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One Generalized Critical Point Theorem and its Applications on Super-quadratic Hamiltonian Systems

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Abstract. In this paper, we prove a generalized critical point theorem under the condition (C), which is weaker than the (PS) condition. As its applications, we obtain the existence of the solutions for the Hamiltonian systems with a new super-quadratic conditions generalizing one in papers [2] and [12].

1. Introduction

Let $p, q \in C^1(\mathbb{R}, \mathbb{R}^n)$, $z = (p, q)$ and $H \in C^1(\mathbb{R}\times\mathbb{R}^{2n}, \mathbb{R})$, then we consider the Hamiltonian system

$$\begin{cases} \dot{p} = -H'_q(t, z), \\ \dot{q} = H'_p(t, z), \end{cases} \tag{1.1}$$

which also can be written as $\dot{z} = JH'_z(t, z)$, where $H'_z = \frac{\partial H}{\partial z} = (H'_p, H'_q) = (\frac{\partial H}{\partial p}, \frac{\partial H}{\partial q})$ and $J = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$ with I_n being the $n \times n$ identity matrix. For simplicity of notations, we denote $(p, q) = (p_1, \ldots, p_n, q_1, \ldots, q_n)$, $(p_i, q_i) = (0, \ldots, p_i, \ldots, 0, \ldots, q_i, \ldots, 0)$, and whenever without confusion we use the same symbols p_i, q_i to represent the vectors $(0, \ldots, p_i, \ldots, 0)$, $(0, \ldots, q_i, \ldots, 0)$ and the numbers p_i, q_i.

In the pioneer work of paper [10], using minimax method, Rabinowitz established the existence of periodic solutions of the autonomous Hamiltonian systems with a classical super-quadratic condition, that is,

(S) there exist constants $\widehat{\theta} \in (0, \frac{1}{2})$ and $R > 0$ such that

$$\widehat{\theta} H'_z(t, z) \cdot z \geq H(t, z) > 0, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^{2n} \text{ with } |z| \geq R.$$

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Besides the minimax method, several different methods have been introduced to study system (1.1). In paper [1], Ambrosetti and Mancini got the solutions of minimal period for convex super-quadratic Hamiltonian systems via the "Fenche dual" of Hamiltonian functions introduced by Clarke in [4]. The Maslov-type index theory was applied to study the minimal periodic solutions of the classical super-quadratic Hamiltonian system in [5,7].

Meanwhile, generalized super-quadratic conditions covering the condition (S) raised in many literatures, such as [2,6,8,12] and references therein. Zhang and Guo [12] considered the existence of periodic solutions of the Hamiltonian systems with the super-quadratic condition (S_1), that is,

(S_1) there exist constants $c_1, c_2, \sigma, \tau > 0$ and $\beta, \mu, \nu > 1$ with $\frac{1}{\mu} + \frac{1}{\nu} < 1$ such that

$$\frac{1}{\mu} H'_p(t,z) \cdot p + \frac{1}{\nu} H'_q(t,z) \cdot q - \left(\frac{1}{\mu} + \frac{1}{\nu} \right) H(t,z) \geq c_1 |z|^{\beta} - c_2, \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n}, \tag{1.2}$$

$$\frac{H(t,z)}{|p|^{1+\frac{\sigma}{\tau}} + |q|^{1+\frac{\tau}{\sigma}}} \to +\infty, \quad \text{as } |z| \to \infty \text{ uniformly in } t. \tag{1.3}$$

An and Wang [2] considered the existence and multiplicity of periodic solutions of the Hamiltonian systems with the super-quadratic condition (S_2), that is,

(S_2) there exists a vector field $\widehat{V}(z)$ with form

$$\widehat{V}(z) = \begin{pmatrix} \frac{1}{\widehat{\alpha}_1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\widehat{\alpha}_n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{\widehat{\beta}_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{1}{\widehat{\beta}_n} \end{pmatrix} z$$

and a constant $R > 0$ such that for $|z| \geq R$, $t \in \mathbb{R}$, $0 < H(t,z) \leq H'_z(t,z) \cdot \widehat{V}(z)$, where α_i and β_i are positive numbers satisfying $\frac{1}{\alpha_i} + \frac{1}{\beta_i} = \epsilon < 1$, $i = 1, 2, \ldots, n$.

An and Wang [2, Lemma 2.2] also showed that condition (S_2) implies that there exist constants $a_1, a_2 > 0$ such that

$$H(t,z) \geq a_1 \sum_{i=1}^{n} \left(|p_i|^{\widehat{\alpha}_i} + |q_i|^{\widehat{\beta}_i} \right) - a_2, \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n}. \tag{1.4}$$

The (PS) condition plays an important role in the critical point theory, and has a weaker version called the condition (C). We recall the (PS) condition and the condition (C) as follows.

Definition 1.1. Let E be a real Banach space, $I \in C^1(E, \mathbb{R})$, we shall say a functional I satisfies the (PS) condition, if any sequence $\{z_m\}$ satisfying that $\{I(z_m)\}$ is bounded and $I'(z_m) \to \mathbf{0}$ has a convergent subsequence as $m \to +\infty$.

Definition 1.2. Let E be a real Banach space, $I \in C^1(E, \mathbb{R})$, we shall say a functional I satisfies the condition (C), if any sequence $\{u_m\}$, such that $\{I(u_m)\}$ is bounded and $\|I'(u_m)\|(1 + \|u_m\|) \to 0$, has a convergent subsequence as $m \to +\infty$.

The second and third authors proved the existence results via a Generalized Mountain Pass Theorem under (PS) condition in [12]. In Section 2, we will prove a generalized critical point theorem under the condition (C) instead of the (PS) condition. In Section 3, as the applications of the generalized critical point theorem to Hamiltonian systems, we generalize the existence results of periodic solutions for system (1.1) in [12] with the following conditions.

