TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 4, pp. 769–785, August 2016
DOI: 10.11650/tjm.20.2016.7135
This paper is available online at http://journal.tms.org.tw

Counting Lines on Quartic Surfaces

Víctor González-Alonso* and Sławomir Rams

Abstract. We prove the sharp bound of at most 64 lines on projective quartic surfaces $S \subset \mathbb{P}^3(\mathbb{C})$ (resp. affine quartics $S \subset \mathbb{C}^3$) that are not ruled by lines. We study configurations of lines on certain non-$K3$ surfaces of degree four and give various examples of singular quartics with many lines.

1. Introduction

The main aim of this note is to study the configurations of lines on projective and affine complex quartic surfaces. More precisely, we prove the following theorem.

Theorem 4.5. (a) *Let $S \subset \mathbb{P}^3(\mathbb{C})$ be a quartic surface with at most isolated singularities, none of which is a quadruple point. Then S contains at most* 64 *lines.*

(b) *Let $S \subset \mathbb{C}^3$ be an affine quartic surface that is not ruled by lines. Then S contains at most* 64 *lines.*

It should be pointed out that the classification of projective complex quartic surfaces that are ruled by lines (i.e., contain infinitely many lines) has been already known to K. Rohn (see [13] for a correct classification in modern language), whereas a smooth projective complex quartic surface with exactly 64 lines has been found by F. Schur in [18].

Configurations of lines on surfaces in $\mathbb{P}^3(\mathbb{C})$ have been already studied in the middle of the nineteenth century. After Cayley, Salmon, Clebsch and others were able to give a complete picture of configurations of lines on cubic surfaces, various classes of degree-4 surfaces in $\mathbb{P}^3(\mathbb{C})$ were studied (see e.g. [2,5,13,16] and the bibliography in the last paper). An attempt to prove a sharp bound on the number of lines on smooth quartic surfaces in $\mathbb{P}^3(\mathbb{C})$ was first made by Segre [19]. Unfortunately, his proof contains serious gaps despite its many ingenious ideas. These gaps were filled in by [14]. The proof of [14] heavily depends on the geometry of $K3$ surfaces and can be generalized to the case of all complex quartic $K3$ surfaces [20] thus yielding the general bound of at most 64 lines in the case of complex quartic surfaces with at most ADE singularities.

Received December 17, 2015; Accepted March 7, 2016.
Communicated by Yoichi Miyaoka.
2010 *Mathematics Subject Classification*. Primary: 14J25, 14N25; Secondary: 14N20, 14J70.
Key words and phrases. Line, Quartic surface, Non-ADE singularity.
*Corresponding author.

In view of the results of [14, 20] there are two natural strategies to prove Theorem 4.5: either the study of the codimension of the various components of the appropriate incidence variety to prove that the degree of the covering map is attained along a component of the Noether-Lefschetz locus that contains smooth quartics (resp. quartics with at most ADE points), or an analysis of configurations of lines on non-$K3$ quartic surfaces. We follow the latter, because it sheds some light on the geometry of line configurations on non-$K3$ quartics. In particular, we show the following proposition.

Proposition 4.4. *Let $S \subset \mathbb{P}^3(\mathbb{C})$ be a non-$K3$ quartic surface. If S is not ruled by lines, then S contains at most 48 lines.*

The classification of degree-four surfaces having at least one nonsimple point has been achieved in [3], where the existence of 2523 various constellations of singularities on such surfaces is shown. Therefore, an attempt to investigate the configurations of lines via lattice-theory techniques and study of divisors on their smooth models seems pretty futile. Instead, we apply the fact that after a coordinate change every quartic with a nonsimple double point P that contains a line belongs to one of the three families (Q4), (Q5), (Q6) (see [21, Theorem 8.1]), with the line lying on a fixed coordinate plane (Lemma 2.3). The latter condition turns out to be very useful for finite-field tests and search for examples.

The organization of the paper is as follows. After recalling some well-known facts on non-$K3$ quartic surfaces with isolated singularities in Section 2, we study the configurations of lines on some of them in Section 3 and give certain examples of non-$K3$ quartics with many lines. Finally in Section 4 we collect classical results on quartics with non-isolated singularities, present the proofs of Proposition 4.4, Theorem 4.5 and give an explicit example of a quartic surface with isolated singularities that contains 39 lines (Example 4.8). This is the best explicit example of a singular quartic with many lines known so far. The families (Q4), (Q5), (Q6) were defined in the unpublished note [21]. Since the author of [21] does not consider the lines away from the non-simple singularity, we sketch a proof of the version of [21, Theorem 8.1] we need.

Finally, let us mention that, apart from being an interesting subject on their own [1, 4, 12, 14, 20], configurations of rational curves on surfaces play an important role in current researches (see e.g. [6, 8, 10]).

Conventions: In this note we work over the field $\mathbb{C}$ of complex numbers. All quartic surfaces we consider are assumed not to be cones. By abuse of notation, whenever it leads to no ambiguity, we use the same symbol to denote a homogeneous polynomial and the set of its zeroes.

2. Non-$K3$ quartics with isolated singularities

In this section we recall some properties of quartic surfaces in $\mathbb{P}^3(\mathbb{C})$ with isolated singularities, which we will use in the sequel.

The quartic surfaces with an isolated triple point were studied intensively in [16]. Let $S \subset \mathbb{P}^3(\mathbb{C})$ be an irreducible quartic with the isolated triple point $O := (0 : 0 : 0 : 1)$. Obviously, such a surface is given by an equation of the form

$$wQ_3(x, y, z) + Q_4(x, y, z) = 0,$$

where Q_3 and Q_4 are homogeneous forms of degree 3 and 4 without common factors. Thus, the quartic S does not contain more than 12 lines that pass through O. Indeed, if a line $\ell \ni O$ on S is defined by two linear forms $L_1(x, y, z) = L_2(x, y, z) = 0$, then $Q_3, Q_4 \in (L_1, L_2)\mathbb{C}[x, y, z]$, so that ℓ is necessarily contained in the complete intersection $Q_3 = Q_4 = 0$ of degree 12.

