text{or} \quad \begin{cases} 1 < p < \infty,\ 2 \leq q < \infty; \\ \beta > \frac{1}{2},\ \gamma_1 = \gamma_2 - 2\beta + 1; \\ m > \max\left\{p, \frac{n}{2\beta}\right\}; \\ 0 < m' < \min\left\{1, \frac{p}{2\beta}\right\}; \\ \beta - 1 + \frac{n}{2p} < \gamma_2 \leq \frac{n}{p}. \end{cases}$$

Here and henceforth, let

$$u(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} u^{\epsilon}_{j,k}(t) \Phi^{\epsilon}_{j,k}(x) \quad \text{and} \quad v(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} v^{\epsilon}_{j,k}(t) \Phi^{\epsilon}_{j,k}(x).$$

We can see that

$$e^{-(t-s)(-\Delta)^{\beta}}\frac{\partial}{\partial x_\ell}(uv)(s,t,x)=\sum_{j'\in\mathbb{Z}}\sum_{i=1}^{5}I_{j'}^{i,\ell}(s,t,x),$$

where

(3.2)

$$\begin{aligned}
I_{j'}^{1,\ell}(u,v)(s,t,x)&=\sum_{\epsilon',k'}\sum_{k''}u_{j',k'}^{\epsilon'}(s)v_{j'-3,k''}^{0}(s)e^{-(t-s)(-\Delta)^{\beta}}\frac{\partial}{\partial x_\ell}\left(\Phi_{j',k'}^{\epsilon'}(x)\Phi_{j'-3,k''}^{0}(x)\right),\\
I_{j'}^{2,\ell}(u,v)(s,t,x)&=\sum_{\epsilon',k'}\sum_{\epsilon'',k''}u_{j',k'}^{\epsilon'}(s)v_{j',k''}^{\epsilon''}(s)e^{-(t-s)(-\Delta)^{\beta}}\frac{\partial}{\partial x_\ell}\left(\Phi_{j',k'}^{\epsilon'}(x)\Phi_{j',k''}^{\epsilon''}(x)\right),\\
I_{j'}^{3,\ell}(u,v)(s,t,x)&=\sum_{0<|j'-j''|\le 3}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}u_{j',k'}^{\epsilon'}(s)v_{j'',k''}^{\epsilon''}(s)e^{-(t-s)(-\Delta)^{\beta}}\frac{\partial}{\partial x_\ell}\left(\Phi_{j',k'}^{\epsilon'}(x)\Phi_{j'',k''}^{\epsilon''}(x)\right),\\
I_{j'}^{4,\ell}(u,v)(s,t,x)&=\sum_{\epsilon',k'}\sum_{k''}v_{j',k'}^{\epsilon'}(s)u_{j'-3,k''}^{0}(s)e^{-(t-s)(-\Delta)^{\beta}}\frac{\partial}{\partial x_\ell}\left(\Phi_{j',k'}^{\epsilon'}(x)\Phi_{j'-3,k''}^{0}(x)\right).
\end{aligned}$$

Write

$$I_\ell^{i}(s,t,x)=\sum_{j'\in\mathbb{Z}}I_{j'}^{i,\ell}(s,t,x)\quad\text{and}\quad I_\ell^{i}(u,v)(t,x)=\int_0^t I_\ell^{i}(s,t,x)\,ds.$$

For $\ell=1,2,\dots,n$, the bilinear terms $B_\ell(u,v)$ can be decomposed as

$$\begin{aligned}
B_\ell(u,v)(t,x)&=\int_0^t e^{-(t-s)(-\Delta)^{\beta}}\frac{\partial}{\partial x_\ell}(uv)\,ds=:\int_0^t\sum_{j'\in\mathbb{Z}}\sum_{i=1}^{4}I_{j'}^{i,\ell}(s,t,x)\,ds\\
&=:\sum_{i=1}^{4}\int_0^t I_\ell^{i}(s,t,x)\,ds=:\sum_{i=1}^{4}I_\ell^{i}(u,v)(t,x).
\end{aligned}\tag{3.3}$$

In order to estimate $B(u,v)$, we further decompose the terms $I_\ell^{i}(u,v)(t,x)$, $i=1,2,3,4$, respectively.

Decomposition of $I_\ell^1(u,v)$. The term $I_\ell^1(u,v)$ is decomposed according to two cases.

Case $[I_\ell^1]_1$: $t\ge 2^{-2j\beta}$. For this case, we write $I_\ell^1(u,v)$ as the sum of the following three terms:

$$\begin{aligned}
I_\ell^1(u,v)(t,x)&=\sum_{\epsilon',j',k'}\sum_{k''}\left(\int_0^{2^{-1-2j'\beta}}+\int_{2^{-1-2j'\beta}}^{t/2}+\int_{t/2}^{t}\right)\\
&\qquad\times\left\{u_{j',k'}^{\epsilon'}(s)v_{j'-3,k''}^{0}(s)e^{-(t-s)(-\Delta)^{\beta}}\frac{\partial}{\partial x_\ell}\left(\Phi_{j',k'}^{\epsilon'}(x)\Phi_{j'-3,k''}^{0}(x)\right)\right\}ds\\
&=:I_\ell^{1,1}(u,v)(t,x)+I_\ell^{1,2}(u,v)(t,x)+I_\ell^{1,3}(u,v)(t,x).
\end{aligned}$$

For $i=1,2,3$, denote

$$I_\ell^{1,i}(u,v)(t,x)=\sum_{(\epsilon,j,k)\in\Lambda_n}a_{j,k}^{\epsilon,i}(t)\Phi_{j,k}^{\epsilon}(x).$$

Case $[I_\ell^1]_2$: $t < 2^{-2j\beta}$. For this case, we denote $a^{\epsilon,4}_{j,k}(t) = a^{\epsilon}_{j,k}(t)$ and then have

$$I_\ell^1(u,v)(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} a^{\epsilon,4}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x).$$

Decomposition of $I_\ell^2(u,v)$. We decompose $I_\ell^2(u,v)$ based on the relation between t and $2^{-2j\beta}$.

Case $[I_\ell^2]_1$: $t \geq 2^{-2j\beta}$. Naturally, $I_\ell^2(u,v)(t,x)$ can be divided into the following three terms:

$$\begin{aligned} I_\ell^2(u,v)(t,x) &= \sum_{j'}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}\left(\int_0^{2^{-1-2j'\beta}} + \int_{2^{-1-2j'\beta}}^{t/2} + \int_{t/2}^{t}\right) \\ &\quad \times \left\{u^{\epsilon'}_{j',k'}(s)v^{\epsilon''}_{j',k''}(s)e^{-(t-s)(-\Delta)^\beta}\frac{\partial}{\partial x_\ell}\left(\Phi^{\epsilon'}_{j',k'}(x)\Phi^{\epsilon''}_{j',k''}(x)\right)\right\}ds \\ &=: I_\ell^{2,1}(u,v)(t,x) + I_\ell^{2,2}(u,v)(t,x) + I_\ell^{2,3}(u,v)(t,x). \end{aligned}$$

Case $[I_\ell^2]_2$: $t \leq 2^{-2j\beta}$. $I_\ell^2(u,v)$ can be decomposed into the sum of $II^4(u,v)$ and $II^5(u,v)$, where

$$\begin{aligned} I_\ell^2(u,v)(t,x) &= \sum_{j'}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}\left(\int_0^{2^{-2j'\beta}} + \int_{2^{-2j'\beta}}^{t}\right) \\ &\quad \times \left\{u^{\epsilon'}_{j',k'}(s)v^{\epsilon''}_{j',k''}(s)e^{-(t-s)(-\Delta)^\beta}\frac{\partial}{\partial x_\ell}\left(\Phi^{\epsilon'}_{j',k'}(x)\Phi^{\epsilon''}_{j',k''}(x)\right)\right\}ds \\ &=: I_\ell^{2,4}(u,v)(t,x) + I_\ell^{2,5}(u,v)(t,x). \end{aligned}$$

For $i = 1,2,3,4,5$, set

$$I_\ell^{2,i}(u,v)(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,i}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x).$$

Decompositions of $I_\ell^3(u,v)$. Without loss of generality, we assume that $0 < j' - j'' \leq 3$. Similarly, we have the following two cases.

Case $[I_\ell^3]_1$: $t \geq 2^{-2j\beta}$. This $I_\ell^3(u,v)$ can be divided into the following three terms:

$$\begin{aligned} I_\ell^3(u,v)(t,x) &= \sum_{0<j'-j''\leq 3}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}\left(\int_0^{2^{-1-2j'\beta}} + \int_{2^{-1-2j'\beta}}^{t/2} + \int_{t/2}^{t}\right) \\ &\quad \times \left\{u^{\epsilon'}_{j',k'}(s)v^{\epsilon''}_{j'',k''}(s)e^{-(t-s)(-\Delta)^\beta}\frac{\partial}{\partial x_l}\left(\Phi^{\epsilon'}_{j',k'}(x)\Phi^{\epsilon''}_{j'',k''}(x)\right)\right\}ds \\ &=: I_\ell^{3,1}(u,v)(t,x) + I_\ell^{3,2}(u,v)(t,x) + I_\ell^{3,3}(u,v)(t,x). \end{aligned}$$

Case $[I_\ell^3]_2$: $t \leq 2^{-2j\beta}$. We divide $I_\ell^3(u,v)$ as follows.

$$\begin{aligned} I_\ell^3(u,v)(t,x) = & \sum_{0<j'-j''\leq 3} \sum_{\epsilon',k'} \sum_{\epsilon'',k''} \left(\int_0^{2^{-2j'\beta}} + \int_{2^{-2j'\beta}}^t \right) \\ & \times \left\{ u_{j',k'}^{\epsilon'}(s) v_{j'',k''}^{\epsilon''}(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_\ell} \left(\Phi_{j',k'}^{\epsilon'}(x) \Phi_{j'',k''}^{\epsilon''}(x) \right) \right\} ds \\ =: & I_\ell^{3,4}(u,v)(t,x) + I_\ell^{3,5}(u,v)(t,x). \end{aligned}$$

For $i = 1,2,3,4,5$, denote

$$I_\ell^{3,i}(u,v)(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} b_{j,k}^{\epsilon,i}(t) \Phi_{j,k}^\epsilon(x).$$

Decomposition of $I_j^{4,\ell}(u,v)$. It is easy to see that the terms $I_j^{1,\ell}(u,v)$ and $I_j^{4,\ell}(u,v)$ are symmetric associated with u and v. Hence for $I_\ell^4(u,v)$, we have a similar decomposition.