Theorem 1.3. *The system* (1.1) *possesses a nontrivial T-periodic solution, if H satisfies*

(H1) $H \in C^1(\mathbb{R} \times \mathbb{R}^{2n}, [0, +\infty))$ *is T-periodic with respect to t;*

(H2) *there exist constants $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n > 1$ such that*

$$\frac{H(t,z)}{\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}} + |q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)} \to 0, \quad \text{as } |z| \to 0 \text{ uniformly in } t;$$

(H3) $$\frac{H(t,z)}{\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}} + |q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)} \to +\infty, \quad \text{as } |z| \to +\infty \text{ uniformly in } t;$$

(H4) *there exist a vector field $V(z)$ and constants $c_1, c_2 > 0$ and $\beta > 1$ such that*

$$H'_z(t,z) \cdot V(z) - H(t,z) \geq c_1 |z|^\beta - c_2, \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n},$$

where

$$V(z) = \begin{pmatrix} \frac{1}{\alpha_1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\alpha_n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{1}{\beta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{1}{\beta_n} \end{pmatrix} z$$

with $\alpha_i, \beta_i > 0$ satisfying $\frac{1}{\alpha_i} + \frac{1}{\beta_i} = 1$ $(i = 1, 2, \ldots, n)$;

(H5) *there exists a constant* $\lambda \in \left(\max\left\{\frac{\sigma_1}{\tau_1}, \ldots, \frac{\sigma_n}{\tau_n}, \frac{\tau_1}{\sigma_1}, \ldots, \frac{\tau_n}{\sigma_n}\right\}, 1+\beta\right)$ *such that*

$$\left|H_z'(t,z)\right| \le c_2(|z|^\lambda + 1), \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n},$$

where $\sigma_1, \ldots, \sigma_n$, $\tau_1, \ldots, \tau_n$ *and* c_2 *are as above.*

Remark 1.4. Suppose $\min\{\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n\} \ge \max\{\sigma_1', \ldots, \sigma_n', \tau_1', \ldots, \tau_n'\} > 0$ and $\sigma > 0$, then we have

$$\sum_{i=1}^n (|p_i|^{\sigma_i} + |q_i|^{\tau_i}) \ge \frac{1}{2n} \sum_{i=1}^n (|p_i|^{\sigma_i'} + |q_i|^{\tau_i'}), \quad \text{where } |z| \ge \sqrt{2n}, \tag{1.5}$$

and

$$\frac{1}{2n} \sum_{i=1}^n (|p_i|^\sigma + |q_i|^\sigma) \le |z|^\sigma \le (2n)^\sigma \sum_{i=1}^n (|p_i|^\sigma + |q_i|^\sigma), \tag{1.6}$$

both of which will be used later.

Proof. Set $L = \max\{|p_1|, \ldots, |p_n|, |q_1|, \ldots, |q_n|\}$. By $|z| > \sqrt{2n}$, it is obvious that $L \ge 1$, so we have that

$$\frac{1}{2n} \sum_{i=1}^n (|p_i|^{\sigma_i'} + |q_i|^{\tau_i'}) \le L^{\max\{\sigma_1', \ldots, \sigma_n', \tau_1', \ldots, \tau_n'\}} \le \sum_{i=1}^n (|p_i|^{\sigma_i} + |q_i|^{\tau_i}).$$

Similarly, we get that

$$\frac{1}{2n} \sum_{i=1}^n (|p_i|^\sigma + |q_i|^\sigma) \le |z|^\sigma \le (2n)^{\frac{\sigma}{2}} \sum_{i=1}^n (|p_i|^\sigma + |q_i|^\sigma). \qquad \square$$

Remark 1.5. (1) If $\alpha_i = \beta_i$, $\sigma_i = \tau_i$ ($i = 1, 2, \ldots, n$), then $\alpha_i = \beta_i = 2$, so (H4) and (H3) become the super-quadratic condition in [6], that is, there exist constants $d_1, d_2 > 0$ and $\widehat{\beta} > 1$ such that

$$H_z'(t,z) \cdot z - 2H(t,z) \ge d_1 |z|^{\widehat{\beta}} - d_2, \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n},$$
$$\frac{H(t,z)}{|z|^2} \to +\infty, \quad \text{as } |z| \to +\infty.$$

(2) If $\sigma_1 = \cdots = \sigma_n = \sigma$, $\tau_1 = \cdots = \tau_n = \tau$, by (1.6), we have that

$$\begin{aligned} \frac{1}{\max\left\{n^{1+\frac{\sigma}{\tau}}, n^{1+\frac{\tau}{\sigma}}\right\}} \left(|p|^{1+\frac{\sigma}{\tau}} + |q|^{1+\frac{\tau}{\sigma}}\right) &\le \sum_{i=1}^n \left(|p_i|^{1+\frac{\sigma}{\tau}} + |q_i|^{1+\frac{\tau}{\sigma}}\right) \\ &\le n\left(|p|^{1+\frac{\sigma}{\tau}} + |q|^{1+\frac{\tau}{\sigma}}\right), \end{aligned}$$

which implies that (H3) is equivalent to (1.3). At the same time, (H4) becomes (1.2) if $\alpha_i = \frac{\mu}{\mu+\nu}$, $\beta_i = \frac{\nu}{\mu+\nu}$ ($i = 1, 2, \ldots, n$), so (S_1) is a special case of (H3) and (H4), and Theorem 1.3 is an improvement of [12, Theorem 1.1].

Claim 1.6. Function

$$H(t,z)=\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\ln^{\eta}(1+|z|^{\xi})$$

satisfies (H1)–(H5), but dissatisfies (S_1) and (S_2), where $\xi,\eta,\sigma_i,\tau_i>1$, σ_i, τ_i satisfy that $\frac{1}{\sigma_i}+\frac{1}{\tau_i}=\epsilon$ $(i=1,2,\ldots,n)$ and $\max\left\{\left|\frac{\sigma_1}{\tau_1}-\frac{\tau_1}{\sigma_1}\right|,\ldots,\left|\frac{\sigma_n}{\tau_n}-\frac{\tau_n}{\sigma_n}\right|\right\}<2$, and there exist two integers i_1 and i_2 $(1\le i_1,\, i_2\le n)$ such that $\frac{\sigma_{i_1}}{\tau_{i_1}}\neq\frac{\sigma_{i_2}}{\tau_{i_2}}$, $V(z)=\operatorname{diag}\left\{1+\frac{\sigma_1}{\tau_1},\ldots,1+\frac{\sigma_n}{\tau_n},1+\frac{\tau_1}{\sigma_1},\ldots,\frac{\tau_n}{\sigma_n}\right\}$.

Proof. It is obvious that H satisfies (H1)–(H3).