In the meantime, if a line $\ell' \subset S$ does not pass through O, then the plane spanned by ℓ' and O cuts out of S the line ℓ' plus a plane cubic with a triple point at O or, equivalently, a union of three lines ℓ_1, ℓ_2, ℓ_3 passing through O. Thus, noting that the linear projection π from $S \setminus O$ onto the plane $P = \{w = 0\}$ maps each line $\ell_i \ni O$ to a point p_i on P, we have a mapping from the set of lines ℓ' off O on S to the set of collinear triplets chosen out of the 12 or less points on P. This correspondence reduces the enumeration of ℓ' off O to a special case of the so-called *orchard planting problem*: given n points on a plane, find the maximum number of collinear triplets of these prescribed points. Rohn claims that the number is 19 for $n = 12$, and his claim is easily verified by a direct computer check. Thus we have the following

Lemma 2.1. [16, p. 59] *If S is an irreducible quartic with an isolated triple point, then S contains at most* 31 *lines.*

The example below (cf. [16]) shows that Rohn's bound is sharp.

Example 2.2. The surface given by

$$wQ_3 + Q_4 = w\left((x + y + z)^3 + xyz\right) + (x + y + z)(x - y)(y - z)(z - x) = 0$$

is singular only at the triple point $O = (0 : 0 : 0 : 1)$ and contains 31 lines. The twelve lines through O intersect the plane $\{w = 0\}$ at the points

$$\begin{aligned}
&p_1 = (0 : 1 : -1), && p_5 = (2 : 2 : -5 + \sqrt{7}), && p_9 = (2 : -5 - \sqrt{7} : 2),\\
&p_2 = (1 : 0 : -1), && p_6 = (2 : 2 : -5 - \sqrt{7}), && p_{10} = (-1 : 1 : 1),\\
&p_3 = (1 : -1 : 0), && p_7 = (1 : -1 : 1), && p_{11} = (-5 + \sqrt{7} : 2 : 2),\\
&p_4 = (1 : 1 : -1), && p_8 = (2 : -5 + \sqrt{7} : 2), && p_{12} = (-5 - \sqrt{7} : 2 : 2).
\end{aligned} \tag{2.1}$$

An elementary computation shows that there are exactly 19 triplets of collinear points among (2.1), so there are precisely 19 lines in $S \setminus \{O\}$.

Normal quartic surfaces have been extensively studied by several authors. For example, Degtyarev [3] asserts that there are 2523 possible constellations of their singularities. In what follows, we use the following result of Wall:

Lemma 2.3. [21, Theorem 8.1] *Let $(x : y : z : w)$ be linear coordinates of $\mathbb{P}^3(\mathbb{C})$. Let $S \subset \mathbb{P}^3(\mathbb{C})$ be a normal quartic surface with a singularity at $O = \{x = y = z = 0\}$. Assume that O is of multiplicity two but non-rational. Then:*

(1) *The tangent cone $C_O S$ of S at O is a double plane, which is given by $z^2 = 0$ after a suitable coordinate change.*

(2) *After a linear coordinate change, the defining equation of S is given by either*

$$w^2z^2 + wzQ_2 + H_4 = 0 \tag{2.2}$$

or

$$w^2z^2 + w(y^3 + zQ_2) + H_4 = 0 \tag{2.3}$$

where $Q_2 = \sum_{i+j=2} q_{ij}x^iy^j$ and $H_4 = \sum_{i+j+k=4} h_{ijk}x^iy^jz^k$ are quadratic and quartic polynomials of x, y and x, y, z respectively.

(3) *Assume that S is defined by (2.2). Then $C_O S \cap S$, set-theoretically given by $z = H_4 = 0$, is a union of four or less lines through O. Any line on S passing through O is one of the components of $C_O S \cap S$. In particular, if a line ℓ' on S is away from O, then ℓ' meets exactly one of these lines through O.*

(4) *Assume that S is defined by (2.3), then one of the following holds:*

$$h_{400} = h_{310} = q_{20} = 0 \quad \text{and} \quad h_{301} \neq 0, \tag{2.4}$$

$$h_{400} = \frac{1}{4}q_{20}^2, \quad h_{310} = \frac{1}{2}q_{20}q_{11}, \quad q_{20} \neq 0 \quad \text{and} \quad h_{301} = 0. \tag{2.5}$$

Moreover, if $\ell \subset S$ is a line that is not contained in the tangent cone $C_O S$, then the coordinate change that leads to (2.3) can be chosen in such a way that ℓ is contained in the plane $\{x = 0\}$.

Proof. Since O is not of A_n type, its tangent cone $C_O S$ must be a double plane. We can hence suppose $O = (0 : 0 : 0 : 1)$ and $C_O S = \{z^2 = 0\}$. In these coordinates, the equation of S must be of the form

$$w^2z^2 + wG_3(x, y, z) + G_4(x, y, z).$$

Write now $G_3(x,y,z) = P_3(x,y) + zQ_2(x,y) + 2z^2L_1(x,y,z)$, where the subindices indicate the degree of the homogeneous polynomials. Replacing the coordinate w by $\widetilde{w} = w + L_1$, the equation becomes

$$\widetilde{w}^2z^2 + \widetilde{w}(P_3 + zQ_2) + (G_4 - L_1P_3 - zL_1Q_2 - z^2L_1^2).$$

If $P_3 \equiv 0$ (i.e., z divides G_3), then we already have an equation of type (2.2). If otherwise P_3 does not vanish identically, we claim that it must be the cube of a linear factor because O is not of D_n type. Hence after a linear change of coordinates involving only (x, y) we may assume $P_3 = y^3$, obtaining an equation of type (2.3). Indeed, in local analytic coordinates around O, S has equation

$$z^2 + (P_3(x,y) + zQ_2(x,y)) + H_4(x,y,z) = u(x,y,z)(z^2 + zA(x,y) + B(x,y))$$

for some power series A, B, u with $u(0,0,0) = 1$, showing that S is locally around O the double cover of (an open set of) the plane, branched along the curve $\Delta : A^2 - 4B = 0$. Therefore O will be of ADE type if and only if Δ has a simple singularity. More precisely, O will be of type D_n if Δ has a triple point with two or three different tangents.