Case $[I_\ell^4]_1$: $t \geq 2^{-2j\beta}$. For this case, we write $I_\ell^4(u,v)$ as the sum of the following three terms:

$$\begin{aligned} I_\ell^4(u,v)(t,x) = & \sum_{\epsilon',j',k'} \sum_{k''} \left(\int_0^{2^{-1-2j'\beta}} + \int_{2^{-1-2j'\beta}}^{t/2} + \int_{t/2}^t \right) \\ & \times \left\{ v_{j',k'}^{\epsilon'}(s) u_{j'-3,k''}^0(s) e^{-(t-s)(-\Delta)^\beta} \frac{\partial}{\partial x_\ell} \left(\Phi_{j',k'}^{\epsilon'}(x) \Phi_{j'-3,k''}^0(x) \right) \right\} ds \\ =: & I_\ell^{4,1}(u,v)(t,x) + I_\ell^{4,2}(u,v)(t,x) + I_\ell^{4,3}(u,v)(t,x). \end{aligned}$$

For $i = 1,2,3$, denote

$$I_\ell^{4,i}(u,v)(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} a_{j,k}^{\epsilon,i}(t) \Phi_{j,k}^\epsilon(x).$$

Case $[I_\ell^4]_2$: $t < 2^{-2j\beta}$. For this case, we denote $a_{j,k}^{\epsilon,4}(t) = a_{j,k}^\epsilon(t)$ and then have

$$I_\ell^4(u,v)(t,x) = \sum_{(\epsilon,j,k)\in\Lambda_n} a_{j,k}^{\epsilon,4}(t) \Phi_{j,k}^\epsilon(x).$$

The above decompositions of $B(u,v)$ provide convenience of the bilinear estimates on $\mathbb{F}_{p,q,m,m'}^{\gamma_1,\gamma_2}$. It is easy to see that the argument for $I_\ell^4(u,v)$ is similar to that for $I_\ell^1(u,v)$. Also the treatments of $I_\ell^3(u,v)$ is similar to that of $I_\ell^2(u,v)$. So we only show that

$$\begin{cases} I_\ell^1(u,v)(t,x) = \displaystyle\sum_{(\epsilon,j,k)\in\Lambda_n} a_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x) \in \mathbb{F}_{p,q,m,m'}^{\gamma_1,\gamma_2}, \\ I_\ell^2(u,v)(t,x) = \displaystyle\sum_{(\epsilon,j,k)\in\Lambda_n} b_{j,k}^\epsilon(t) \Phi_{j,k}^\epsilon(x) \in \mathbb{F}_{p,q,m,m'}^{\gamma_1,\gamma_2}. \end{cases} \tag{3.4}$$

By the decompositions obtained above, (3.4) is equivalent to the following estimates:

$$\begin{cases} \text{For } i=1,2,3,4, \displaystyle\sum_{(\epsilon,j,k)\in\Lambda_n} a^{\epsilon,i}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x)\in \mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'};\\ \text{For } i=1,2,3,4,5, \displaystyle\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,i}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x)\in \mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}. \end{cases}$$

The demonstration will be concluded by the following Lemmas 3.2–3.7.

Remark 3.1. In the proofs of Lemmas 3.2–3.7 below, we only give the details for the case $q\geq 2$. The case $1<q<2$ can be dealt with similarly.

3.2. Bilinear estimate on Bloch type tent spaces

Lemma 3.2. *Let* $(\beta,p,q,\gamma_1,\gamma_2,m,m')$ *satisfy* (3.1) *and* $u,v\in\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. *Then*

(i) $\sum_{(\epsilon,j,k)\in\Lambda_n} a^{\epsilon,i}_{j,k}(t)\Phi^{\epsilon}_{j,k}\in\mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}$, $i=1,2,3$;

(ii) $\sum_{(\epsilon,j,k)\in\Lambda_n} a^{\epsilon,4}_{j,k}(t)\Phi^{\epsilon}_{j,k}\in\mathbb{F}^{\gamma_1,\gamma_2,II}_{p,q}$.

Proof. For simplicity, we assume $\|u\|_{\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}}=\|v\|_{\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}}=1$. At first we prove (i). The proof of this part is divided into three parts. For $i=1,2,3$, write

$$I_{a,i}(t):=|Q_r|^{\gamma_2/n-1/p}\,t^m \times\left\|\left(\sum_{j\geq\max\{-\log_2 r,-(\log_2 t)/(2\beta)\}}\sum_{Q_{j,k}\subset Q_r}2^{qj(\gamma_1+n/2+2m\beta)}\left|a^{\epsilon,i}_{j,k}(t)\right|^q\chi_{j,k}(x)\right)^{1/q}\right\|_p.$$

Case 3.2.1: $\sum_{(\epsilon,j,k)\in\Lambda_n} a^{\epsilon,1}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x)\in\mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}$. Because $v\in\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$, by Lemma 2.16, one has $\left|v^0_{j'-3,k''}(s)\right|\lesssim s^{-(\gamma_2-\gamma_1)/(2\beta)}2^{-nj'/2}$ and

$$\begin{aligned}\left|a^{\epsilon,1}_{j,k}(t)\right| &\lesssim 2^{nj/2+j}\sum_{|j-j'|\leq 1}\sum_{\epsilon',k',k''}\int_0^{2^{-1-2j'\beta}}e^{-ct2^{2j\beta}}\frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N}\frac{\left|v^0_{j'-3,k''}(s)\right|}{(1+|2^{j-j'+3}k''-k|)^N}\,ds\\ &\lesssim 2^j\sum_{|j-j'|\leq 1}\sum_{\epsilon',k'}\int_0^{2^{-1-2j'\beta}}\frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N}e^{-ct2^{2j\beta}}s^{1/(2\beta)-1}\,ds.\end{aligned}$$

By $|j-j'|\leq 1$ and Hölder's inequality, it is easy to see that

$$\begin{aligned}&\left(\int_0^{2^{-1-2j'\beta}}\sum_{\epsilon',k'}\frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N}s^{1/(2\beta)-1}\,ds\right)^q\\ &\lesssim\sum_{w\in\mathbb{Z}^n}(1+|w|)^{-N}\sum_{Q_{j',k'}\subset Q^w_{j,k}}\frac{1}{(1+|2^{j-j'}k'-k|)^N}2^{-qj'}\left(\int_0^{2^{-1-2j'\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q s^{qm'}\frac{ds}{s}\right).\end{aligned}$$

Let

$$g_{j'} = 2^{j'(\gamma_1+n/2)} \sum_{Q_{j',k'}\subset Q_r^w} \left(\int_0^{2^{-1-2j'\beta}} \left| u_{j',k'}^{\epsilon'}(s) \right|^q (s2^{2j'\beta})^{qm'} \frac{ds}{s} \right)^{1/q} \chi_{j',k'}(x).$$

We apply Lemma 2.10 to obtain

$$\begin{aligned} I_{a,1}(t) &\lesssim |Q_r|^{\gamma_2/n-1/p}\, t^m \\ &\quad \times \Bigg\| \Bigg[\sum_{j\geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} \sum_{Q_{j,k}\subset Q_r} \sum_{|j-j'|\leq 1} 2^{qj(\gamma_1+n/2+2m\beta)} 2^{qj} e^{-cqt2^{2j\beta}} \\ &\quad \times \sum_{|j-j'|\leq 1} \left(\int_0^{2^{-1-2j'\beta}} \sum_{\epsilon',k'} \frac{\left| u_{j',k'}^{\epsilon'}(s) \right|}{(1+|2^{j-j'}k'-k|)^N} s^{1/(2\beta)-1}\, ds \right)^q \chi_{j,k}(x) \Bigg]^{1/q} \Bigg\|_p \\ &\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \Bigg\| \Bigg[\sum_{j\geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} (M_A(g_{j'})(x))^q \Bigg]^{1/q} \Bigg\|_p \\ &\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \Bigg\| \Bigg[\sum_{j\geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} |g_{j'}(x)|^q \Bigg]^{1/q} \Bigg\|_p \lesssim 1. \end{aligned}$$

Case 3.2.2: $\sum_{(\epsilon,j,k)\in\Lambda_n} a_{j,k}^{\epsilon,2}(t)\Phi_{j,k}^{\epsilon}(x) \in \mathbb{F}_{p,q,m}^{\gamma_1,\gamma_2,I}$. Because $\left| v_{j'-3,k'}^0(s) \right| \lesssim s^{1/(2\beta)-1} 2^{-nj'/2}$, we get

$$\left| a_{j,k}^{\epsilon,2}(t) \right| \lesssim 2^j \sum_{|j-j'|\leq 1} \sum_{\epsilon',k'} \int_{2^{-1-2j'\beta}}^{t/2} \frac{\left| u_{j',k'}^{\epsilon'}(s) \right|}{(1+|2^{j-j'}k'-k|)^N} e^{-ct2^{2j\beta}} s^{1/(2\beta)-1}\, ds.$$

Then

$$\begin{aligned} I_{a,2}(t) &\lesssim |Q_r|^{\gamma_2/n-1/p}\, t^m \\ &\quad \times \Bigg\| \Bigg(\sum_{j\geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2+2m\beta)} \\ &\quad \times \left(2^j \sum_{|j-j'|\leq 1} \sum_{\epsilon',k'} \int_{2^{-1-2j'\beta}}^{t/2} \frac{\left| u_{j',k'}^{\epsilon'}(s) \right|}{(1+|2^{j-j'}k'-k|)^N} e^{-ct2^{2j\beta}} s^{1/(2\beta)-1}\, ds \right)^q \chi_{j,k}(x) \Bigg)^{1/q} \Bigg\|_p \\ &\lesssim |Q_r|^{\gamma_2/n-1/p}\, t^m \Bigg\| \Bigg(\sum_{j\geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2+2m\beta)} 2^{qj} e^{-cqt2^{2j\beta}} \\ &\quad \times \sum_{|j-j'|\leq 1} \left(\sum_{\epsilon',k'} \int_{2^{-1-2j'\beta}}^{t/2} \frac{\left| u_{j',k'}^{\epsilon'}(s) \right|}{(1+|2^{j-j'}k'-k|)^N} s^{1/(2\beta)-1}\, ds \right)^q \chi_{j,k}(x) \Bigg)^{1/q} \Bigg\|_p . \end{aligned}$$

At first we assume $t > r^{2\beta}$. Because $u \in \mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}$, let $r = 2^{-j}$ and $Q_r = Q^w_{j,k}$. For fixed j',

$$2^{-nj(\gamma_2/n-1/p)} s^m \left\| \left(\sum_{Q_{j',k'} \subset Q^w_{j,k}} 2^{qj'(\gamma_1+n/2+2m\beta)} \left| u^{\epsilon'}_{j',k'}(s) \right|^q \chi_{j',k'}(x) \right)^{1/q} \right\|_p \lesssim 1.$$