Set $\beta=\min\left\{1+\frac{\sigma_1}{\tau_1},\ldots,1+\frac{\sigma_n}{\tau_n},1+\frac{\tau_1}{\sigma_1},\ldots,1+\frac{\tau_n}{\sigma_n}\right\}$ and $b=\max\{\frac{\sigma_1}{\tau_1},\ldots,\frac{\sigma_n}{\tau_n},\frac{\tau_1}{\sigma_1},\ldots,\frac{\tau_n}{\sigma_n}\}$, it is obvious that $b<1+\beta$ and $1<\beta<2$.

Step 1. We will check that H satisfies (H4). Set $\alpha_i=1+\frac{\sigma_i}{\tau_i}$ and $\beta_i=1+\frac{\tau_i}{\sigma_i}$ $(i=1,2,\ldots,n)$, for $(t,z)\in\mathbb{R}\times\mathbb{R}^{2n}$ with $|z|\ge\sqrt{2n}$, by (1.5) and (1.6), we have that

$$\begin{aligned}
&H_z'(t,z)\cdot V(z)-H(t,z)\\
&=\sum_{i=1}^{n}\Bigg[\frac{1}{\alpha_i}\Bigg(\left(1+\frac{\sigma_i}{\tau_i}\right)|p_i|^{\frac{\sigma_i}{\tau_i}-1}\ln^{\eta}(1+|z|^{\xi})\\
&\qquad\qquad+\frac{\xi\eta\,|z|^{\xi-2}}{1+|z|^{\xi}}\ln^{\eta-1}(1+|z|^{\xi})\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\Bigg)\cdot|p_i|^2\\
&\qquad+\frac{1}{\beta_i}\Bigg(\left(1+\frac{\tau_i}{\sigma_i}\right)|q_i|^{\frac{\tau_i}{\sigma_i}-1}\ln^{\eta}(1+|z|^{\xi})\\
&\qquad\qquad+\frac{\xi\eta\,|z|^{\xi-2}}{1+|z|^{\xi}}\ln^{\eta-1}(1+|z|^{\xi})\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\Bigg)\cdot|q_i|^2\Bigg]\\
&\quad-\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\ln^{\eta}(1+|z|^{\xi})\\
&=\xi\eta\ln^{\eta-1}(1+|z|^{\xi})\frac{|z|^{\xi}}{1+|z|^{\xi}}\frac{\sum_{i=1}^{n}\left(\frac{1}{\alpha_i}|p_i|^2+\frac{1}{\beta_i}|q_i|^2\right)}{|z|^2}\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\\
&\ge\frac{\xi\eta}{b+1}\ln^{\eta-1}(1+|z|^{\xi})\frac{|z|^{\xi}}{1+|z|^{\xi}}\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\\
&\ge\frac{\xi\eta}{b+1}\cdot\frac{1}{2n}\cdot(\ln 2)^{\eta-1}\cdot\frac{1}{2}\sum_{i=1}^{n}(|p_i|^{\beta}+|q_i|^{\beta})\\
&\ge\frac{\xi\eta(\ln 2)^{\eta-1}}{(2n)^{\beta+1}(2b+2)}|z|^{\beta},
\end{aligned}$$

so (H4) is proved.

Step 2. We will check that H satisfies (H5). Choosing $\lambda \in (b, 1+\beta)$, by Remark 1.4, we have that

$$\begin{aligned}
\frac{\sum_{i=1}^{n}\left(|p_i|^{\frac{\sigma_i}{\tau_i}}+|q_i|^{\frac{\tau_i}{\sigma_i}}\right)\ln^{\eta}(1+|z|^{\xi})}{|z|^{\lambda}} &= \frac{\sum_{i=1}^{n}\left(|p_i|^{\frac{\sigma_i}{\tau_i}}+|q_i|^{\frac{\tau_i}{\sigma_i}}\right)}{|z|^{b}}\cdot\frac{\ln^{\eta}(1+|z|^{\xi})}{|z|^{\lambda-b}} \\
&\le \frac{\sum_{i=1}^{n}\left(|p_i|^{\frac{\sigma_i}{\tau_i}}+|q_i|^{\frac{\tau_i}{\sigma_i}}\right)}{4n^2\sum_{i=1}^{n}\left(|p_i|^{\frac{\sigma_i}{\tau_i}}+|q_i|^{\frac{\tau_i}{\sigma_i}}\right)}\cdot\frac{\ln^{\eta}(1+|z|^{\xi})}{|z|^{\lambda-b}} \\
&\le \frac{\ln^{\eta}(1+|z|^{\xi})}{4n^2\,|z|^{\lambda-b}} \\
&\to 0, \quad \text{as } |z|\to+\infty.
\end{aligned} \tag{1.7}$$

Similarly, we also have

$$\begin{aligned}
&\frac{\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\ln^{\eta-1}(1+|z|^{\xi})\frac{|z|^{\xi-1}}{1+|z|^{\xi}}}{|z|^{\lambda}} \\
&= \frac{\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\ln^{\eta-1}(1+|z|^{\xi})}{|z|^{1+\lambda}}\cdot\frac{|z|^{\xi}}{1+|z|^{\xi}} \\
&\to 0, \quad \text{as } |z|\to+\infty.
\end{aligned} \tag{1.8}$$

Assuming R is sufficiently large, if $|z|>R$, (1.7) and (1.8) imply that

$$\begin{aligned}
\left|H_z'(t,z)\right| &\le \sum_{i=1}^{n}\left[\left(1+\frac{\sigma_i}{\tau_i}\right)|p_i|^{\frac{\sigma_i}{\tau_i}}\ln^{\eta}(1+|z|^{\xi})+\left(1+\frac{\tau_i}{\sigma_i}\right)|q_i|^{\frac{\tau_i}{\sigma_i}}\ln^{\eta}(1+|z|^{\xi})\right] \\
&\quad + 2n\xi\eta\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\ln^{\eta-1}(1+|z|^{\xi})\frac{|z|^{\xi-1}}{1+|z|^{\xi}} \\
&\le (1+b)\sum_{i=1}^{n}\left(|p_i|^{\frac{\sigma_i}{\tau_i}}+|q_i|^{\frac{\tau_i}{\sigma_i}}\right)\ln^{\eta}(1+|z|^{\xi}) \\
&\quad + 2n\xi\eta\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\ln^{\eta-1}(1+|z|^{\xi})\frac{|z|^{\xi-1}}{1+|z|^{\xi}} \\
&\le c_2\,|z|^{\lambda},
\end{aligned}$$

thus (H5) is proved.