Analyzing the jets of A and B we obtain (below h.o.t. stands for "higher order terms")

$$A = Q_2 + \eta_3(x,y) + \text{h.o.t.} \quad \text{and} \quad B = P_3 + \eta_4(x,y) - P_3\eta_2(x,y) + \text{h.o.t.},$$

where $\eta_i(x,y)$ is the coefficient of z^{4-i} in H_4. Hence $A^2 - 4B = -4P_3(x,y) + \text{h.o.t.}$, which shows that O is of type D_n if P_3 is not the cube of a linear form. Assuming from now on that $P_3 = y^3$ and expanding $A^2 - 4B$ a bit further, it can be shown that Δ has a non-simple singularity if and only if

$$q_{20}^2 - 4h_{400} = 2q_{20}q_{11} - 4h_{310} = q_{20}h_{301} = 0. \tag{2.6}$$

If both $q_{20} = h_{301} = 0$ then O is not an isolated singularity, hence only one of the two coefficients vanishes. Clearly, if $q_{20} = 0$ (resp. $h_{301} = 0$) these conditions reduce to (2.4) (resp. (2.5)).

As for the last claim, suppose that S is given by either (2.2) or (2.3), and let $\pi = \{ax + by + cz = 0\}$ be the plane spanned by ℓ and O. It can be checked that no plane of the form $\{y = \lambda z\}$ cuts out a line on S, hence we can assume $a = 1$, i.e., π admits an equation of the form $\widetilde{x} := x + by + cz = 0$. Replacing x by $\widetilde{x}$ introduces higher powers of z in the term linear in w, which can be removed with a linear change of w as at the beginning of the proof. This will change Q_2 and H_4, but the new equation will remain of type (2.2) (resp. (2.3) satisfying the conditions (2.6)). □

We introduce the following notation.

Definition 2.4. We say that the quartic S belongs to the family (Q4) (resp. (Q5), resp. (Q6)) if and only if after a coordinate change it is given by (2.2) (resp. (2.3), (2.4), resp. (2.3), (2.5)) and the point $O = (0:0:0:1)$ is an isolated singularity of S.

If the quartic S belongs to one of the above families, then the (reduced) tangent cone C_OS is the plane $\{z = 0\}$. Obviously, a line $\ell \subset S$ that runs through the double point O is a component of the intersection $C_OS \cap S$. This results in the following simple observation.

Lemma 2.5. (a) *Every quartic S in the family* (Q4) *contains at most four lines through the singularity O, which form a hyperplane section of S.*

(b) *If a surface S belongs to* (Q5)*, then exactly one line ℓ_0 on S contains the singularity O. Moreover the hyperplane section $C_OS \cap S$ consists of either two lines or the line ℓ_0 and an irreducible conic. If the former holds the line ℓ_0 meets exactly one other line on S. In the latter case, the line ℓ_0 intersects no other lines on the quartic S.*

(c) *No quartic S in the family* (Q6) *contains any line through O.*

Proof. (a) The intersection $C_OS \cap S$ is given by the vanishing of $H_4(x, y, 0)$, so S contains at most 4 lines that pass through O.

(b) Put $\ell_0 := \{y = z = 0\}$. An elementary computation (i.e., substitution in (2.4) and its partial derivatives) shows that $\ell_0 \subset S$, the double point O is the only singularity of S on the line ℓ_0 and we have $T_PS = C_OS = \{z = 0\}$ for all points $P \in \ell_0$, $P \neq O$.

Moreover, if h_{220} vanishes, then we have $S.(C_OS) = 3\ell_0 + \ell_1$, where $\ell_1 := \{h_{130}\, x + h_{040}\, y + w = 0\}$. Since every line on S that runs through a point $P \in \ell_0$, $P \neq O$ lies on the tangent plane $T_PS = \{z = 0\}$, the proof (for this case) is complete.

Suppose that h_{220} does not vanish, so we can assume $h_{220} = 1$. Then, one can easily see that the hyperplane section $S.(C_OS)$ consists of the double line $2\ell_0$ and an irreducible conic C. Indeed, the latter can be parametrized (away from the double point O) by the map

$$\mathbb{C} \ni t \mapsto P(t) := (t : 1 : 0 : -t^2 - h_{130}t - h_{040}) \in S. \tag{2.7}$$

Again, any line on S through a point $P \in \ell_0$, $P \neq O$ must lie on C_OS, so there are no such lines in this case.

(c) One can easily see that the hyperplane section $S.(C_OS)$ is a (cuspidal) rational quartic curve, hence it contains no lines. □

Recall that, according to [21] (see also [3]) (minimal smooth models of) the surfaces in the families (Q4), (Q5) are either rational or elliptic ruled.

Suppose that a quartic S belongs to the family (Q6). Obviously, such a surface meets the hyperplane $\{y = tz\}$ along a quartic curve with the singular point $O = (0:0:0:1)$.

Moreover, an elementary computation (see [21, p. 30]) shows that the conditions (2.5) imply that for generic parameter t the projection from the point O endows the resulting quartic curve with the structure of a double cover of $\mathbb{P}^1(\mathbb{C})$ branched at exactly two points. By Riemann-Hurwitz such a curve is rational, which yields the following lemma that we will need in the sequel.

Lemma 2.6. [21, p. 30] *If the quartic S belongs to the family* (Q6)*, then S is a rational surface.*

3. Counting lines intersecting with a prescribed curve on S

In this section *we assume that the quartic $S \subset \mathbb{P}^3(\mathbb{C})$ we consider contains finitely many lines.* By [14, Proposition 1.1], (resp. [15, Theorem 1.1]) a line (resp. a conic) on a smooth quartic $S \subset \mathbb{P}^3(\mathbb{C})$ meets at most 20 other lines (resp. 48 lines) on the surface in question. Moreover, both bounds are sharp (see [14, 15]). In this section we examine the analogous problem for certain lines and conics on some non-$K3$ quartics.

Lemma 3.1. *Let S belong to the family* (Q4) *and let ℓ_0 be a line in $S \cap (C_O S)$. Then ℓ_0 does not intersect with more than seven other lines on S.*

Proof. By Lemma 2.3 we can consider the quartic S that is given by (2.2) with the non-rational double point $O = (0 : 0 : 0 : 1)$. Moreover, after a linear change of coordinates involving only x and y (hence not changing the shape of the equation (2.2)), we can assume that the line $\ell_0 = \{x = z = 0\}$ lies on S (i.e., we have $h_{040} = 0$ in (2.2)). We are to show that the line ℓ_0 meets at most four lines on the quartic S that do not contain the singularity O.