For $s \geq 2^{-1-2j'\beta}$, if $s2^{2j'\beta} \geq 1$, then $\left|u^{\epsilon'}_{j',k'}(s)\right| \lesssim 2^{j'(\gamma_2-\gamma_1-n/2)}(s2^{2j'\beta})^{-m}$. If $s2^{2j'\beta} < 1$,

$$\left|u^{\epsilon'}_{j',k'}(s)\right| \leq 2^{j'(\gamma_2-\gamma_1-n/2)} \leq 2^{j'(\gamma_2-\gamma_1-n/2)}(s2^{2j'\beta})^{-m}.$$

Because $|j-j'| \leq 1$, the above estimate for $\left|u^{\epsilon'}_{j',k'}(s)\right|$ and Hölder's inequality imply that

$$\begin{aligned} I_{a,2}(t) &\lesssim |Q_r|^{\gamma_2/n-1/p} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} \sum_{j' \geq \max\{-\log_2 r_w, -(\log_2 t)/(2\beta)-1\}} 2^{qj'(\gamma_1+n/2)} \\ &\quad \times (t2^{2j'\beta})^{m+1/q} e^{-ct2^{2j'\beta}} 2^{j'(\gamma_2-\gamma_1-n/2)} \left\| \left(\sum_{Q_{j',k'} \subset Q^w_r} \chi_{j',k'}(x) \right)^{1/q} \right\|_p \\ &\lesssim |Q_r|^{\gamma_2/n} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} t^{-\gamma_2/(2\beta)} \sum_{j' \geq -(\log_2 t)/(2\beta)-1} (t2^{2j'\beta})^{m+1/q+\gamma_2/(2\beta)} e^{-ct2^{2j'\beta}} \lesssim 1, \end{aligned}$$

where we have used the fact that $t > r^{2\beta}$ in the last step.

Now we deal with the case $t \leq r^{2\beta}$. Let

$$g_{j'} = 2^{j'(\gamma_1+n/2)} \sum_{Q_{j',k'} \subset Q^w_r} \left(\int_{2^{-1-2j'\beta}}^{t/2} \left| u^{\epsilon'}_{j',k'}(s) \right|^q (s2^{2j'\beta})^{qm} \frac{ds}{s} \right)^{1/q} \chi_{j',k'}(x).$$

Note that $m \gg q/(2\beta)$. By $t < r^{2\beta}$, we use Lemma 2.10 and Hölder's inequality to obtain

$$\begin{aligned} I_{a,2}(t) &\lesssim |Q_r|^{\gamma_2/n-1/p} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} \\ &\quad \times \Bigg\| \Bigg[\sum_{j \geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} \sum_{Q_{j,k} \subset Q_r} 2^{qj(\gamma_1+n/2)} \sum_{|j-j'|\leq 1} 2^{q(j-j')} \\ &\quad \times \sum_{Q_{\epsilon',k'} \subset Q^w_{j,k}} \left(1+\left|2^{j-j'}k'-k\right|\right)^{-N} \left(\int_{2^{-1-2j'\beta}}^{t/2} \left| u^{\epsilon'}_{j',k'}(s) \right|^q (s2^{2j'\beta})^{qm} \frac{ds}{s} \right) \chi_{j,k}(x) \Bigg]^{1/q} \Bigg\|_p \\ &\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \left\| \left(\sum_{j' \geq -\log_2 r_w} (M_A(g_{j'})(x))^q \right)^{1/q} \right\|_p \\ &\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \left\| \left(\sum_{j' \geq -\log_2 r_w} |g_{j'}(x)|^q \right)^{1/q} \right\|_p \lesssim 1. \end{aligned}$$

Case 3.2.3: $\sum_{(\epsilon,j,k)\in\Lambda_n} a^{\epsilon,3}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x) \in \mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}$. We have

$$\left|a^{\epsilon,3}_{j,k}(t)\right| \lesssim 2^j \sum_{|j-j'|\le 1}\sum_{\epsilon',k',k''}\int_{t/2}^{t} e^{-c(t-s)2^{2j\beta}} \frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N} s^{1/(2\beta)-1}\,ds.$$

By Hölder's inequality and the fact $|j-j'|\le 1$, we can use the above estimate to deduce that

$$\begin{aligned} I_{a,3}(t) &\lesssim |Q_r|^{\gamma_2/n-1/p}\, t^m \Bigg\| \Bigg[\sum_{j\ge\max\{-\log_2 r,-(\log_2 t)/(2\beta)\}} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2+2m\beta)} 2^{qj} \\ &\quad \times \sum_{|j-j'|\le 1}\left|\sum_{\epsilon',k'}\int_{t/2}^{t}\frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N} e^{-c(t-s)2^{2j\beta}} s^{1/(2\beta)-1}\,ds\right|^q \chi_{j,k}(x)\Bigg]^{1/q}\Bigg\|_p \\ &\lesssim |Q_r|^{\gamma_2/n-1/p} \Bigg\| \Bigg[\sum_{j\ge\max\{-\log_2 r,-(\log_2 t)/(2\beta)\}} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2+2m\beta)} \\ &\quad \times \sum_{|j-j'|\le 1}\sum_{\epsilon',k'}\left(1+\left|2^{j-j'}k'-k\right|\right)^{-N} \\ &\quad \times \left(\int_{t/2}^{t}\left|u^{\epsilon'}_{j',k'}(s)\right|^q e^{-c(t-s)2^{2j\beta}} s^{qm}\frac{ds}{s}\right)(t2^{2j\beta})^{q/(2\beta)-(q-1)}\chi_{j,k}(x)\Bigg]^{1/q}\Bigg\|_p. \end{aligned}$$

Let

$$\begin{aligned} g_{j'} &= 2^{j'(\gamma_1+n/2+2m\beta)} \\ &\quad \times \sum_{Q_{j',k'}\subset Q^w_r}\left[(t2^{2j\beta})^{q/(2\beta)-(q-1)}\left(\int_{t/2}^{t}\left|u^{\epsilon'}_{j',k'}(s)\right|^q e^{-c(t-s)2^{2j\beta}} s^{qm}\frac{ds}{s}\right)\right]^{1/q}\chi_{j',k'}(x). \end{aligned}$$

Hence Lemma 2.10 implies that

$$\sum_{\epsilon',k'}\frac{t^{q/(2\beta)-(q-1)}}{(1+|2^{j-j'}k'-k|)^N}\left(\int_{t/2}^{t} e^{-c(t-s)2^{2j\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q s^{qm}\frac{ds}{s}\right) \lesssim \sum_{w\in\mathbb{Z}^n}\frac{1}{(1+|w|)^N}(M_A(g_{j'})(x))^q.$$

We can get

$$\begin{aligned} I_{a,3}(t) &\lesssim \sum_{w\in\mathbb{Z}^n}\frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^{N'}} \Bigg\|\Bigg[\sum_{j'\ge\max\{-\log_2 r_w,-(\log_2 t)/(2\beta)-1\}} 2^{qj'(\gamma_1+n/2+2m\beta)}(t2^{2j'\beta})^{q/(2\beta)-q+1} \\ &\quad \times \sum_{Q_{j',k'}\subset Q^w_r}\int_{t/2}^{t} e^{-c(t-s)2^{2j\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q s^{qm}\frac{ds}{s}\chi_{j',k'}(x)\Bigg]^{1/q}\Bigg\|_p. \end{aligned}$$

If $t > r^{2\beta}$, by Lemma 2.16, we can see that $\left|u^{\epsilon'}_{j',k'}(s)\right| \lesssim 2^{j'(\gamma_2-\gamma_1-n/2)}(s2^{2j'\beta})^{-m}$. This gives

$$I_{a,3}(t) \lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \left\| \left[\sum_{j'\geq \max\{-\log_2 r_w, -(\log_2 t)/(2\beta)-1\}} (t2^{2j'\beta})^{q/(2\beta)-(q-1)-1} \right.\right.$$
$$\left.\left. \times\ (t2^{2j'\beta})^{(q\gamma_2)/(2\beta)} t^{-(q\gamma_2)/(2\beta)} \sum_{Q_{j',k'}\subset Q_r^w} \chi_{j',k'}(x) \right]^{1/q} \right\|_p$$
$$\lesssim \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} \sum_{j'\geq \max\{-\log_2 r_w, -(\log_2 t)/(2\beta)-1\}} (t2^{2j'\beta})^{1/(2\beta)-1+\gamma_2/(2\beta)} \lesssim 1,$$

where we have used the facts $t > r^{2\beta}$ and $\gamma_1 < 0$, respectively.

For $t < r^{2\beta}$, because $u \in \mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}$, we have

$$|Q_r|^{\gamma_2/n-1/p}\, s^m \left\| \left(2^{qj'(\gamma_1+n/2+2m\beta)} \left| u^{\epsilon'}_{j',k'}(s) \right|^q \chi_{j',k'}(x) \right)^{1/q} \right\|_p \lesssim 1.$$

Because $Q_{j',k'} \subset Q^w_r$, that is, $2^{-j'} < r_w$, the above estimate implies that

$$\left| u^{\epsilon'}_{j',k'}(s) \right| (s2^{2j'\beta})^m \lesssim |Q_r|^{-\gamma_2/n}\, 2^{-j'(\gamma_1+n/2)}.$$

We can obtain

$$I_{a,3}(t) \lesssim |Q_r|^{\gamma_2/n-1/p} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} \left\| \left[\sum_{j'\geq \max\{-\log_2 r_w, -(\log_2 t)/(2\beta)-1\}} 2^{qj'(\gamma_1+2m\beta+n/2)} \right.\right.$$
$$\times \sum_{Q_{j',k'}\subset Q^w_r} \left(\int_{t/2}^t e^{-c(t-s)2^{2j'\beta}} s^{qm} 2^{-qj'(\gamma_1+n/2)} (s2^{2j'\beta})^{-qm} |Q_r|^{-q\gamma_2/n}\, \frac{ds}{s} \right)$$
$$\left.\left. \times\, (t2^{2j'\beta})^{q/(2\beta)-(q-1)} \chi_{j',k'}(x) \right]^{1/q} \right\|_p$$
$$\lesssim |Q_r|^{-1/p} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N}$$
$$\times \left\| \left[\sum_{j'\geq \max\{-\log_2 r_w, -(\log_2 t)/(2\beta)-1\}} (t2^{2j'\beta})^{q/(2\beta)-q} \sum_{Q_{j',k'}\subset Q^w_r} \chi_{j',k'}(x) \right]^{1/q} \right\|_p \lesssim 1.$$

Now we prove (ii). Write

$$II_{a,4}(t) := |Q_r|^{\gamma_2/n-1/p} \left\| \left(\sum_{-\log_2 r<j<-(\log_2 t)/(2\beta)} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2)} \left| a^{\epsilon,4}_{j,k}(t) \right|^q \chi_{j,k}(x) \right)^{1/q} \right\|_p.$$