Step 3. We will check H dissatisfies (S_1). Choosing arbitrary constants $\mu, \nu > 1$, we know that there exists an integer i_0 such that $\frac{\mu}{\nu} \neq \frac{\sigma_{i_0}}{\tau_{i_0}}$. Without loss of generality, we may assume $\frac{\mu}{\nu} > \frac{\sigma_1}{\tau_1}$. Set $p = (p_1, 0, \dots, 0)$, $q = (0, \dots, 0)$ and $z = (p, q)$, then we have

$$\begin{aligned}
&\frac{1}{\mu} H'_p(t,z) \cdot p + \frac{1}{\nu} H'_q(t,z) \cdot q - \left(\frac{1}{\mu} + \frac{1}{\nu}\right) H(t,z) \\
&= \frac{1}{\mu}\left(1 + \frac{\sigma_1}{\tau_1}\right) |p_1|^{1+\frac{\sigma_1}{\tau_1}} \ln^{\eta}(1 + |p_1|^{\xi}) + \xi\eta\, |p_1|^{1+\frac{\sigma_1}{\tau_1}} \ln^{\eta-1}(1 + |p_1|^{\xi}) \frac{|p_1|^{\xi}}{1 + |p_1|^{\xi}} \\
&\quad - \left(\frac{1}{\mu} + \frac{1}{\nu}\right) |p_1|^{1+\frac{\sigma_1}{\tau_1}} \ln^{\eta}(1 + |p_1|^{\xi}) \\
&= |p_1|^{1+\frac{\sigma_1}{\tau_1}} \ln^{\eta-1}(1 + |p_1|^{\xi}) \left[\left(\frac{\sigma_1}{\mu\tau_1} - \frac{1}{\nu}\right) \ln(1 + |p_1|) + \xi\eta \frac{|p_1|^{\xi}}{1 + |p_1|^{\xi}}\right] \\
&\to -\infty, \quad \text{as } |z| \to +\infty,
\end{aligned}$$

which violates (S_1).

Step 4. We will check H disstatisfies (S_2). Choose constants $\widehat{\alpha}_1, \dots, \widehat{\alpha}_n, \widehat{\beta}_1, \dots, \widehat{\beta}_n$ with $\frac{1}{\widehat{\alpha}_i} + \frac{1}{\widehat{\beta}_i} = \epsilon < 1$ $(i = 1, 2, \dots, n)$. Without loss of generality, we assume $\widehat{\alpha}_1 \geq \widehat{\beta}_1$.

If $\frac{\sigma_1 + \tau_1}{\widehat{\alpha}_1 \tau_1} - 1 < 0$, set $p = (p_1, 0, \dots, 0)$, $q = (0, \dots, 0)$ and $z = (p, q)$, then we have

$$\begin{aligned}
&H'_z(t,z) \cdot \widehat{V}(z) - H(t,z) \\
&= \frac{1}{\widehat{\alpha}_1}\left[\left(1 + \frac{\sigma_1}{\tau_1}\right) |p_1|^{1+\frac{\sigma_1}{\tau_1}} \ln^{\eta}(1 + |p_1|^{\xi}) + \xi\eta\, |p_1|^{1+\frac{\sigma_1}{\tau_1}} \ln^{\eta-1}(1 + |p_1|^{\xi}) \frac{|p_1|^{\xi}}{1 + |p_1|^{\xi}}\right] \\
&\quad - |p_1|^{1+\frac{\sigma_1}{\tau_1}} \ln^{\eta}(1 + |p_1|^{\xi}) \\
&= |p_1|^{1+\frac{\sigma_1}{\tau_1}} \ln^{\eta-1}(1 + |p_1|^{\xi}) \left[\left(\frac{\sigma_1 + \tau_1}{\widehat{\alpha}_1 \tau_1} - 1\right) \ln(1 + |p_1|) + \xi\eta \frac{|p_1|^{\xi}}{1 + |p_1|^{\xi}}\right] \\
&\to -\infty, \quad \text{as } |z| \to +\infty,
\end{aligned}$$

which violates (S_2).

If $\frac{\sigma_1 + \tau_1}{\widehat{\alpha}_1 \tau_1} - 1 \geq 0$, which implies that $\frac{\sigma_1 + \tau_1}{\sigma_1} \leq \frac{\widehat{\alpha}_1}{\widehat{\alpha}_1 - 1}$, then we have

$$\frac{\sigma_1 + \tau_1}{\widehat{\beta}_1 \sigma_1} - 1 = \left(\epsilon - \frac{1}{\widehat{\alpha}_1}\right) \frac{\sigma_1 + \tau_1}{\sigma_1} - 1 \leq \left(\epsilon - \frac{1}{\widehat{\alpha}_1}\right) \frac{\widehat{\alpha}_1}{\widehat{\alpha}_1 - 1} - 1 < 0.$$

Similarly, set $p = (0, \dots, 0)$, $q = (q_1, 0, \dots, 0)$ and $z = (p, q)$, then we have

$$\begin{aligned}
&H'_z(t,z) \cdot \widehat{V}(z) - H(t,z) \\
&= |q_1|^{1+\frac{\tau_1}{\sigma_1}} \ln^{\eta-1}(1 + |q_1|^{\xi}) \left[\left(\frac{\sigma_1 + \tau_1}{\widehat{\beta}_1 \tau_1} - 1\right) \ln(1 + |q_1|) + \xi\eta \frac{|q_1|^{\xi}}{1 + |q_1|^{\xi}}\right] \\
&\to -\infty, \quad \text{as } |z| \to +\infty,
\end{aligned}$$

which violates (S_2). Thus we complete the proof. $\square$

Claim 1.7. Condition (S_2) implies (H3) and (H4). That is, the super-quadratic conditions are generalized in our paper.