Consider the pencil $|\mathcal{O}_S(1) - \ell_0|$, i.e., the cubics C_λ residual to ℓ_0 in the intersection of S with the plane $\Pi_\lambda := \{x = \lambda z\}$, where $\lambda \in \mathbb{C}$. By direct computation, the intersection of C_λ with ℓ_0 is given by

$$y^2(q_{02}w + (h_{031} + \lambda h_{130})y). \tag{3.1}$$

Suppose $q_{02} = 0$. By direct computation, if h_{031}, h_{130} vanish, then $\ell_0 \subset \mathrm{sing}(S)$, which is excluded by our assumptions. Moreover, for $h_{130} = 0$, the equation (3.1) yields that all lines on S that meet ℓ_0 run through the double point O. If otherwise $h_{130} \neq 0$, we infer from (3.1) that at most one cubic C_λ can contain lines on S that meet ℓ_0 away from O.

Thus we have shown Lemma 3.1 when q_{02} vanishes and we can now assume $q_{02} = 1$. By (3.1) we have

$$C_\lambda.\ell_0 = 2O + P_\lambda.$$

Therefore, the cubic C_λ is smooth at the point $P_\lambda = (0 : 1 : 0 : -h_{031} - \lambda h_{130})$ and the tangent line $T_{P_\lambda} C_\lambda$ can be the only line in the hyperplane section $S \cap \Pi_\lambda$ that does not contain the double point O. For the given parameter λ the line in question can be parametrized as $(y:z) \mapsto (\lambda z : y : z : L_\lambda(y,z))$. Substituting the above parametrization into the equation of C_λ we arrive at a polynomial of the form

$$z^2(\mathfrak{b}_3(\lambda)y + \mathfrak{b}_4(\lambda)z), \tag{3.2}$$

where $\deg(\mathfrak{b}_j) = j$. By definition $T_{P_{\lambda_0}}C_{\lambda_0} \subset S$ if and only if λ_0 is a common zero of $\mathfrak{b}_3$, $\mathfrak{b}_4$. Since S contains finitely many lines, the polynomials $\mathfrak{b}_3$ and $\mathfrak{b}_4$ cannot simultaneously vanish (identically). Therefore the number of lines in question is bounded by $\deg(\mathfrak{b}_4) = 4$ and the proof is complete. □

Lemma 3.1 yields a bound on the number of lines on quartics in the family (Q4).

Proposition 3.2. *If S belongs to the family* (Q4)*, then it contains at most* 20 *lines.*

Proof. By Lemmas 2.5(a) and 3.1 the hyperplane section $S \cap \{z = 0\}$ consists of at most four lines, each of which meets at most four lines on S that do not contain O. □

The proof of Proposition 3.2 gives a direct way of finding all surfaces S in (Q4) that contain exactly 20 lines. For such a quartic, the hyperplane section $S \cap C_O S$ must consist of exactly four lines. For each of the lines the polynomial $\mathfrak{b}_4$ has four different zeroes and $\mathfrak{b}_3 \equiv 0$. The latter condition for a given line in the hyperplane section results in a system of (huge) equations involving the coefficients of the quartic. In this way one obtains the following simple example which shows that the bound of Proposition 3.2 is sharp.

Example 3.3. Let S be the quartic surface given by

$$\begin{aligned} w^2z^2 + wz(x+y)^2 + x^3y + xy^3 + x^3z - 3x^2z^2 + 4xz^3 \\ + 3x^2yz - 6xyz^2 + 3xy^2z + 4yz^3 - 3y^2z^2 + y^3z. \end{aligned}$$

Obviously, S belongs to the family (Q4). Moreover, a standard Gröbner basis computation shows that $(0:0:0:1)$ is the unique singularity of S. We claim that S contains exactly 20 lines.

Indeed, the hyperplane section $S \cap \{z = 0\}$ consists of the four lines

$$x = 0,\ y = 0,\ x = (\pm i)y.$$

For the first line (i.e., $x = 0$), one can easily imitate the proof of Lemma 3.1 and check that $\mathfrak{b}_3 \equiv 0$ whereas $\mathfrak{b}_4$ has four different roots (cf. (3.2)). This yields that the line in question meets exactly seven other lines on S. A similar reasoning shows that the line $x = iy$ intersects precisely seven other lines on S. Since the quartic S is invariant under the automorphism $(x:y:z:w) \mapsto (y:x:z:w)$, the claim follows.

In the following remark we collect certain facts from [15] that we will use in the sequel.

Remark 3.4. Recall that we assumed the quartic surface S not to be ruled by lines. Given a smooth point $P \in S$ and a line $\ell \subset X$ that runs through the point P, it is clear that ℓ is contained both in the tangent space $T_P S$ and in the quadric V_P defined by the Hessian matrix of (the equation) S at P. Moreover, if $T_P S$ is no component of V_P (e.g. when $\mathrm{rk}(V_P) \geq 3$), then the intersection $T_P \cap V_P$ consists of (at most) two lines (the so-called *principal lines*). Suppose that the point P varies along a rational curve C that is parametrized by the map $t \mapsto P(t)$ and put

$$Z := Z_C := \overline{\bigcup_t (T_{P(t)} \cap V_{P(t)})}. \tag{3.3}$$

By definition the variety Z contains all the lines on S that meet the rational curve in question. Moreover, Z is a surface that intersects S properly (indeed, by construction each component of Z is ruled), when the condition

$$T_P \not\subset V_P \quad \text{for general } P \in C \tag{3.4}$$

is satisfied. In this case, the number of lines on S that meet the rational curve is bounded by $4\deg(Z)$.

Obviously, as t varies, $T_{P(t)}$ and $V_{P(t)}$ are given by the equations

$$\mathfrak{l}_t(x_1, \ldots, x_4) = \sum_{i=1}^{4} \frac{\partial S}{\partial x_i}(P(t))x_i \quad \text{and} \quad \mathfrak{q}_t(x_1, \ldots, x_4) = \sum_{i=j=1}^{4} \frac{\partial^2 S}{\partial x_i \partial x_j}(P(t))x_i x_j, \tag{3.5}$$

and the resultant $\mathfrak{R}_t := \mathrm{Res}_t(\mathfrak{l}_t, \mathfrak{q}_t)$ of (3.5) with respect to the parameter t belongs to the ideal $\mathcal{I}(Z)$. Computing $\mathfrak{R}_t$ by means of the Sylvester matrix leads to

$$\deg(\mathfrak{R}_t) \leq \deg_t(\mathfrak{q}_t)\deg_x(\mathfrak{l}_t) + \deg_t(\mathfrak{l}_t)\deg_x(\mathfrak{q}_t) = \deg_t(\mathfrak{q}_t) + 2\deg_t(\mathfrak{l}_t). \tag{3.6}$$

After those preparations we can examine lines on quartics in the family (Q5).