For this case, we can see that

$$\begin{aligned}\left|a_{j,k}^{\epsilon,4}(t)\right| &\lesssim 2^{nj/2+j} \sum_{|j-j'|\le 1} \sum_{\epsilon',k',k''} \int_0^t \frac{\left|u_{j',k'}^{\epsilon'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N} \frac{\left|v_{j'-3,k''}^{0}(s)\right|}{(1+|k'-k''|)^N} e^{-c(t-s)2^{2j\beta}}\, ds \\ &\lesssim 2^j \sum_{|j-j'|\le 1} \sum_{\epsilon',k'} \int_0^t \frac{\left|u_{j',k'}^{\epsilon'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N} e^{-c(t-s)2^{2j\beta}} s^{1/(2\beta)-1}\, ds.\end{aligned}$$

By Hölder's inequality and the facts that $t \le 2^{-2j\beta}$ and $|j-j'|\le 1$, we obtain

$$\begin{aligned}&II_{a,4}(t)\\ &\lesssim |Q_r|^{\gamma_2/n-1/p} \Bigg\| \Bigg[\sum_{-\log_2 r<j'\le -(\log_2 t)/(2\beta)} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2)} 2^{qj} \\ &\quad\times \sum_{|j-j'|\le 1} \left| \sum_{\epsilon',k'} \int_0^t e^{-c(t-s)2^{2j\beta}} \frac{\left|u_{j',k'}^{\epsilon'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N} s^{1/(2\beta)-1}\, ds \right|^q \chi_{j,k}(x) \Bigg]^{1/q} \Bigg\|_p \\ &\lesssim |Q_r|^{\gamma_2/n-1/p} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} \Bigg\| \Bigg\{ \sum_{-\log_2 r<j'\le -(\log_2 t)/(2\beta)} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2)} \sum_{|j-j'|\le 1} \\ &\quad\times \Bigg[\sum_{Q_{\epsilon',k'}\subset Q_{j,k}^w} \frac{1}{(1+|2^{j-j'}k'-k|)^N} \left(\int_0^t \left|u_{j',k'}^{\epsilon'}(s)\right|^q (s2^{2j'\beta})^{qm'} \frac{ds}{s} \right)^{1/q} \Bigg]^q \chi_{j,k}(x) \Bigg\}^{1/q} \Bigg\|_p .\end{aligned}$$

Let

$$g_{j'}(x) = 2^{j'(\gamma_1+n/2)} \sum_{Q_{j',k'}\subset Q_r^w} \left(\int_0^t \left|u_{j',k'}^{\epsilon'}(s)\right|^q (s2^{2j'\beta})^{qm'} \frac{ds}{s} \right)^{1/q} \chi_{j',k'}(x).$$

Finally, by Lemma 2.10, we have

$$\begin{aligned}II_{a,4}(t) &\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \\ &\qquad\times \Bigg\| \Bigg\{ \sum_{-\log_2 r<j'\le -(\log_2 t)/(2\beta)} \sum_{|j-j'|\le 1} 2^{q(j-j')(\gamma_1+n/2)} \left[M_A(g_{j'})(x)\right]^q \chi_{j,k}(x) \Bigg\}^{1/q} \Bigg\|_p \\ &\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \Bigg\| \Bigg\{ \sum_{-\log_2 r_w\le j'\le -(\log_2 t)/(2\beta)-1} |g_{j'}(x)|^q \Bigg\}^{1/q} \Bigg\|_p \lesssim 1,\end{aligned}$$

where in the last inequality, we have used the fact $t \le 2^{-2j'\beta}$. □

Lemma 3.3. *Let* $(\beta,p,q,\gamma_1,\gamma_2,m,m')$ *satisfy* (3.1) *and* $u,v\in \mathbb{F}_{p,q,m,m'}^{\gamma_1,\gamma_2}$. *Then*

$$\sum_{(\epsilon,j,k)\in\Lambda_n} b_{j,k}^{\epsilon,i}(t)\Phi_{j,k}^{\epsilon}(x) \in \mathbb{F}_{p,q,m}^{\gamma_1,\gamma_2,I}, \quad i=1,2,3.$$

Proof. For $i=1,2,3$, write

$$I_{b,i}(t) := |Q_r|^{\gamma_2/n-1/p}\, t^m \times \left\| \left(\sum_{j\geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2+2m\beta)} \left|a^{\epsilon,i}_{j,k}(t)\right|^q \chi_{j,k}(x) \right)^{1/q} \right\|_p .$$

We divide the proof into three cases.

Case 3.3.1: Under $2<q<\infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,1}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x) \in \mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}$. For $0<s<2^{-1-2j'\beta}$ and $r=2^{-j}$, the fact $v\in \mathbb{F}^{\gamma_1,\gamma_2,II}_{p,q}$ implies that

$$\left|v^{\epsilon''}_{j',k''}(s)\right| \lesssim 2^{j(\gamma_2-n/p)} 2^{-j'(\gamma_1+n/2-n/p)}.$$

By Hölder's inequality, we have

$$\begin{aligned}
\left|b^{\epsilon,1}_{j,k}(t)\right| &\lesssim 2^{nj/2+j} e^{-ct2^{2j\beta}} \sum_{j<j'+2} \sum_{\epsilon',k'} \sum_{\epsilon'',k''} \int_0^{2^{-1-2j'\beta}} \frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N} \frac{\left|v^{\epsilon''}_{j',k''}(s)\right|}{(1+|k'-k''|)^N}\, ds \\
&\lesssim \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} 2^{nj/2+j} 2^{j(\gamma_2-n/p)} e^{-ct2^{2j\beta}} \sum_{j<j'+2} 2^{-j'(\gamma_1+n/2-n/p)} 2^{-2j'\beta(1-1/q)} \\
&\quad \times \sum_{Q_{j',k'}\subset Q^w_{j,k}} \left(1+\left|2^{j-j'}k'-k\right|\right)^{-N} \left(\int_0^{2^{-1-2j'\beta}} \left|u^{\epsilon'}_{j',k'}(s)\right|^q ds \right)^{1/q},
\end{aligned}$$

Let

$$g_{j'} = \sum_{Q_{j',k'}\subset Q^w_{j,k}} \left(1+\left|2^{j-j'}k'-k\right|\right)^{-N} \left(\int_0^{2^{-1-2j'\beta}} \left|u^{\epsilon'}_{j',k'}(s)\right|^q ds \right)^{1/q}.$$

Hence Cauchy-Schwartz's inequality gives

$$\begin{aligned}
I_{b,1}(t) \lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \Bigg\| \Bigg\{ &\sum_{j\geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} 2^{qj(\gamma_1+n+1+2m\beta)} (t2^{2j\beta})^{qm} \\
&\times e^{-ct2^{2j\beta}} 2^{qj(\gamma_2-n/p)} \left[\sum_{j<j'+2} 2^{-j'(\gamma_1+n/2-n/p)} 2^{-2j'\beta(1-1/q)} |g_{j'}| \right]^q \chi_{j,k}(x) \Bigg\}^{1/q} \Bigg\|_p .
\end{aligned}$$

Let

$$f_{j'} := \sum_{Q_{j',k'}\subset Q^w_{j,k}} \frac{2^{j'(\gamma_1+n/2-2\beta/q)}}{(1+|2^{j-j'}k'-k|)^N} \left(\int_0^{2^{-1-2j'\beta}} \left|u^{\epsilon'}_{j',k'}(s)\right|^q (s2^{2j'\beta})^{m'}\, ds \right)^{1/q} \chi_{j',k'}(x).$$

Then

$$
\begin{aligned}
I_{b,1}(t) &\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \left\| \left\{ \sum_{j\geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} 2^{qj(\gamma_1+n+1+2m\beta+\gamma_2-n/p)} \right.\right. \\
&\quad \left.\left. \times \sum_{j<j'+2} 2^{(\delta+qn)(j'-j)} 2^{-qj'(2\gamma_1+n-n/p+2\beta)} (M_A(f_{j'})(x))^q \chi_{j,k}(x) \right\}^{1/q} \right\|_p \\
&\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \sum_{j<j'+2} 2^{(j'-j)[\delta/q-(\gamma_1+\gamma_2+1-n/p)]} \\
&\quad \times \left\| \left\{ \sum_{j'\geq -\log_2 r_w} \sum_{Q_{j',k'}\subset Q_r^w} 2^{qj'(\gamma_1+n/2)} \int_0^{2^{-1-2j'\beta}} \left| u_{j',k'}^{\epsilon'}(s)\right|^q (s2^{2j'\beta})^{m'}\, ds\, \chi_{j',k'}(x) \right\}^{1/q} \right\|_p \\
&\lesssim \|u\|_{\mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}}.
\end{aligned}
$$

Case 3.3.2: Under $2<q<\infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,2}_{j,k}(t)\Phi^\epsilon_{j,k}(x) \in \mathbb{F}^{\gamma_1,\gamma_2,I}_{p,q,m}$. At first, we assume that $t>r^{2\beta}$. Because $\left|v^{\epsilon''}_{j',k''}(s)\right| \lesssim 2^{j(\gamma_2-n/p)}2^{-j'(\gamma_1+n/2-n/p)}(s2^{2j'\beta})^{-m}$, we obtain that

$$
\begin{aligned}
\left|b^{\epsilon,2}_{j,k}(t)\right| &\lesssim 2^{nj/2+j} e^{-ct2^{2j\beta}} \sum_{j<j'+2}\sum_{\epsilon',k'}\sum_{\epsilon'',k''} \int_{2^{-1-2j'\beta}}^{t/2} \frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N} \frac{\left|v^{\epsilon''}_{j',k''}(s)\right|}{(1+|k'-k''|)^N}\, ds \\
&\lesssim 2^{nj/2+j} e^{-ct2^{2j\beta}} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} \sum_{w'\in\mathbb{Z}^n} (1+|w-w'|)^{-N} 2^{2j(\gamma_2-n/p)} \\
&\quad \times \sum_{j<j'+2} 2^{-2j'(\gamma_1+n/2-n/p)} 2^{n(j'-j)} \int_{2^{-1-2j'\beta}}^{t/2} (s2^{2j'\beta})^{-2m}\, ds \\
&\lesssim 2^{-nj/2+j} 2^{2\gamma_2 j} 2^{-2j\gamma_1} e^{-ct2^{2j\beta}} 2^{-2j\beta},
\end{aligned}
$$

where we have used the fact that $\gamma_1+\beta-n/p>0$. By $\gamma_2<n/p$, the above estimate implies that

$$
I_{b,2}(t) \lesssim |Q_r|^{\gamma_2/n-1/p}\, t^m \sum_{j\geq \max\{-\log_2 r, -(\log_2 t)/(2\beta)\}} 2^{j(2\gamma_2-\gamma_1+2m\beta+1-2\beta)} e^{-ct2^{2j\beta}} 2^{-nj/p} \lesssim 1.
$$