Proof. Set $\sigma_i = \widehat{\alpha}_i$, $\tau_i = \widehat{\beta}_i$, let

$$\omega(z) = \sum_{i=1}^{n} \left(|p_i|^{1+\frac{\sigma_i}{\tau_i}} + |q_i|^{1+\frac{\tau_i}{\sigma_i}} \right) \quad \text{and} \quad \widehat{\omega}(z) = \sum_{i=1}^{n} \left(|p_i|^{\widehat{\alpha}_i} + |q_i|^{\widehat{\beta}_i} \right),$$

we claim that $\frac{\widehat{\omega}(z)}{\omega(z)} \to +\infty$, as $|z| \to +\infty$. If not, we have that

$$\liminf_{|z|\to+\infty} \frac{\widehat{\omega}(z)}{\omega(z)} = a < +\infty.$$

Then there exist a constant $R \in \mathbb{N}^*$ and a consequence $\{z_m\}$ such that $|z_m| \to +\infty$ as $m \to +\infty$, and $\widehat{\omega}(z_m) < (a+1)\omega(z_m)$ for $m > R$. Meanwhile, note that $\frac{1}{\widehat{\alpha}_i} + \frac{1}{\widehat{\beta}_i} < 1$ $(i = 1, 2, \ldots, n)$, we have that $1 + \frac{\widehat{\alpha}_i}{\widehat{\beta}_i} = \widehat{\alpha}_i \left(\frac{1}{\widehat{\alpha}_i} + \frac{1}{\widehat{\beta}_i} \right) < \widehat{\alpha}_i$, $1 + \frac{\widehat{\beta}_i}{\widehat{\alpha}_i} < \widehat{\beta}_i$ $(i = 1, 2, \ldots, n)$. Set $z_m = (p_1^m, \ldots, p_n^m, q_1^m, \ldots, q_n^m)$, it is obvious that

$$\begin{aligned} 0 &> \widehat{\omega}(z_m) - (a+1)\omega(z_m) \\ &= \sum_{i=1}^{n} \left[\left(|p_i^m|^{\widehat{\alpha}} - (a+1) |p_i^m|^{1+\frac{\widehat{\alpha}_i}{\widehat{\beta}_i}} \right) + \left(|q_i^m|^{\widehat{\beta}_i} - (a+1) |q_i^m|^{1+\frac{\widehat{\beta}_i}{\widehat{\alpha}_i}} \right) \right] \\ &\to +\infty, \quad \text{as } m \to +\infty, \end{aligned}$$

which is a contradiction. So (S_2) implies (H3).

Next, set $\alpha_i = \frac{\widehat{\alpha}_i + \widehat{\beta}_i}{\widehat{\beta}_i}$, $\beta_i = \frac{\widehat{\alpha}_i + \widehat{\beta}_i}{\widehat{\alpha}_i}$ and $\beta = \min \left\{ \widehat{\alpha}_1, \ldots, \widehat{\alpha}_n, \widehat{\beta}_1, \ldots, \widehat{\beta}_n \right\}$, by (1.4), (1.5) and (1.6), note that $\frac{1}{\varepsilon} = \frac{\widehat{\alpha}_i \widehat{\beta}_i}{\widehat{\alpha}_i + \widehat{\beta}_i}$ $(i = 1, 2, \ldots, n)$, we have that

$$\begin{aligned} H_z'(t,z) \cdot V(z) - H(t,z) &= \frac{1}{\epsilon} \left(H_z'(t,z) \cdot \widehat{V}(z) - \epsilon H(t,z) \right) \\ &\geq \frac{1-\epsilon}{\varepsilon} H(t,z) \\ &\geq \frac{a_1(1-\epsilon)}{\varepsilon} \sum_{i=1}^{n} \left(|p_i|^{\widehat{\alpha}_i} + |q_i|^{\widehat{\beta}_i} \right) - \frac{a_2(1-\epsilon)}{\varepsilon} \\ &\geq \frac{a_1(1-\epsilon)}{2n\varepsilon} \sum_{i=1}^{n} \left(|p_i|^{\beta} + |q_i|^{\beta} \right) - \frac{a_2(1-\epsilon)}{\varepsilon} \\ &\geq \frac{a_1(1-\epsilon)}{\varepsilon(2n)^{\beta+1}} |z|^{\beta} - \frac{a_2(1-\epsilon)}{\varepsilon}, \end{aligned}$$

for $(t,z) \in \mathbb{R} \times \mathbb{R}^{2n}$ with $|z| \geq \sqrt{2n}$. So, (S_2) indicates (H4). $\square$

2. One deformation theorem and generalized critical point theorem

Firstly, we introduce some notations. We denote by E a real Banach space, by E^* its dual, and by $(\cdot,\cdot)$ the pairing between E^* and E. Let $B_R(u)$ denote the open ball in E centered at u with radius $R > 0$. For some $c \in \mathbb{R}$, we set $A_c = \{u \in E \mid I(u) \leq c\}$, $K_c = \{u \in E \mid I'(u) = \mathbf{0}, I(u) = c\}$ and $\overline{E} = \{u \in E \mid I'(u) \neq \mathbf{0}\}$.

Lemma 2.1. [3] *If functional $I \in C^1(E,\mathbb{R})$, then there exists a locally Lipschitzian continuous mapping $\phi\colon \overline{E} \to E$ satisfying the conditions*

$$\|\phi(u)\| \leq \frac{2}{\|I'(u)\|} \quad \textit{and} \quad (I'(u),\phi(u)) \geq 1,\ \forall\, u \in \overline{E}. \tag{2.1}$$

Remark 2.2. From the proof of [3, Lemma 2.4], we know that the above mapping ϕ is odd in u, if $I(u)$ is even in u.

The following Theorem 2.3 is similar to [11, Theorem A.4] except for condition (C), also similar to [3, Theorem 2.1] except for the following result (4) and (8).

Theorem 2.3. *Let $I \in C^1(E,\mathbb{R})$ and satisfy the condition* (C). *If $c \in \mathbb{R}$, $\overline{\varepsilon} > 0$ small enough, and N is any neighborhood of K_c, then there exists an $\varepsilon \in (0,\overline{\varepsilon})$ and $\eta \in C([0,1]\times E, E)$ such that*

(1) $\eta(0,u) = u$, $\forall\, u \in E$,

(2) $\eta(t,u) = u$, $\forall\, t \in [0,1]$ *and* $I(u) \notin [c-\overline{\varepsilon}, c+\overline{\varepsilon}]$,

(3) $\eta(t,u)$ *is a homeomorphism of E onto E for each* $t \in [0,1]$,

(4) $\|\eta(t,u) - u\| \leq k_1 + k_2\|u\|$, *where $t \in [0,1]$, $k_1 > 0$ and $k_2 > 0$ are constants independent of u, thus $\eta\colon [0,1]\times E \to E$ is a bounded mapping,*

(5) $I(\eta(t,u)) \leq I(u)$, $\forall\, t \in [0,1]$ *and* $u \in E$,

(6) $\eta(1, A_{c+\varepsilon} \setminus N) \subset A_{c-\varepsilon}$,

(7) *if* $K_c = \varnothing$, *then* $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$,

(8) *if $I(u)$ is even in u, then $\eta(t,u)$ is odd in u.*

Proof. The idea comes from [8] and [11, pp. 82–85]. We assume $K_c \neq \varnothing$.