Lemma 3.5. *Let S be given by* (2.3), (2.4), *and let $\ell_0 \subset S$ be the unique line through O.*

(a) *If $S.(C_O S) = 2\ell_0 + C$, then the irreducible conic C meets at most* 27 *lines on S.*

(b) *Suppose $S.(C_O S) = 3\ell_0 + \ell_1$. Then the line ℓ_1 intersects at most* 23 *lines on S.*

Proof. (a) Keeping the notation of Lemma 2.5(b) and its proof, we may assume $h_{220} = 1$. In order to count the lines that meet the conic C away from the double point O we consider the variety Z_C (see (3.3)).

One can easily see that S is smooth along the image of the parametrization (2.7). We substitute (2.7) into the Hessian of S and check that the condition (3.4) is fulfilled. One can also see that $\deg_t(\mathfrak{l}_t) \leq 3$ and $\deg_t(\mathfrak{q}_t) = 4$, so (3.6) yields the inequality $\deg(\mathfrak{R}_t) \leq 10$.

By direct computation, the resultant $\mathfrak{R}_t$ is divisible by z^2 and all partial derivatives of $(\mathfrak{R}_t/z^2)$ vanish along the line ℓ_0 and the conic C. Thus, both rational curves appear in the intersection cycle

$$S.\,\mathrm{V}(\mathfrak{R}_t/z^2) \tag{3.7}$$

with multiplicity at least two. Moreover, since ℓ_0 is the only line on $S \cap \mathrm{V}(z)$ (see Lemma 2.5(b)), we have $Z_C \subset \mathrm{V}(\mathfrak{R}_t/z^2)$ and the number of lines on S that meet C is bounded by $27 = 32 - 4 - 1$.

(b) Recall that $h_{220} = 0$ in this case (see the proof of Lemma 2.5(b)). We parametrize the line ℓ_1 away from the point $(1 : 0 : 0 : -h_{130}) \in \ell_0$ and put $P(t) := (t : 1 : 0 : -h_{130}t - h_{040})$ for $t \in \mathbb{C}$ to study the variety Z_{ℓ_1} (see (3.3)).

A direct computation shows that the quartic S is smooth along the image of the parametrization given above and there are at most two points $P \in \ell_1$ such that $\mathrm{rk}(V_P) < 3$. One can easily check that $\deg_t(\mathfrak{l}_t) = 3$ and $\deg_t(\mathfrak{q}_t) = 2$. Moreover, one has $z^2 \mid \mathfrak{R}_t$ and the degree-6 polynomial $(\mathfrak{R}_t/z^2)$ vanishes along the lines ℓ_0, ℓ_1. In this case, however, it is no longer singular along ℓ_0 (but its partial derivatives vanish along ℓ_1). Finally, we consider the cycle (3.7) to obtain the bound of at most 23 lines on S that meet ℓ_1. □

Recall that in this section all quartic surfaces S are assumed to contain finitely many lines. As an immediate consequence, we obtain the following proposition.

Proposition 3.6. *If a quartic S belongs to the family* (Q5)*, then the number of lines contained in S is* 27 *or less.*

Proof. By Lemma 2.5(b), every line on S meets the curve residual to the line ℓ_0 in the hyperplane section $S \cap C_O S$. Therefore, the claim follows directly from Lemma 3.5. □

This bound could probably be improved by studying the infinitesimal neighbourhood of C (resp. ℓ_1) in S. However, since it is already better than Rohn's bound in the case where S contains a triple point, we will not pursue further improvements.

Finally, we examine lines on quartic surfaces that contain a line of double points.

Lemma 3.7. *If a quartic S contains a line ℓ_0 of double points, then ℓ_0 meets at most* 16 *other lines on the surface S.*

Proof. We can assume that $\ell_0 = \mathrm{V}(x, y)$, the quartic S is given by the polynomial

$$x^2\, Q_{20}(x, y, z, w) + xy\, Q_{11}(x, y, z, w) + y^2\, Q_{02}(x, y, z, w) = 0 \tag{3.8}$$

and the conic C_0 residual to the line ℓ_0 in the hyperplane section $S \cap \mathrm{V}(x)$ is irreducible. Then, all singular conics in $|\mathcal{O}_S(1) - 2\ell_0|$ are given by vanishing of the determinant of the Hessian matrix of $(S(x, \lambda x, z, w)/x^2)$. One can easily check that the determinant in question is of degree at most 8 with respect to the parameter λ, so S contains at most 16 lines that meet the line ℓ_0. $\square$

Lemma 3.8. *If a quartic S contains a line ℓ_0 of double points, then a general conic C_0 in $|\mathcal{O}_S(1) - 2\ell_0|$ meets at most* 19 *lines on S.*

Proof. We keep the notation of the proof of Lemma 3.7 and consider the variety Z_{C_0} (see (3.3)).

Suppose that the conic C_0 meets the line ℓ_0 transversally. Without loss of generality we can assume that C_0 can be parametrized (away from one point on the double line ℓ_0) by $P(t) := (0 : t : 1 : t^2)$ for $t \in \mathbb{C}$. Then (cf. (3.5)) we have $\deg_t(\mathfrak{l}_t) = 5$ and $\deg_t(\mathfrak{q}_t) = 4$. Moreover, the form $\mathfrak{l}_t$ is divisible by t (recall that $P(0) \in \operatorname{sing}(S)$). Thus $\widetilde{\mathfrak{R}}_t := \operatorname{Res}_t(\widetilde{\mathfrak{l}}_t, \mathfrak{q}_t)$, where $\widetilde{\mathfrak{l}}_t := \mathfrak{l}_t/t$, belongs to the ideal of the variety Z_{C_0} (cf. Remark 3.4) and (3.6) gives $\deg(\widetilde{\mathfrak{R}}_t) \le 12$. Furthermore, a direct computation shows that $x^2 \mid \widetilde{\mathfrak{R}}_t$, all partial derivatives of the polynomial $(\widetilde{\mathfrak{R}}_t/x^2)$ vanish along the line of singularities of S and the non-degeneracy condition (3.4) is satisfied. Therefore, a general conic $C_0 \in |\mathcal{O}_S(1) - 2\ell_0|$ meets the surface $\mathrm{V}(\widetilde{\mathfrak{R}}_t/x^2)$ in at most 18 points, two of which belong to the line ℓ_0. This yields the bound of at most 17 lines on S that meet the curve C_0.