Now, we suppose $t\leq r^{2\beta}$. Because $v\in \mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}\cap \mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}$, if $Q_{j',k'}\subset Q^w_{j,k}$,

$$
\left(\int_{2^{-1-2j'\beta}}^{r^{2\beta}} \left|v^{\epsilon''}_{j',k''}(s)\right|^q (s2^{2j'\beta})^{qm} \frac{ds}{s} \right)^{1/q} \lesssim r2^{j(\gamma_2-n/p)} 2^{-j'(\gamma_1+n/2-n/p)}.
$$

It can be deduced that

$$\begin{aligned}\left|b_{j,k}^{\epsilon,2}(t)\right| &\lesssim 2^{j(n/2+1)}e^{-ct2^{2j\beta}}\sum_{j<j'+2}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}\int_{2^{-1-2j'\beta}}^{t/2}\frac{\left|u_{j',k'}^{\epsilon'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N}\frac{\left|v_{j',k''}^{\epsilon''}(s)\right|}{(1+|k'-k''|)^N}\,ds\\ &\lesssim 2^{j(n/2+1+\gamma_2-n/p)}e^{-ct2^{2j\beta}}\sum_{w\in\mathbb{Z}^n}(1+|w|)^{-N}\sum_{j<j'+2}2^{-j'(2\beta+\gamma_1+n/2-n/p)}\\ &\quad\times\sum_{Q_{j',k'}\subset Q_{j,k}^w}\left(1+\left|2^{j-j'}k'-k\right|\right)^{-N}\left(\int_{2^{-2j'\beta}}^{r^{2\beta}}\left|u_{j',k'}^{\epsilon'}(s)\right|^q(s2^{2j'\beta})^{qm}\frac{ds}{s}\right)^{1/q}.\end{aligned}$$

Let

$$g_{j'}:=\sum_{Q_{j',k'}\subset Q_{j,k}^w}\left(1+\left|2^{j-j'}k'-k\right|\right)^{-N}\left(\int_{2^{-2j'\beta}}^{r^{2\beta}}\left|u_{j',k'}^{\epsilon'}(s)\right|^q(s2^{2j'\beta})^{qm}\frac{ds}{s}\right)^{1/q}.$$

This implies that

$$\begin{aligned}I_{b,2}(t) &\lesssim t^m\sum_{w\in\mathbb{Z}^n}\frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N}\left\|\left\{\sum_{j\geq\max\{-\log_2 r,-(\log_2 t)/(2\beta)\}}\sum_{Q_{j,k}\subset Q_r}2^{qj(\gamma_1+n+1+2m\beta)}\right.\right.\\ &\quad\left.\left.\times\ 2^{qj(\gamma_2-n/p)}e^{-ct2^{2j\beta}}\sum_{j<j'+2}2^{\delta(j'-j)}2^{-qj'(2\beta+\gamma_1+n/2-n/p)}\left|g_{j'}\right|^q\chi_{j,k}(x)\right\}^{1/q}\right\|_p\\ &\lesssim t^m\sum_{w\in\mathbb{Z}^n}\frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N}\left\|\left\{\sum_{j\geq\max\{-\log_2 r,-(\log_2 t)/(2\beta)\}}2^{qj(\gamma_1+n+1)}2^{qj(\gamma_2-n/p)}\right.\right.\\ &\quad\left.\left.\times\sum_{j<j'+2}2^{\delta(j'-j)}2^{-qj'(\gamma_1+n/2-n/p+2\beta)}\left|g_{j'}\right|^q\right\}^{1/q}\right\|_p.\end{aligned}$$

Let

$$f_{j'}=\sum_{Q_{j',k'}\subset Q_r^w}2^{j'(\gamma_1+n/2)}\left(\int_{2^{-1-2j'\beta}}^{r^{2\beta}}\left|u_{j',k'}^{\epsilon'}(s)\right|^q(s2^{2j'\beta})^{qm}\frac{ds}{s}\right)^{1/q}\chi_{j',k'}(x).$$

By Lemma 2.10, we can see that $\left|g_{j'}\right|^q\lesssim 2^{-qj'(\gamma_1+n/2)}2^{qn(j'-j)}(M_A(f_{j'})(x))^q$ and hence

$$\begin{aligned}I_{b,2}(t) &\lesssim \sum_{w\in\mathbb{Z}^n}\frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N}\\ &\quad\times\left\|\left\{\sum_{j\geq\max\{-\log_2 r,-(\log_2 t)/(2\beta)\}}\sum_{j<j'+2}2^{(j'-j)[\delta-(\gamma_1+\gamma_2+1-n/p)]}(M_A(f_{j'})(x))^q\right\}^{1/q}\right\|_p\\ &\lesssim \sum_{w\in\mathbb{Z}^n}\frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N}\sum_{j<j'+2}2^{(j'-j)[\delta-(\gamma_1+\gamma_2+1-n/p)]}\left\|\left\{\sum_{j'\geq-\log_2 r_w}|f_{j'}(x)|^q\right\}^{1/q}\right\|_p\\ &\lesssim \|u\|_{\mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}}+\|u\|_{\mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}}\end{aligned}$$

Case 3.3.3: Under $2 < q < \infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b_{j,k}^{\epsilon,3}(t)\Phi_{j,k}^{\epsilon}(x) \in \mathbb{F}_{p,q,m}^{\gamma_1,\gamma_2,I}$. Because $v \in \mathbb{F}_{p,q,m}^{\gamma_1,\gamma_2,I} \cap \mathbb{F}_{p,q}^{\gamma_1,\gamma_2,II}$, then

$$\left(\int_{t/2}^{t} \left|v_{j',k''}^{\epsilon''}(s)\right|^q ds\right)^{1/q} \lesssim 2^{j(\gamma_2-n/p)}2^{-j'(\gamma_1+n/2-n/p)}(t2^{2j'\beta})^{-m}t^{1/q}.$$

By the above estimate and Hölder's inequality, we can get

$$\begin{aligned}\left|b_{j,k}^{\epsilon,3}(t)\right| &\lesssim 2^{nj/2+j} \sum_{j<j'+2}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}\int_{t/2}^{t} \frac{\left|u_{j',k'}^{\epsilon'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N}\frac{\left|v_{j',k''}^{\epsilon''}(s)\right|}{(1+|k'-k''|)^N}e^{-c(t-s)2^{2j\beta}}\,ds\\
&\lesssim 2^{j(n/2+1+\gamma_2-n/p)}t^{1-1/q}\sum_{w\in\mathbb{Z}^n}(1+|w|)^{-N}\sum_{j<j'+2}2^{-j'(\gamma_1+n/2-n/p)}(t2^{2j'\beta})^{-m}\\
&\quad\times\sum_{\epsilon',k'}\left(1+\left|2^{j-j'}k'-k\right|\right)^{-N}\left(\int_{t/2}^{t}\left|u_{j',k'}^{\epsilon'}(s)\right|^q ds\right)^{1/q}.\end{aligned}$$

By $\gamma_2 \le n/p$ and $j \ge -\log_2 r$, we can get

$$\begin{aligned}I_{b,3}(t) &\lesssim \sum_{w\in\mathbb{Z}^n}\frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N}t^m\left\|\left\{\sum_{j\ge\max\{-\log_2 r,-(\log_2 t)/(2\beta)\}}2^{qj(\gamma_1+n+1+2m\beta+\gamma_2-n/p)}t^{q-1}\right.\right.\\
&\quad\times\sum_{j<j'+2}2^{\delta(j'-j)}2^{-qj'(\gamma_1+n/2-n/p)}(t2^{2j'\beta})^{-qm}\\
&\quad\times\left.\left.\left[\sum_{\epsilon',k'}\frac{1}{(1+|2^{j-j'}k'-k|)^N}\left(\int_{t/2}^{t}\left|u_{j',k'}^{\epsilon'}(s)\right|^q ds\right)^{1/q}\right]^q\chi_{j,k}(x)\right\}^{1/q}\right\|_p\\
&\lesssim \sum_{j'\ge-(\log_2 t)/(2\beta)-2}(t2^{2j'\beta})^{1-m}\sum_{j<j'+2}2^{(j'-j)[\delta/q-(\gamma_1+\gamma_2+1-2n/p+2m\beta)]}\lesssim 1.\end{aligned}$$

This completes the proof. □

Lemma 3.4. *Let* $(\beta,p,q,\gamma_1,\gamma_2,m,m')$ *satisfy* (3.1) *and* $u,v\in\mathbb{F}_{p,q,m,m'}^{\gamma_1,\gamma_2}$. *Then*

$$\sum_{(\epsilon,j,k)\in\Lambda_n} b_{j,k}^{\epsilon,i}(t)\Phi_{j,k}^{\epsilon}(x)\in\mathbb{F}_{p,q}^{\gamma_1,\gamma_2,II},\quad i=4,5.$$

Proof. For $i=4,5$, write

$$II_{b,i}(t) := |Q_r|^{\gamma_2/n-1/p}\left\|\left(\sum_{-\log_2 r<j<-(\log_2 t)/(2\beta)}\sum_{Q_{j,k}\subset Q_r}2^{qj(\gamma_1+n/2)}\left|b_{j,k}^{\epsilon,i}(t)\right|^q\chi_{j,k}(x)\right)^{1/q}\right\|_p.$$

We still divide the proof into two cases.