First of all, we observe that K_c is compact via condition (C). Let M_σ denote the σ-neighborhood of K_c, i.e., $M_\sigma = \{u \in E \mid \|u - K_c\| < \sigma\}$. We choose σ suitable small such that $M_\sigma \subset N$, therefore it suffices to prove (6) with N replaced by M_σ. Choosing $R > 0$

large enough such that $(A_{c+\widehat{\varepsilon}} \setminus A_{c-\widehat{\varepsilon}}) \cap (B_R(\mathbf{0}) \setminus M_{\sigma/8}) \neq \varnothing$, from the condition (C), we can claim that there exist constants $b > 0$ and $\widehat{\varepsilon} > 0$ such that

$$\left\|I'(u)\right\| > b, \quad u \in (A_{c+\widehat{\varepsilon}} \setminus A_{c-\widehat{\varepsilon}}) \cap (B_R(\mathbf{0}) \setminus M_{\sigma/8}). \tag{2.2}$$

If (2.2) does not hold, then there exists a sequence $\{u_m\}$ such that

$$u_m \in (A_{c+1/m} \setminus A_{c-1/m}) \cap (B_R(\mathbf{0}) \setminus M_{\sigma/8}) \quad \text{and} \quad \left\|I'(u_m)\right\| \to 0 \text{ as } m \to +\infty.$$

Therefore we can get that $\{I(u_m)\}$ is bounded and $(1 + \|u_m\|) \|I'(u_m)\| \to 0$ as $m \to +\infty$. By the condition (C), there exists a subsequence of $\{u_m\}$ converging to $u \in K_c \setminus M_{\sigma/8}$. But $K_c \setminus M_{\sigma/8} = \varnothing$, hence (2.2) holds. Similarly, we can get

$$\left\|I'(u)\right\| > b, \quad u \in (A_{c+\widehat{\varepsilon}} \setminus A_{c-\widehat{\varepsilon}}) \cap (M_\sigma \setminus M_{\sigma/8}) \tag{2.3}$$

and

$$\left\|I'(u)\right\| > 0, \quad u \in (A_{c+2\widehat{\varepsilon}} \setminus A_{c-2\widehat{\varepsilon}}) \setminus M_{\sigma/10}. \tag{2.4}$$

Since (2.2), (2.3) and (2.4) still hold if $\widehat{\varepsilon}$ decreases, we can assume

$$0 < \widehat{\varepsilon} < \min\left\{\frac{b\sigma}{8}, \sigma, \overline{\varepsilon}, \frac{1}{2}\right\}. \tag{2.5}$$

Choosing any $\varepsilon \in (0, \widehat{\varepsilon})$, we set $A = \{u \in E \mid I(u) \geq c + \widehat{\varepsilon} \text{ or } I(u) \leq c - \widehat{\varepsilon}\}$, $B = \{u \in E \mid c - \varepsilon \leq I(u) \leq c + \varepsilon\}$, and the function

$$f(u) = \frac{\|u - A\|}{\|u - A\| + \|u - B\|},$$

then $f = 0$ on A, $f = 1$ on B, and $0 \leq f(u) \leq 1$, $\forall\, u \in E$. It is obvious that f is locally Lipschitzian continuous. Similarly, there is a Lipschitzian continuous function $g(u) = \frac{\|u - M_{\sigma/8}\|}{\|u - M_{\sigma/8}\| + \|u - E \setminus M_{\sigma/4}\|}$ with $0 \leq g(u) \leq 1$, $\forall\, u \in E$. Note that if I is even, sets A, B and M_σ will be symmetric with respect to the origin, so f and g are even functions. Set $\Psi(u) = f(u) \cdot g(u)$, $\forall\, u \in E$ then

$$\Psi(u) = \begin{cases} 0, & \text{if } u \notin I^{-1}((c - \widehat{\varepsilon}, c + \widehat{\varepsilon})) \text{ or } u \in M_{\sigma/8}, \\ 1, & \text{if } u \in I^{-1}([c - \varepsilon, c + \varepsilon]) \text{ and } u \notin M_{\sigma/4}. \end{cases} \tag{2.6}$$

Furthermore, consider the mapping $V_0 \colon E \to E$ defined by

$$V_0(u) = \begin{cases} -\Psi(u)\phi(u), & u \in \overline{E}, \\ \mathbf{0}, & u \notin \overline{E}, \end{cases}$$

where ϕ is defined by Lemma 2.1. Obviously, V_0 is locally Lipschitzian continuous in E and V_0 is odd if I is even. By the first inequality in (2.1), we have

$$\|V_0(u)\| \le \frac{2}{\|I'(u)\|}, \quad \forall\, u \in \overline{E}. \tag{2.7}$$

Next, we shall show that there exist two constants $k_1 > 0$ and $k_2 > 0$ such that

$$\|V_0(u)\| \le k_1 + k_2 \|u\|. \tag{2.8}$$

If $u \notin I^{-1}((c-\widehat{\varepsilon}, c+\widehat{\varepsilon})) \setminus M_{\sigma/8}$, then $V_0(u) = \mathbf{0}$, so (2.8) is trivial. Thus, we can suppose that $u \in I^{-1}((c-\widehat{\varepsilon}, c+\widehat{\varepsilon})) \setminus M_{\sigma/8}$. For R in (2.2), if $\|u\| \le R$, then $\|V_0(u)\|$ is bounded via (2.3) and (2.7). If $\|u\| \ge R$, we can claim that there exists a constant $\delta > 0$ such that $\|I'(u)\| \ge \delta/\|u\|$. Otherwise, there exists a sequence $\{u_m\}$ such that $u_m \in I^{-1}((c-\widehat{\varepsilon}, c+\widehat{\varepsilon})) \setminus M_{\sigma/8}$, $\|u_m\| > m$ and $\|u_m\|\,\|I'(u_m)\| < 1/m$ (m large enough), we can get that the sequence $\{u_m\}$ has a convergent subsequence via the condition (C), which contradicts to $\|u_m\| > m$. So we get $\|V_0(u)\| \le \frac{2}{\delta}\|u\|$ via (2.7). So we conclude that (2.8) holds everywhere.