Suppose now that the conic C_0 is tangent to the line ℓ_0. We can assume that C_0 is parametrized (away from the point on the line ℓ_0) by $P(t) := (0 : 1 : t^2 : t)$, where $t \in \mathbb{C}$. In this case we have $\deg_t(\mathfrak{l}_t) \le 4$ and $\deg_t(\mathfrak{q}_t) \le 4$. Moreover, by a direct computation, the resultant $\mathfrak{R}_t$ is again divisible by x^2, the non-degeneracy condition (3.4) is fullfilled and the surface $\mathrm{V}(\mathfrak{R}_t/x^2)$ is singular along the line ℓ_0. Thus a general conic $C_0 \in |\mathcal{O}_S(1) - 2\ell_0|$ meets at most 19 lines on S. $\square$

Lemma 3.9. *If a quartic S is not ruled by lines and contains a line of double points, then there are at most* 35 *lines on S.*

Proof. The claim follows directly from Lemmas 3.7 and 3.8. $\square$

Example 3.10. Consider the quartic S given by (3.8), where we put

$$\begin{aligned}
Q_{20} &:= wz + yw + yz - \frac{10}{3}xy - 4y^2,\\
Q_{11} &:= z(w + z),\\
Q_{02} &:= \frac{1}{2}w^2 + \frac{1}{4}wz + \frac{1}{2}z^2 + xw + 3xz + \frac{5}{4}yw + \frac{5}{4}yz - \frac{1}{2}xy.
\end{aligned}$$

The hyperplane section $\mathrm{V}(y)$ splits into x^2zw. One can check that the line of singularities $\ell_0 := \mathrm{V}(y, x)$ meets 16 other lines on S. Moreover, the line $\mathrm{V}(y, z)$ (resp. $\mathrm{V}(y, w)$) intersects eight (resp. six) other lines on S. Altogether, the surface S contains 27 lines.

4. Counting lines on quartic surfaces

Recall that in this note all quartic surfaces are assumed not to be cones.

In order to count lines on the singular quartics we will use the technique invented already by Salmon (see e.g. [17, p. 277], [7, § 13.2], [10, § 8] and the bibliography therein).

Lemma 4.1. [10, Theorem 13] *Let $S \subset \mathbb{P}^3(\mathbb{C})$ be a quartic surface with isolated singularities, none of which is a quadruple point. Then there exists a degree-*20 *hypersurface $\mathcal{F}_S$ in $\mathbb{P}^3(\mathbb{C})$ such that $S \not\subseteq \mathcal{F}_S$ and*

$$\mathcal{F}_S \cap S = \left\{ P \in S \mid \textit{there exists a line } \ell \subset \mathbb{P}^3 \textit{ with } i_P(S, \ell) \geq 4 \right\}.$$

Proof. By [13, § 3] no quartic surface S with isolated singularities (none of which is a quadruple point) contains infinitely many lines. The claim follows directly from [10, Theorem 13] (and the discussion that precedes it in [10]). □

Lemma 4.2. *Let $S \subset \mathbb{P}^3(\mathbb{C})$ be a non-$K3$ quartic surface with isolated singularities, none of which is a quadruple point. Then S does not contain more than* 48 *lines.*

Proof. It results from [13, § 3] that the surface S contains only finitely many lines. If S has a triple point, the claim follows from Lemma 2.1. By Propositions 3.2 and 3.6 we can assume that S belongs to the family (Q6) and it belongs to neither (Q4) nor (Q5). In particular, Lemma 2.5 implies that

(4.1) any line on S runs at worse through rational double points of S.

Suppose S contains a line ℓ_0. At first we construct a smooth model of S that we use in the proof. The pencil $|\mathcal{O}_S(1) - \ell_0|$ induces a rational map $\psi_{\ell_0} : S \dashrightarrow \mathbb{P}^1$ whose general fiber is a smooth elliptic curve. Let $\widetilde{S}$ be the minimal resolution of singularities of S. By (4.1), the rational map ψ_{ℓ_0} lifts to a morphism $\widetilde{\psi}_{\ell_0} : \widetilde{S} \to \mathbb{P}^1$.

By Lemma 2.3 we can assume that S is given by (2.3), (2.5) and the line ℓ_0 is contained in the plane $\mathrm{V}(x)$. Let $\widetilde{C}$ be the strict transform of the cubic C that is residual to the line ℓ_0 in the hyperplane section $S \cap \mathrm{V}(x)$. A local computation with help of (2.3) shows that $\widetilde{C}$ is a (-1)-curve. Thus the genus-one fibration $\widetilde{\psi}_{\ell_0}$ is not minimal, but only its fibers induced by cubics through non-rational double points of S contain (-1)-curves. We blow-down the strict transforms of these cubics to obtain a minimal genus-one fibration $\phi_{\ell_0} : S_{\ell_0} \to \mathbb{P}^1$. Observe that the smooth surface S_{ℓ_0} is rational (see Lemma 2.6). Moreover, the lines on S that meet (resp. do not intersect) the line ℓ_0 become components of singular fibers (resp. sections) of the genus-one fibration ϕ_{ℓ_0}.

Claim 1. *The quartic S contains at most* 11 *pairwise disjoint lines.*

Indeed, suppose that S contains twelve pairwise disjoint lines $\ell_0, \ell_1, \ldots, \ell_{11}$ and consider the rational elliptic surface S_{ℓ_0}. Let $C \subset S$ be a cubic that is coplanar with ℓ_0 and runs through a non-rational double point of the quartic surface. Since the lines $\ell_1, \ldots, \ell_{11}$ meet C in different points, their proper transforms $\widehat{\ell_i} \subset S_{\ell_0}$ meet transversely in exactly n points, where n is the number of non-rational singularities of S. Moreover, each $\widehat{\ell_i}$ is a section of ϕ_{ℓ_0}, hence it is a (-1)-curve. From the equality $\widehat{\ell_i}^2 = \widetilde{\ell_i}^2 - n = -2 - n$ (where $\widetilde{\ell_i} \subset \widetilde{S}$ denotes the strict transform on the minimal resolution of singularities), we infer that S has one non-rational singularity and that $\widehat{\ell_i}.\widehat{\ell_j} = 1$ whenever $i \neq j$. In this way we can check that the intersection matrix $[\widehat{\ell_i}.\widehat{\ell_j}]_{i,j=1,\ldots,11}$ has rank 11, which is impossible because for the Picard number of the rational elliptic surface S_{ℓ_0} one has $\rho(S_{\ell_0}) = 10$. Contradiction.