Case 3.4.1: Under $2\le q<\infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,4}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x)\in \mathbb{F}^{\gamma_1,\gamma_2,II}_{p,q}$. We can see that

$$\begin{aligned}\left|b^{\epsilon,4}_{j,k}(t)\right| &\lesssim 2^{nj/2+j}\sum_{j<j'+2}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}\int_0^{2^{-2j'\beta}}\frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N}\frac{\left|v^{\epsilon''}_{j',k''}(s)\right|}{(1+|k'-k''|)^N}\,ds\\ &\lesssim 2^{nj/2+j}\sum_{j<j'+2}2^{-2j'\beta(1-2/q)}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}\frac{1}{(1+|2^{j-j'}k'-k|)^N}\frac{1}{(1+|k'-k''|)^N}\\ &\quad\times\left(\int_0^{2^{-2j'\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q ds\right)^{1/q}\left(\int_0^{2^{-2j'\beta}}\left|v^{\epsilon''}_{j',k''}(s)\right|^q ds\right)^{1/q}.\end{aligned}$$

Let

$$f_{j'}=2^{j'(\gamma_1+n/2)}\sum_{Q_{j',k'}\subset Q^w_r}\left(\int_0^{2^{-2j'\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q(s2^{2j'\beta})^{qm'}\frac{ds}{s}\right)^{1/q}\chi_{j',k'}(x).$$

Because

$$\left(\int_0^{2^{-2j'\beta}}\left|v^{\epsilon''}_{j',k''}(s)\right|^q ds\right)^{1/q}\lesssim 2^{j\gamma_2-n/p}2^{-j'(\gamma_1+n/2-n/p)}2^{-2j'\beta/q},$$

by Lemma 2.10, we can obtain

$$\begin{aligned}II_{b,4}(t) &\lesssim |Q_r|^{\gamma_2/n-1/p}\Bigg\|\Bigg\{\sum_{-\log_2 r<j<-(\log_2 t)/(2\beta)}\sum_{Q_{j,k}\subset Q_r}2^{qj(\gamma_1+n+1+\gamma_2-n/p)}\\ &\quad\times\left[\sum_{j<j'+2}\sum_{\epsilon',k'}\frac{2^{-j'(2\beta-2\beta/q+\gamma_1+n/2-n/p)}}{(1+|2^{j-j'}k'-k|)^N}\left(\int_0^{2^{-2j'\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q ds\right)^{1/q}\right]^q\chi_{j,k}(x)\Bigg\}^{1/q}\Bigg\|_p\\ &\lesssim\sum_{w\in\mathbb{Z}^n}\frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^{-N}}\Bigg\|\Bigg\{\sum_{-\log_2 r<j<-(\log_2 t)/(2\beta)}2^{qj(\gamma_1+\gamma_2+1+n-n/p)}\\ &\quad\times\sum_{j<j'+2}2^{(\delta+qn)(j'-j)}2^{-qj'[2\gamma_1+n-n/p+2\beta]}(M_A(f_{j'})(x))^q\Bigg\}^{1/q}\Bigg\|_p\\ &\lesssim\|u\|_{\mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}}.\end{aligned}$$

Case 3.4.2: Under $2\le q<\infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,5}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x)\in \mathbb{F}^{\gamma_1,\gamma_2,II}_{p,q}$. By a simple computation, we have

$$\left(\int_{2^{-2j'\beta}}^t\left|u^{\epsilon'}_{j',k'}(s)\right|^q s^{qm}\frac{ds}{s}\right)^{1/q}\lesssim 2^{-2j'\beta m}\left(\int_{2^{-2j'\beta}}^t\left|u^{\epsilon'}_{j',k'}(s)\right|^q(s2^{2j'\beta})^{qm}\frac{ds}{s}\right)^{1/q}$$

and

$$\left(\int_{2^{-2j'\beta}}^t\left|v^{\epsilon''}_{j',k''}(s)\right|^q s^{qm}\frac{ds}{s}\right)^{1/q}\lesssim 2^{-2j'\beta m}2^{-j'(\gamma_1+n/2-n/p)}2^{j(\gamma_2-n/p)}.$$

Similarly, the above estimate and Holder's inequality imply that

$$\begin{aligned}\left|b_{j,k}^{\epsilon,5}(t)\right| &\lesssim 2^{nj/2+j}\sum_{j<j'+2}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}\int_{2^{-2j'\beta}}^{t}\frac{\left|u_{j',k'}^{\epsilon'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N}\frac{\left|v_{j',k''}^{\epsilon''}(s)\right|}{(1+|k'-k''|)^N}\,ds\\ &\lesssim 2^{nj/2+j}2^{j(\gamma_2-n/p)}\sum_{j<j'+2}2^{-j'(\gamma_1+n/2-n/p)}2^{-2j'\beta}\\ &\quad\times\sum_{\epsilon',k'}(1+|2^{j'-j}k'-k|)^{-N}\left(\int_{2^{-2j'\beta}}^{t}\left|u_{j',k'}^{\epsilon'}(s)\right|^q(s2^{2j'\beta})^{qm}\frac{ds}{s}\right)^{1/q}.\end{aligned}$$

Let

$$f_{j'}=2^{j'(\gamma_1+n/2)}\sum_{Q_{j',k'}\subset Q_r^w}\left(\int_{2^{-2j'\beta}}^{t}\left|u_{j',k'}^{\epsilon'}(s)\right|^q(s2^{2j'\beta})^{qm'}\frac{ds}{s}\right)^{1/q}\chi_{j',k'}(x).$$

Similar to Case 3.4.1, we can obtain

$$\begin{aligned}II_{b,5}(t) &\lesssim \sum_{w\in\mathbb{Z}^n}\frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N}\left\|\left\{\sum_{-\log_2 r<j<-(\log_2 t)/(2\beta)}\sum_{Q_{j,k}\subset Q_r}2^{qj(\gamma_1+\gamma_2+1+n-n/p)}\right.\right.\\ &\quad\left.\left.\times\sum_{j<j'+2}2^{(\delta+qn)(j'-j)}2^{-qj'(2\gamma_1+n-n/p+2\beta)}(M_A(f_{j'})(x))^q\chi_{j,k}(x)\right\}^{1/q}\right\|_p\\ &\lesssim 1,\end{aligned}$$

which completes the proof. □

3.3. Bilinear estimates on L^p type tent spaces

Lemma 3.5. *Let* $(\beta,p,q,\gamma_1,\gamma_2,m,m')$ *satisfy* (3.1) *and* $u,v\in\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. *Then*

(i) $\sum_{(\epsilon,j,k)\in\Lambda_n}a_{j,k}^{\epsilon,i}(t)\Phi_{j,k}^{\epsilon}\in\mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}$, $i=1,2,3$;

(ii) $\sum_{(\epsilon,j,k)\in\Lambda_n}a_{j,k}^{\epsilon,4}(t)\Phi_{j,k}^{\epsilon}\in\mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}$.

Proof. Now we prove (i). For $i=1,2,3$, write

$$III_{a,i}:=|Q_r|^{\gamma_2/n-1/p}\left\|\left(\sum_{(\epsilon,j,k)\in\Lambda_Q^n}2^{qj(\gamma_1+n/2+2m\beta)}\int_{2^{-2j\beta}}^{r^{2\beta}}\left|a_{j,k}^{\epsilon,i}(t)\right|^q t^{qm}\frac{dt}{t}\chi_{j,k}(x)\right)^{1/q}\right\|_p.$$

We still divide the proof into three cases:
Case 3.5.1: Under $2\le q<\infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n}a_{j,k}^{\epsilon,1}(t)\Phi_{j,k}^{\epsilon}(x)\in\mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}$;
Case 3.5.2: Under $2\le q<\infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n}a_{j,k}^{\epsilon,2}(t)\Phi_{j,k}^{\epsilon}(x)\in\mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}$;
Case 3.5.3: Under $2\le q<\infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n}a_{j,k}^{\epsilon,3}(t)\Phi_{j,k}^{\epsilon}(x)\in\mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}$.
For simplicity, we only give the proof of Case 3.5.1. The proofs of Cases 3.5.2 and 3.5.3 are similar.

Now we prove Case 3.5.1. By $u, v \in \mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$, we can see that

$$\begin{aligned}\left|a^{\epsilon}_{j,k}(t)\right| &\lesssim 2^{nj/2+j} \sum_{|j-j'|\leq 1} \sum_{\epsilon',k',k''} \int_0^{2^{-1-2j'\beta}} \frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N} \frac{\left|v^{0}_{j'-3,k''}(s)\right|}{(1+|2^{j-j'+3}k''-k|)^N} e^{-ct2^{2j\beta}}\, ds \\ &\lesssim 2^{j} \sum_{|j-j'|\leq 1} \sum_{\epsilon',k',k''} \int_0^{2^{-1-2j'\beta}} \left|u^{\epsilon'}_{j',k'}(s)\right| e^{-ct2^{2j\beta}} (1+|2^{j-j'}k'-k|)^{-N} s^{1/(2\beta)-1}\, ds.\end{aligned}$$

We can get

$$\begin{aligned} III_{a,1} \lesssim |Q_r|^{\gamma_2/n-1/p} \Bigg\| \Bigg[&\sum_{(\epsilon,j,k)\in\Lambda^Q_n} 2^{qj(\gamma_1+n/2+2m\beta)} 2^{qj} e^{-ct2^{2j\beta}} \int_{2^{-2j\beta}}^{r^{2\beta}} \sum_{|j-j'|\leq 1} \\ &\times \left(\sum_{\epsilon',k'} \int_0^{2^{-1-2j'\beta}} \frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N} s^{1/(2\beta)-1}\, ds \right)^q t^{qm} \frac{dt}{t} \chi_{j,k}(x) \Bigg]^{1/q} \Bigg\|_p .\end{aligned}$$

Let

$$g_{j'} = 2^{j'(\gamma_1+n/2)} \sum_{Q_{j',k'}\subset Q^w_r} \left(\int_0^{2^{-1-2j'\beta}} \left|u^{\epsilon'}_{j',k'}(s)\right|^q (s2^{2j'\beta})^{qm'} \frac{ds}{s} \right)^{1/q} \chi_{j',k'}(x).$$

For $|j-j'|\leq 1$, we have, by Lemma 2.10 and Hölder's inequality for k' to deduce that

$$\begin{aligned} III_{a,1} &\lesssim |Q_r|^{\gamma_2/n-1/p} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} \left\| \left\{ \sum_{j'\geq -\log_2 r_w} (M_A(g_{j'})(x))^q \right\}^{1/q} \right\|_p \\ &\lesssim |Q_r|^{\gamma_2/n-1/p} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} \left\| \left\{ \sum_{j'\geq -\log_2 r_w} |g_{j'}(x)|^q \right\}^{1/q} \right\|_p \\ &\lesssim \|u\|_{\mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}} + \|u\|_{\mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}} .\end{aligned}$$

Now we prove (ii). Write

$$IV_{a,4} := |Q_r|^{\gamma_2/n-1/p} \left\| \left[\sum_{j\geq -\log_2 r} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2+2m'\beta)} \int_0^{2^{-2j\beta}} \left|a^{\epsilon,4}_{j,k}(t)\right|^q t^{qm'} \frac{dt}{t} \chi_{j,k}(x) \right]^{1/q} \right\|_p .$$