Consider the following initial value problem,

$$\begin{cases} \dfrac{\mathrm{d}\eta(t,u)}{\mathrm{d}t} = V_0(\eta(t,u)), \\ \eta(0,u) = u. \end{cases} \tag{2.9}$$

The basic existence-uniqueness theorem for ordinary differential equations implies that for each $u \in E$, (2.9) has a unique solution defined for t in a maximal interval (t^-, t^+). As usual argument, we have $t^{\pm} = \pm\infty$.

The continuous dependence of solution of (2.9) on the initial value u implies $\eta \in C([0,1]\times E, E)$ and (2.9) implies (1) holds. Since $\overline{\varepsilon} > \widehat{\varepsilon}$, $V_0(u) = \mathbf{0}$ on A, so (2) is true. The semigroup property for solutions of (2.9) gives (3). Integrating (2.9) on $[0,t] \subseteq [0,1]$, using (2.8) and (1), we have

$$\|\eta(t,u) - \eta(0,u)\| \le (k_1 + k_2\|u\|)\,|t| \le k_1 + k_2 \|u\|.$$

Hence (4) holds. An argument similar to that in the proof of [3, Theorem 2.1] shows that (5) and (6) hold. If $I(u)$ is even in u, we know that $V_0(u)$ is odd in u. We can get $\eta(t,u)$ is also odd in u via the basic existence-uniqueness theorem for ordinary differential equations, hence (8) holds. □

Remark 2.4. (2.1) can be replaced by

$$\|\phi(u)\| \le \frac{\alpha}{\|I'(u)\|}, \tag{2.10}$$

$$(I'(u), \phi(u)) \ge \beta, \tag{2.11}$$

where $\alpha > \beta > 0$. Moreover, the proof of Theorem 2.3 is essentially unchanged aside from replacing (2.5) by $0 < \widehat{\varepsilon} < \min\left\{\overline{\varepsilon}, \sigma, \frac{b\sigma}{4\alpha}, \frac{1}{\alpha}\right\}$.

The following result is similar to [11, Proposition A.18], the difference is the (PS) condition replaced by the condition (C).

Lemma 2.5. *Suppose E is a real Hilbert space, $I \in C^1(E, \mathbb{R})$ satisfies the condition* (C)*, where $I(u) = \frac{1}{2}(Lu, u) + \varphi(u)$, L is self-adjoint and φ' is compact. Then*

$$\eta(t, u) = \exp(\theta(t, u)L)u + K(t, u),$$

where $\theta \in C([0, 1] \times E, [0, 1/b^2])$ and $K\colon [0, 1] \times E \to E$ is compact.

Proof. The idea comes from [11]. Because the (PS) condition is replaced with the condition (C), we must modify the proof in [11].

The mapping η is determined as the solution of the initial value problem

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\Psi(\eta)\phi(\eta), \quad \eta(0, u) = u, \tag{2.12}$$

where Ψ is the mapping defined in (2.6), $0 \leq \Psi(\eta) \leq 1$ and ϕ is the mapping defined in Remark 2.4 with $\alpha = 2$, $\beta = 1/2$.

Case 1. If $u \notin D := \left\{u \in E \mid I(u) \in [c - \widehat{\varepsilon}, c + \widehat{\varepsilon}] \text{ and } u \notin M_{\sigma/8}\right\}$, then $\Psi(u) = 0$. From the basic existence-uniqueness theorem for ordinary differential equations, we know that $\eta(t, u) \equiv u \notin D$, $\forall\, t \in \mathbb{R}$. Thus, the orbit $\eta(t, u)$ cannot enter D for $t \in \mathbb{R}$.

Case 2. If $u \in D$, we can claim that the orbit $\eta(t, u)$ cannot leave D for $t \in \mathbb{R}$. Otherwise, for some t_0, $\eta(t_0, u) \notin D$. Setting $\eta(t_0, u) = u_0$, we can check that $\overline{\eta}(t, u) \equiv u_0$ is a solution to the ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}\eta}{\mathrm{d}t} = -\Psi(\eta)\phi(\eta), \\ \eta(t_0, u) = u_0. \end{cases}$$

From the basic existence-uniqueness theorem for ordinary differential equations, we have the solution $\eta(t, u) \equiv u_0 \notin D$, $\forall\, t \in \mathbb{R}$. This contradicts to the fact that $\eta(0, u) = u \in D$.

Considering Case 2, ϕ need only be defined on D. We note that $\|I'(u)\| > 0$ on D via (2.4). We claim such a ϕ can be chosen so that

$$\phi(u) = \frac{Lu + W(u)}{\|I'(u)\|^2}, \quad \forall\, u \in D, \tag{2.13}$$

where $W\colon E \to E$ is compact. We will prove (2.13) in Lemma 2.7 later. Assuming this for the moment, (2.12) becomes

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} + \Psi(\eta)\frac{L\eta}{\|I'(\eta)\|^2} = -\Psi(\eta)\frac{W(\eta)}{\|I'(\eta)\|^2}.$$

Considering η in the argument of Ψ, I' and W as being known, η satisfies an inhomogeneous linear equation and therefore it can be represented as

$$\eta(t,u) = \exp\left(\left(-\int_0^t \frac{\Psi(\eta(s,u))}{\|I'(\eta(s,u))\|^2}\,\mathrm{d}s\right) L\right) u + \overline{K}(t,u), \quad \forall\, u \in D,$$

where $\overline{K}\colon [0,1]\times D \to E$ is defined by

$$\begin{aligned}\overline{K}(t,u) = &-\exp\left(\left(-\int_0^t \frac{\Psi(\eta(s,u))}{\|I'(\eta(s,u))\|^2}\,\mathrm{d}s\right) L\right)\\ &\times \int_0^t \left[\exp\left(\left(\int_0^\tau \frac{\Psi(\eta(s,u))}{\|I'(\eta(s,u))\|^2}\,\mathrm{d}s\right) L\right) \frac{\Psi(\eta(\tau,u))W(\eta(\tau,u))}{\|I'(\eta(\tau,u))\|^2}\right] \mathrm{d}\tau.\end{aligned}$$

Now, we define a functional ψ on $[0,1]\times E$ as follows:

$$\psi(t,u) = \begin{cases} \dfrac{\Psi(\eta(t,u))}{\|I'(\eta(t,u))\|^2}, & u \in D,\\ 0, & \text{otherwise}.\end{cases}$$

By the definition of Ψ and (2.4), it is obvious that $\psi \in C([0,1]\times E, E)$.