Claim 2. *If a line $\ell_0 \subset S$ meets at least* 12 *other lines, then S contains at most* 22 *lines.*

Each line $\ell \subset S$ meeting ℓ_0 induces a component of a singular fiber of the genus-one fibration on S_{ℓ_0}. Since $e(S_{\ell_0}) = 12$, there can be at most 12 such lines. Moreover, these 12 lines on S induce four I_3-fibers of the fibration ϕ_{ℓ_0}. If there are no other lines on S, then we are done. Otherwise, ϕ_{ℓ_0} has a section and S_{ℓ_0} is the extremal rational surface $X_{3,3,3,3}$. By [11, p. 77], the fibration ϕ_{ℓ_0} has nine sections and the Claim 2 follows.

Claim 3. *If there are no four coplanar lines on S, then each line on S meets at most* 6 *other lines on the quartic surface.*

Indeed, each line that meets the line $\ell_0 \subset S$ gives a contribution of at least 2 to the Euler number $e(S_{\ell_0}) = 12$, so we have at most 6 such lines.

Claim 4. *If there are four coplanar lines on S, then S contains at most* 32 *lines.*

By Claim 2 we can assume that each line on S meets at most 11 other lines. Since the sum of Euler numbers of all singular fibers is 12, the case of exactly 11 lines that meet a given line on S is also ruled out. Since each line on S intersects one of the four coplanar lines, Claim 4 follows.

After those preparations we can complete the proof of Lemma 4.2. By Claims 3 and 4 we can assume that S contains no four coplanar lines and each line meets at most 6 other lines on S. If there exist coplanar lines ℓ_1, ℓ_2 on S such that the irreducible conic $C \in |\mathcal{O}_S(1) - (\ell_1 + \ell_2)|$ is not contained in the degree-20 hypersurface $\mathcal{F}_S$, then one can easily check that the number of lines on S is bounded by

$$\deg(\mathcal{F}_S|_{C \setminus (\ell_1 \cup \ell_2)}) + 6 + 6 = 48.$$

Otherwise, each pair of coplanar lines on S induces a conic in the degree-80 cycle $(\mathcal{F}_S \cdot S)$. If the number of the residual conics in $\mathcal{F}_S$ does exceed 15 we obtain again that S contains at most 48 lines. Suppose we have at most 15 conics in $\mathcal{F}_S$ and at least 49 lines on S. Then there are at least $49 - 30 = 19$ pairewise disjoint lines on S, which contradicts to Claim 1. $\square$

Configurations of lines on irreducible quartic surfaces with non-isolated singularities were a subject of interest of classical algebraic geometry. In the lemma below we recall some well-known facts (see e.g. [5, 13]).

Lemma 4.3. *Let $S \subset \mathbb{P}^3(\mathbb{C})$ be an irreducible quartic.*

(a) *If S contains either a twisted cubic of double points or a line of triple points or two skew lines of singular points, then it is ruled by lines.*

(b) [2, p. 143] *If an irreducible quartic S is singular along a conic, then it contains at most 16 lines.*

Proof. (a) Observe that if S is singular along a twisted cubic C_3, then S is covered by secants of C_3. Indeed, let $P \in S$ be a general (smooth) point, and let ℓ be a secant (or tangent) of C_3 that runs through the point P (recall that $\mathbb{P}^3(\mathbb{C})$ is the union of secants and tangents of C_3). Since $\ell.S \geq 5$, we obtain $\ell \subset S$ and the claim follows.

To prove the claim in other cases consider the pencil of hyperplanes that contain the line (resp. one of the lines) of singular points, and observe that each element of the pencil contains another line.

(b) Suppose that S is singular along a smooth conic C and put $\Pi := \mathrm{span}(C)$. By [5, § 8.6], the system $|\mathcal{I}_C(2)|$ of quadrics in $\mathbb{P}^3$ induces a rational map $\phi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$, which is not defined at C, contracts the plane Π to a point $P \in \mathbb{P}^4$, and is one-to-one outside Π. Its image is a smooth quadric threefold $Q_1 \subset \mathbb{P}^4$. Since $S \in \left|\mathcal{I}_C^2(2)\right|$, there exists another quadric threefold $Q_2 \subset \mathbb{P}^4$, such that $\phi(S) = Q_1 \cap Q_2$ (see also [5, § 7.2.1]). One can easily check that ϕ maps lines on S to lines on the Fano surface $Q_1 \cap Q_2$. The latter contains at most 16 lines (see e.g. [5, § 8.6.3]) which yields the claim. □

The above lemma immediately yields the following proposition.

Proposition 4.4. *Let $S \subset \mathbb{P}^3(\mathbb{C})$ be a non-$K3$ quartic surface. If S is not ruled by lines, then S contains at most 48 lines.*

Proof. By Lemma 4.2, we can assume that S has non-isolated singularities. It results from Bertini (see [9, Theorem 6.3.4]) that a (reduced) curve contained in $\mathrm{sing}(S)$ has degree at most 3. By Lemma 4.3 we can assume that S contains a line of double points. Lemma 3.9 completes the proof. □

Finally, we can prove the main result of this note. Recall that cones are excluded from the quartic surfaces we consider.

Theorem 4.5. (a) *Let $S \subset \mathbb{P}^3(\mathbb{C})$ be a quartic surface with at most isolated singularities, none of which is a quadruple point. Then S contains at most 64 lines.*

(b) *Let $S \subset \mathbb{C}^3$ be an affine quartic surface that is not ruled by lines. Then S contains at most* 64 *lines.*

Proof. (a) If S has at most A-D-E singularities, the claim follows from [20, Theorem 1.1]. Otherwise, it is not a $K3$-surface, so Proposition 4.4 completes the proof.

(b) Consider the projective closure $\overline{S} \subset \mathbb{P}^3(\mathbb{C})$. If it has at most isolated singularities we can apply (a). Otherwise, we use Proposition 4.4. □

Both bounds are sharp, because the Schur quartic contains 64 lines [18]. The results of Section 3 yield the following observation

Observation 4.6. If a non-$K3$ quartic surface $S \subset \mathbb{P}^3(\mathbb{C})$ with isolated singularities and without quadruple points contains more than 31 lines, then S is given by (2.3) and (2.5).