Take μ such that $qm'+q-1-q/(2\beta) \leq q\mu < q-1$. Similarly we can get

$$\begin{aligned} IV_{a,4} \lesssim |Q_r|^{\gamma_2/n-1/p} \Bigg\| \Bigg[&\sum_{j\geq -\log_2 r} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2+2m'\beta)} \int_0^{2^{-2j\beta}} 2^{qj} \sum_{|j-j'|\leq 2} \\ &\times \left| \sum_{\epsilon',k'} \frac{1}{(1+|2^{j-j'}k'-k|)^N} \int_0^t \left|u^{\epsilon'}_{j',k'}(s)\right| e^{-c(t-s)2^{2j\beta}} s^{1/(2\beta)-1}\, ds \right|^q t^{qm'} \frac{dt}{t} \chi_{j,k}(x) \Bigg]^{1/q} \Bigg\|_p\end{aligned}$$

$$\lesssim |Q_r|^{\gamma_2/n-1/p} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N}$$
$$\times \left\| \left[\sum_{j\geq -\log_2 r} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2)} \sum_{|j-j'|\leq 1} \sum_{Q_{j',k'}\subset Q^w_{j,k}} (1+|2^{j-j'}k'-k|)^{-N} t^{q-1-q\mu} \right.\right.$$
$$\left.\left. \times \int_0^{2^{-2j\beta}} \left|u^{\epsilon'}_{j',k'}(s)\right|^q s^{q(1/(2\beta)-1+\mu)} \left(\int_s^{2^{-2j\beta}} 2^{qj(1+2m'\beta)} t^{qm'+q-q\mu-2}\, dt\right) ds\, \chi_{j,k}(x) \right]^{1/q} \right\|_p .$$

Let

$$g_{j'}(x) = 2^{j'(\gamma_1+n/2)} \sum_{Q_{j',k'}\subset Q^w_r} \left(\int_0^{2^{-2j\beta}} \left|u^{\epsilon'}_{j',k'}(s)\right|^q (s2^{2j'\beta})^{qm'} \frac{ds}{s}\right)^{1/q} \chi_{j',k'}(x).$$

By $s2^{2j\beta'} \leq 1$ and $qm'+q-1-q/(2\beta) \leq q\mu$, a simple computation yields

$$IV_{a,4} \lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \left\| \left[\sum_{j\geq -\log_2 r} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2)} \sum_{|j-j'|\leq 1} \sum_{Q_{j',k'}\subset Q^w_{j,k}} \frac{t^{q-1-q\mu}}{(1+|2^{j-j'}k'-k|)^N} \right.\right.$$
$$\left.\left. \times \int_0^{2^{-2j\beta}} \left|u^{\epsilon'}_{j',k'}(s)\right|^q s^{q(1/(2\beta)-1+\mu)+1} \frac{ds}{s}\, 2^{qj'(2\beta\mu+2\beta/q-2\beta+1)} \chi_{j,k}(x) \right]^{1/q} \right\|_p$$
$$\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \left\| \left\{ \sum_{j\geq -\log_2 r} \sum_{Q_{j,k}\subset Q_r} \sum_{|j-j'|\leq 1} 2^{q(j-j')(\gamma_1+n/2)} \left[M_A(g_{j'})(x)\right]^q \chi_{j,k}(x) \right\}^{1/q} \right\|_p$$
$$\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \left\| \left\{ \sum_{j'\geq -\log_2 r_w} |g_{j'}(x)|^q \right\}^{1/q} \right\|_p$$
$$\lesssim \|u\|_{\mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}} .$$

This completes the proof. □

Lemma 3.6. *Let* $(\beta, p, q, \gamma_1, \gamma_2, m, m')$ *satisfy* (3.1) *and* $u, v \in \mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. *Then*

$$\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,i}_{j,k} \in \mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}, \quad i = 1,2,3.$$

Proof. For $i = 1,2,3$, write

$$III_{b,i} := |Q_r|^{\gamma_2/n-1/p} \left\| \left\{ \sum_{j\geq -\log_2 r} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2+2m\beta)} \int_{2^{-2j'\beta}}^{r^{2\beta}} \left|b^{\epsilon,1}_{j,k}(t)\right|^q t^{qm} \frac{dt}{t} \chi_{j,k}(x) \right\}^{1/q} \right\|_p .$$

We divide the proof into three cases:

Case 3.6.1: Under $2 \leq q < \infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,1}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x) \in \mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}$;

Case 3.6.2: Under $2 \leq q < \infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,2}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x) \in \mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}$;
Case 3.6.3: Under $2 \leq q < \infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,3}_{j,k}(t)\Phi^{\epsilon}_{j,k}(x) \in \mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}$.
For simplicity, we only give the proof of Case 3.6.1. Cases 3.6.2 and 3.6.3 can be dealt with similarly. Now we prove Case 3.6.1. Because $s2^{2j'\beta} \leq 1$, we have

$$\left|u^{\epsilon'}_{j',k'}(s)\right| + \left|v^{\epsilon''}_{j',k''}(s)\right| \lesssim 2^{-(n/2+\gamma_1-\gamma_2)j'}$$

and

$$\left(\int_0^{2^{-1-2j'\beta}} \left|v^{\epsilon''}_{j',k''}(s)\right|^q ds\right)^{1/q} \lesssim 2^{-2j'\beta/q}2^{-j'(\gamma_1+n/2-n/p)}2^{j(\gamma_2-n/p)}.$$

Then Holder's inequality implies that

$$\begin{aligned}\left|b^{\epsilon,1}_{j,k}(t)\right| &\lesssim 2^{nj/2+j}e^{-ct2^{2j\beta}} \sum_{j<j'+2}\sum_{\epsilon',k'}\sum_{\epsilon'',k''}\int_0^{2^{-1-2j'\beta}} \frac{\left|u^{\epsilon'}_{j',k'}(s)\right|}{(1+|2^{j-j'}k'-k|)^N}\frac{\left|v^{\epsilon''}_{j',k''}(s)\right|}{(1+|k'-k''|)^N}\,ds\\ &\lesssim 2^{nj/2+j}e^{-ct2^{2j\beta}}2^{j(\gamma_2-n/p)}\sum_{w\in\mathbb{Z}^n}(1+|w|)^{-N}\sum_{j<j'+2}\sum_{\epsilon',k'}(1+|2^{j-j'}k'-k|)^{-N}\\ &\quad\times 2^{-2j'\beta/q}2^{-j'(\gamma_1+n/2-n/p)}2^{-2j'\beta(1-2/q)}\left(\int_0^{2^{-1-2j'\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q ds\right)^{1/q}.\end{aligned}$$

The above estimate implies that

$$\begin{aligned}III_{b,1} \lesssim \sum_{w\in\mathbb{Z}^n}\frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N}\Bigg\|\Bigg\{&\sum_{j\geq-\log_2 r}\sum_{Q_{j,k}\subset Q_r}2^{qj(\gamma_1+n+1)}\int_{2^{-2qj\beta}}^{r^{2\beta}}e^{-cqt2^{2j\beta}}2^{j(\gamma_2-n/p)}\\ &\times\sum_{j<j'+2}2^{\delta(j'-j)}2^{-qj'(\gamma_1+n/2-n/p)}2^{-2j'\beta(q-1)}\Bigg[\sum_{Q_{j',k'}\subset Q^w_{j,k}}(1+|2^{j-j'}k'-k|)^{-N}\\ &\times\left(\int_0^{2^{-1-2j'\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q ds\right)^{1/q}\Bigg]^q(t2^{2j\beta})^{qm}\frac{dt}{t}\,\chi_{j,k}(x)\Bigg\}^{1/q}\Bigg\|_p.\end{aligned}$$

Let

$$g_{j'} = \left[\sum_{Q_{j',k'}\subset Q^w_{j,k}}(1+|2^{j-j'}k'-k|)^{-N}\left(\int_0^{2^{-1-2j'\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q ds\right)^{1/q}\right]^q$$

and

$$f_{j'} = 2^{j'(\gamma_1+n/2)}\sum_{Q_{j',k'}\subset Q^w_r}\left(\int_0^{2^{-1-2j'\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q (s2^{2j'\beta})^{qm'}\frac{ds}{s}\right)^{1/q}.$$

Then

$$\begin{aligned}|g_{j'}| &\lesssim 2^{-2j'\beta}\left[\sum_{Q_{j',k'}\subset Q^w_{j,k}}\frac{1}{(1+|2^{j-j'}k'-k|)^N}\left(\int_0^{2^{-1-2j'\beta}}\left|u^{\epsilon'}_{j',k'}(s)\right|^q(s2^{2j'\beta})^{qm'}\frac{ds}{s}\right)^{1/q}\right]^q\\ &\lesssim 2^{-2j'\beta}2^{-qj'(\gamma_1+n/2)}2^{qn(j'-j)}(M_A(f_{j'})(x))^q.\end{aligned}$$

Hence

$$III_{b,1} \lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \left\| \left\{ \sum_{j\geq \log_2 r} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n+1+\gamma_2-n/p)} \sum_{j<j'+2} 2^{(\delta+qn)(j'-j)} \right.\right.$$
$$\left.\left. \times\ 2^{-qj'(\gamma_1+n/2-n/p+2\beta+\gamma_1+n/2)} (M_A(f_{j'})(x))^q \chi_{j,k}(x) \right\}^{1/q} \right\|_p$$
$$\lesssim \sum_{w\in\mathbb{Z}^n} \frac{|Q_r|^{\gamma_2/n-1/p}}{(1+|w|)^N} \left\| \left\{ \sum_{j'\geq -\log_2 r_w} \sum_{j<j'+2} 2^{q(j'-j)[\delta/q-(\gamma_1+\gamma_2+1-n/p)]} (M_A(f_{j'})(x))^q \right\}^{1/q} \right\|_p$$
$$\lesssim \|u\|_{\mathbb{F}^{\gamma_1,\gamma_2,III}_{p,q,m}}.$$

The proof is finished. □

Lemma 3.7. *Let* $(\beta, p, q, \gamma_1, \gamma_2, m, m')$ *satisfy* (3.1) *and* $u, v \in \mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'}$. *Then*

$$\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,i}_{j,k}(t)\Phi^\epsilon_{j,k}(x) \in \mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}, \quad i = 4, 5.$$

Proof. For $i = 4, 5$, write

$$II_{b,i} := |Q_r|^{\gamma_2/n-1/p} \left\| \left\{ \sum_{j\geq -\log_2 r} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n/2)} \int_0^{2^{-2j'\beta}} \left| b^{\epsilon,i}_{j,k}(t) \right|^q (t2^{2j'\beta})^{qm'} \frac{dt}{t} \chi_{j,k}(x) \right\}^{1/q} \right\|_p.$$

We still divide the proof into two cases:
Case 3.7.1: Under $2 \leq q < \infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,4}_{j,k}(t)\Phi^\epsilon_{j,k}(x) \in \mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}$;
Case 3.7.2: Under $2 \leq q < \infty$, $\sum_{(\epsilon,j,k)\in\Lambda_n} b^{\epsilon,5}_{j,k}(t)\Phi^\epsilon_{j,k}(x) \in \mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}$.
We only prove Case 3.7.1. The treatment for Case 3.7.2 is similar. We have

$$\left| b^{\epsilon,4}_{j,k}(t) \right| \lesssim 2^{nj/2+j} 2^{j(\gamma_2-n/p)} \sum_{j<j'+2} 2^{-2j'\beta(1-2/q)} 2^{-j'(\gamma_1+n/2-n/p)} 2^{-4j'\beta/q}$$
$$\times \sum_{\epsilon',k'} \frac{1}{(1+|2^{j-j'}k'-k|)^N} \left(\int_0^{2^{-2j'\beta}} \left| u^{\epsilon'}_{j',k'}(s) \right|^q (s2^{2j'\beta})^{qm'} \frac{ds}{s} \right)^{1/q}.$$

Then by $\int_0^{2^{-2j'\beta}} (t2^{2j'\beta})^{qm'} \frac{dt}{t} \lesssim 1$, we apply Lemma 2.10 to derive that

$$II_{b,4} \lesssim |Q_r|^{\gamma_2/n-1/p} \sum_{w\in\mathbb{Z}^n} (1+|w|)^{-N} \left\| \left\{ \sum_{j\geq -\log_2 r} \sum_{Q_{j,k}\subset Q_r} 2^{qj(\gamma_1+n+1+\gamma_2-n/p)} \right.\right.$$
$$\times \sum_{j<j'+2} 2^{\delta(j'-j)} 2^{-2qj'\beta(1-2/q)} 2^{-qj'(\gamma_1+n/2-n/p)} 2^{-4j'\beta}$$
$$\left.\left. \times \left[\sum_{Q_{j',k'}\subset Q^w_r} \frac{1}{(1+|2^{j-j'}k'-k|)^N} \left(\int_0^{2^{-2j'\beta}} \left| u^{\epsilon'}_{j',k'}(s) \right|^q (s2^{2j'\beta})^{qm'} \frac{ds}{s} \right)^{1/q} \right]^q \chi_{j,k}(x) \right\}^{1/q} \right\|_p$$
$$\lesssim \|u\|_{\mathbb{F}^{\gamma_1,\gamma_2,IV}_{p,q,m'}}.$$

This completes the proof of Lemma 3.7. □

4. Well-posedness of magneto-hydrodynamic equations with data in $\dot{F}^{\gamma_1,\gamma_2}_{p,q}$

4.1. Generalized magneto-hydrodynamic equations

In this section, we consider the incompressible fractional magneto-hydrodynamic $(\text{FMHD})_\beta$ equations in $\mathbb{R}^{n+1}_+$, $n \geq 2$:

$$\begin{cases} \partial_t u + (-\triangle)^\beta u + u \cdot \nabla u + \nabla p - b \cdot \nabla b = 0, \\ \partial_t b + (-\triangle)^\beta b + u \cdot \nabla b - b \cdot \nabla u = 0, \\ u|_{t=0} = u_0, \ b|_{t=0} = b_0 \end{cases} \tag{4.1}$$

with $\nabla \cdot u = \nabla \cdot b = 0$ and $\beta \in (1/2, 1)$. Here $(-\triangle)^\beta$ is the fractional Laplacian with respect to x defined by

$$\widehat{(-\triangle)^\beta} u(t,\xi) = |\xi|^{2\beta}\, \widehat{u}(t,\xi).$$

In equations (4.1), $u(t,x) = (u_1(t,x), \dots, u_n(t,x))$ and $b(t,x) = (b_1(t,x), \dots, b_n(t,x))$ are the flow velocity and the magnetic field at point (x,t), respectively. $u_0(x)$ and $b_0(x)$ are the initial velocity and magnetic field and satisfy $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$, respectively.

Theorem 4.1. *Let*

$$\begin{cases} 1 < p, q < \infty, \\ \beta > 1/2, \ \gamma_1 = \gamma_2 - 2\beta + 1, \\ m > \max\{p, n/(2\beta)\}, \ 0 < m' < \min\{1, p/(2\beta)\}. \end{cases} \tag{4.2}$$

If the index (β, p, γ_2) obeys

$$1 < q \leq 2 \quad \& \quad 2\beta - \frac{2}{q} + \frac{n}{p}\left(1 - \frac{1}{q}\right) < \gamma_2 \leq \frac{n}{p}$$

or

$$2 < q < \infty \quad \& \quad \beta - 1 + \frac{n}{2p} < \gamma_2 \leq \frac{n}{p},$$

then (4.1) has a unique global mild solution in $(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n$ for any initial data (u_0, b_0) with $\|(u_0, b_0)\|_{(\dot{F}^{\gamma_1,\gamma_2}_{p,q})^n}$ being small.

Proof. By Lemmas 3.2–3.7, we have known that the bilinear operator

$$B(u,v) = \int_0^t e^{-(t-s)(-\Delta)^\beta} P \nabla \cdot (u \otimes v)\, ds \tag{4.3}$$

is bounded from $(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n \times (\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n$ to $(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n$.

The solution (u, b) to equations (4.1) can be written as

$$u(t,x) = e^{-t(-\triangle)^\beta} u_0(x) - B(u,u) + B(b,b) := F_1(u,b)$$

and

$$b(t,x) = e^{-(-\triangle)^{\beta}} b_0(x) - B(u,b) + B(b,u) := F_2(u,b).$$

We rewrite the solution (u,b) as

$$\begin{pmatrix} u \\ b \end{pmatrix} = \begin{pmatrix} F_1(u,b) \\ F_2(u,b) \end{pmatrix} := F(u,b).$$

If (4.3) holds, we have

$$\|F_1(u,b)\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \leq C_1 \|u_0\|_{(\dot{F}^{\gamma_1,\gamma_2}_{p,q})^n} + C_2 \|(u,b)\|^2_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n}$$

and

$$\begin{aligned}
&\left\|F_1(u,b)(t) - F_1(u',b')\right\|_{(X^{\gamma_1,\gamma_2,\tau}_{p,m,m'})^n} \\
&\lesssim \left\|B(u,u) - B(u',u')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} + \left\|B(b,b) - B(b',b')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \\
&\lesssim \left(\|u\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} + \left\|u'\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n}\right) \left\|u - u'\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \\
&\quad + \left(\|b\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} + \left\|b'\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n}\right) \left\|b - b'\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \\
&\lesssim \left\|(u,b) - (u',b')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \left(\|(u,b)\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} + \left\|(u',b')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n}\right).
\end{aligned}$$

Similarly, it can be obtained that

$$\|F_2(u,b)\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \leq C_1 \|b_0\|_{(\dot{F}^{\gamma_1,\gamma_2}_{p,q})^n} + C_2 \|(u,b)\|^2_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n}$$

and

$$\begin{aligned}
&\left\|F_2(u,b)(t) - F_2(u',b')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \\
&\lesssim \left\|(u,b) - (u',b')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \left(\|(u,b)\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} + \left\|(u',b')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n}\right).
\end{aligned}$$

Thus, we have

$$\|F(u,b)\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \leq C \|(u_0,b_0)\|_{(\dot{B}^{\gamma_1,\gamma_2}_{p,p})^n} + C \|(u,b)\|^2_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n}$$

and

$$\begin{aligned}
&\left\|F(u,b) - F(u',b')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \\
&\lesssim \left\|(u,b) - (u',b')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \left(\|(u,b)\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} + \left\|(u',b')\right\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n}\right).
\end{aligned}$$

Let R be a constant satisfying $R \leq 2C \|(u_0,b_0)\|_{(B^{\gamma_1,\gamma_2}_{p,p})^n}$. Then F is a contraction mapping from

$$E = \left\{(u,b) \in (\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n : \|(u,b)\|_{(\mathbb{F}^{\gamma_1,\gamma_2}_{p,q,m,m'})^n} \leq R\right\}$$

into itself. Thus, there exists a fixed point (u,b) of the operator F. $\square$

4.2. Quasi-geostrophic equations

In this section, we study the well-posedness and regularity of quasi-geostrophic equation

$$\begin{cases} \partial_t\theta = -(-\Delta)^{\beta}\theta + \partial_1(\theta R_2\theta) - \partial_2(\theta R_1\theta), \\ \theta(0,x) = \theta_0(x), \end{cases} \tag{4.4}$$

with initial data in the space $\dot{B}^{\gamma_1,\gamma_2}_{p,p}$, where $\beta \in (1/2, 1)$.

The equations $(\mathrm{DQG})_\beta$ are important models in the atmosphere and ocean fluid dynamics. It was proposed by P. Constantin and A. Majda, etc. that the equations $(\mathrm{DQG})_\beta$ can be regarded as low-dimensional models for mathematical study of singularity in smooth solutions of unforced incompressible three-dimensional fluid equations. See e.g. [12, 21, 23, 44, 45] and the references therein.

Owing to the importance in mathematical and geophysical fluid dynamics mentioned above, the equations $(\mathrm{DQG})_\beta$ have been intensively studied. Many important progress has been made. We refer the reader to [5, 6, 8, 10, 11, 13, 22, 37, 55, 56] for details.

The solution to equations (4.4) can be represented as

$$u(t,\cdot) = e^{-t(-\Delta)^{\beta}}u_0 + B(u,u),$$

where the bilinear form $B(u,v)$ is defined by

$$B(u,v) = \int_0^t e^{-(t-s)(-\Delta)^{\beta}}(\partial_1(vR_2u) - \partial_2(vR_1u))\,ds.$$

By the space $\mathbb{F}^{\gamma_1,\gamma_2,\tau}_{p,m,m'}$ introduced in Section 3, we consider the well-posedness and regularity of the quasi-geostrophic equations (4.4).

Theorem 4.2. *Given $1/2 < \beta < 1$, $\gamma_1 = \gamma_2 - 2\beta + 1$. If the index $(p, \gamma_1, \gamma_2, m, m')$ satisfies the conditions (4.2), the quasi-geostrophic dissipative equation (4.4) has a unique global mild solution in $(\mathbb{F}^{\gamma_1,\gamma_2}_{p,m,m'})^2$ for all small initial data $a(x)$ with $\nabla \cdot a = 0$ and $\|a\|_{(\dot{F}^{\gamma_1,\gamma_2}_{p,p})^2}$ small enough.*

Proof. By Picard's contraction principle, it suffices to verify that the bilinear operator

$$B(u,v) = \int_0^t e^{-(t-s)(-\Delta)^{\beta}}(\partial_1(vR_2u) - \partial_2(vR_1u))\,ds$$

is bounded from $(\mathbb{F}^{\gamma_1,\gamma_2}_{p,m,m'})^2 \times (\mathbb{F}^{\gamma_1,\gamma_2}_{p,m,m'})^2$ to $(\mathbb{F}^{\gamma_1,\gamma_2}_{p,m,m'})^2$. The proof is similar to that of Theorem 4.1. We omit the details. □

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