Finite field computations combined with results of Section 3 (in particular Example 2.2) support the following conjecture.

Conjecture 4.7. (a) *If a non-$K3$ quartic surface $S \subset \mathbb{P}^3(\mathbb{C})$ contains more than* 31 *lines, then it is ruled by lines.*

(b) *Let $S \subset \mathbb{P}^3(\mathbb{C})$ be a quartic surface that is not ruled by lines. If* $\operatorname{sing}(S) \neq \emptyset$ *then S contains strictly less than* 64 *lines.*

Finally, recall that the maximal number of lines on quartic hypersurfaces in $\mathbb{P}^3(\mathbb{C})$ with non-empty singular locus (that are not ruled by lines) is unknown. The example below shows that the number in question is attained either by a surface in (Q6) or by a non-smooth $K3$-quartic.

Example 4.8. One can easily check that the set of singularities of the surface given by

$$S : x^4 + xz^3 + y^2zw + xw^3 = 0$$

consists of one singular point of type A_3 and of three A_1-points. Studying the lines that meet the plane $\mathrm{V}(x)$ one finds that S contains exactly 39 lines. This is the largest number of lines known on an explicitly given singular complex quartic surface in $\mathbb{P}^3(\mathbb{C})$ until now. While working on this note we were informed by D. Veniani that a Torelli-theorem argument can be applied to show the existence of a complex quartic surface with non-empty singular locus and 40 lines.

Acknowledgments

We thank Wolf Barth, Alex Degtyarev and Matthias Schütt for helpful discussions. We are extremely grateful to Duco van Straten for calling our attention to various papers of A. Clebsch. The first author was funding by projects ERC StG 279723 (SURFARI) and MTM2012-38122-C03-01/FEDER (of the Spanish "Ministerio de Economía y Competitividad") is gratefully acknowledged. The second author was partially supported by National Science Centre, Poland, grant 2014/15/B/ST1/02197.

References

[1] S. Boissière and A. Sarti, *Counting lines on surfaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), no. 1, 39–52.

[2] A. Clebsch, *Ueber die Flächen vierter Ordnung, welche eine Doppelcurve zweiten Grades besitzen*, J. Reine Angew. Math. **1868** (1868), no. 69, 142–184. `http://dx.doi.org/10.1515/crll.1868.69.142`

[3] A. I. Degtyarev, *Classification of quartic surfaces that have a nonsimple singular point*, Math. USSR-Izv. **35** (1990), no. 3, 607–627.

[4] A. I. Degtyarev, I. Itenberg and A. Sinan Sertöz, *Lines on quartic surfaces*, preprint (2016), arXiv:math/1601.04238.

[5] Igor V. Dolgachev, *Classical algebraic geometry: A modern view*, Cambridge University Press, Cambridge, 2012.

[6] M. Dumnicki, B. Harbourne, T. Szemberg and H. Tutaj-Gasińska, *Linear subspaces, symbolic powers and Nagata type conjectures*, Adv. Math. **252** (2014), 471–491.

[7] D. Eisenbud and J. Harris, 3264 *and all that: Intersection theory in algebraic geometry*, Draft available at `http://isites.harvard.edu/fs/docs/icb.topic720403.files/book.pdf`, 2012.

[8] J. Harris and Y. Tschinkel, *Rational points on quartics*, Duke Math. J. **104** (2000), no. 3, 477–500. `http://dx.doi.org/10.1215/s0012-7094-00-10436-x`

[9] J.-P. Jouanolou, *Théorèmes de Bertini et applications*, Progress in Mathematics **42**, Birkhäuser Boston, Boston, MA, 1983.

[10] J. Kollár, *Szemerédi-Trotter-type theorems in dimension* 3, Adv. Math. **271** (2015), 30-61. `http://dx.doi.org/10.1016/j.aim.2014.11.014`

[11] R. Miranda, *The Basic Theory of Elliptic Surfaces*, Dottorato di Ricerca in Mathematica, ETS Editrice, Pisa, 1989.

[12] Y. Miyaoka, *Counting lines and conics on a surface*, Publ. Res. Inst. Math. Sci. **45** (2009), no. 3, 919–923.

[13] I. Polo-Blanco, M. van der Put and J. Top, *Ruled quartic surfaces, models and classification*, Geom. Dedicata **150** (2011), no. 1, 151–180. http://dx.doi.org/10.1007/s10711-010-9500-0

[14] S. Rams and M. Schütt, 64 *lines on smooth quartic surfaces*, Math. Ann. **362** (2015), no. 1-2, 679–698. http://dx.doi.org/10.1007/s00208-014-1139-y

[15] ———, 112 *lines on smooth quartic surfaces (characteristic* 3*)*, Q. J. Math. **66** (2015), no. 3, 941–951. http://dx.doi.org/10.1093/qmath/hav018

[16] K. Rohn, *Ueber die Flächen vierter Ordnung mit dreifachem Punkte*, Math. Ann. **24** (1884), no. 1, 55–151. http://dx.doi.org/10.1007/bf01446444

[17] G. Salmon, *A treatise on the analytic geometry of three dimensions. Vol. II.*, Fifth edition, Longmans and Green, London, 1915.

[18] F. Schur, *Ueber eine besondre Classe von Flächen vierter Ordnung*, Math. Ann. **20** (1882), no. 2, 254–296. http://dx.doi.org/10.1007/bf01446525

[19] B. Segre, *The maximum number of lines lying on a quartic surface*, Quart. J. Math., Oxford Ser. **14** (1943), no. 1, 86–96. http://dx.doi.org/10.1093/qmath/os-14.1.86

[20] D. C. Veniani, *The maximum number of lines lying on a K3 quartic surface*, preprint (2015), arXiv:math/1502.04510.

[21] C. T. C. Wall, *Sextic curves and quartic surfaces with higher singularities*, preprint 1998, available at http://www.liv.ac.uk/$\sim$ctcw/Other.html.

Víctor González-Alonso
Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
E-mail address: gonzalez@math.uni-hannover.de

Sławomir Rams
Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
and
Institute of Mathematics, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland
E-mail address: slawomir.rams@uj.edu.pl