So we have that $\eta(t,u)$ has the following form

$$\eta(t,u) = \exp\left(\left(\int_0^t -\psi(s,u)\,\mathrm{d}s\right) L\right) u + K(t,u), \quad u \in E,$$

where $K\colon [0,1]\times E \to E$ is defined by

$$\begin{aligned}K(t,u) = &-\exp\left(\left(\int_0^t -\psi(s,u)\,\mathrm{d}s\right) L\right)\\ &\times \int_0^t \left[\exp\left(\left(\int_0^\tau \psi(\tau,u)\,\mathrm{d}s\right) L\right) \psi(\tau,u) W(\eta(\tau,u))\right] \mathrm{d}\tau.\end{aligned}$$

To see that $K\colon [0,1]\times E \to E$ is compact, suppose $F \subset E$ is bounded. Without loss of generality, we may assume $F = B_{R_1}(\mathbf{0})$ for every fixed $R_1 > 0$. From Theorem 2.3(4), $\eta([0,1]\times B_{R_1}(\mathbf{0})) \subset B_{R_2}(\mathbf{0})$, where $R_2 = R_1 + k_1 + k_2 R_1$. Therefore $W(\eta([0,1]\times B_{R_1}(\mathbf{0}))) \subset W(B_{R_2}(\mathbf{0})) \subset \overline{W(B_{R_2}(\mathbf{0}))}$.

(i) If $D \cap B_{R_1}(\mathbf{0}) = \varnothing$, we know that $K(t, B_{R_1}(\mathbf{0})) = \mathbf{0}$.

(ii) If $D \cap B_{R_1}(\mathbf{0}) \neq \varnothing$, from (i), we have that $K(t, B_{R_1}(\mathbf{0})) = K(t, D\cap B_{R_1}(\mathbf{0})) \cup \{\mathbf{0}\}$, we only need check $K(t, D\cap B_{R_1}(\mathbf{0}))$ is compact. It is similar to (2.2), we can get

$$\|I'(u)\| \geq b, \quad \forall\, u \in D \cap B_{R_1}(\mathbf{0}). \tag{2.14}$$

For any fixed $t \in [0,1]$, we set $Y_t = \left\{\exp((\gamma - \gamma_t)L)wz \;\middle|\; \gamma, w \in [0, 1/b^2], z \in \overline{W(B_{R_2}(\mathbf{0}))}\right\}$, where γ_t is a constant and $\gamma_t \in [0, 1/b^2]$. Since the mapping

$$(\gamma, w, z) \to \exp((\gamma - \gamma_t)L)wz$$

is a continuous function on the compact set $[0,1/b^2]^2 \times \overline{W(B_{R_2}(\mathbf{0}))}$, its range Y_t is compact. Therefore the closed convex hull $\widehat{Y_t}$ of Y_t is also compact. For every fixed $u \in D \cap B_{R_1}(\mathbf{0})$ and $\forall\, \tau \in [0,t]$, we have $\int_0^\tau \varphi(s,u)\,\mathrm{d}s \in [0,1/b^2]$ and $\int_0^t \varphi(s,u)\,\mathrm{d}s \in [0,1/b^2]$ via the definition of the functional Ψ and (2.14), we can get

$$z_{t,u}(\tau) := \exp\left(\left(\int_0^\tau \psi(s,u)\,\mathrm{d}s - \int_0^t \psi(s,u)\,\mathrm{d}s\right) L\right) \psi(\tau,u) W(\eta(\tau,u)) \in Y_t.$$

Hence, $\int_0^t z_{t,u}(\tau)\,\mathrm{d}\tau \in \widehat{Y_t}$. From (i) and (ii), we can get K is compact.

Finally, we can choose $\theta(t,u) = -\int_0^t \psi(s,u)\,\mathrm{d}s$, it is obvious that $\theta \in C([0,1] \times E, [0,1/b^2])$. $\square$

Next, we will prove that (2.13) holds, to this end, we first prove the following lemma.

Lemma 2.6. *Let E be a real Hilbert space and operator $T\colon E \to E$ be compact. Then given any γ, there exists a mapping $\widehat{T}\colon E \to E$ such that $\widehat{T}$ is compact, locally Lipschitzian continuous, and*

$$\left\|T(u) - \widehat{T}(u)\right\| \le \frac{\gamma}{1+\|u\|}, \quad \forall\, u \in E.$$

Proof. The proof is similar to [11, Proposition A.23], except that we need replace the open covering $\{S_u \mid u \in E\}$ with the open covering $\left\{\overline{S}_u \mid u \in E\right\}$, where $\overline{S}_u := B_1(u) \cap \left\{v \in E \;\middle|\; \|T(u) - T(v)\| < \frac{\gamma}{1+R_u}, R_u = \sup_{v \in B_1(u)} \{\|v\|\}\right\}$. $\square$

To complete the proof of Lemma 2.5, we need the following lemma.

Lemma 2.7. *Suppose E is a real Hilbert space, $I \in C^1(E,\mathbb{R})$ satisfies the condition* (C), *$I(u) = \frac{1}{2}(Lu,u) + \varphi(u)$, L is self-adjoint and φ' is compact. Then there exists a locally Lipschitzian continuous mapping $\phi\colon D \to E$ defined by*

$$\phi(u) = \frac{Lu + W(u)}{\|I'(u)\|^2},$$

where $W\colon E \to E$ is compact and ϕ satisfies (2.10) *and* (2.11) *with $\alpha = 2$ and $\beta = 1/2$.*

Proof. It is similar to (2.2), there exist constants $h > 0$ and $\widehat{\varepsilon}_0 > 0$ such that

$$(1 + \|u\|)\left\|I'(u)\right\| \ge h, \quad \forall\, u \in (A_{c+\widehat{\varepsilon}_0} \setminus A_{c-\widehat{\varepsilon}_0}) \setminus M_{\sigma/8}. \tag{2.15}$$

Since (2.15) still holds if $\widehat{\varepsilon}_0$ decreases, we can set $\widehat{\varepsilon}_0 = \widehat{\varepsilon}$, so we have that

$$(1 + \|u\|)\left\|I'(u)\right\| \ge h, \quad \forall\, u \in D. \tag{2.16}$$

Next, we will check $\phi(u) = \frac{Lu+W(u)}{\|I'(u)\|^2}$ satisfies (2.10) and (2.11) with $\alpha = 2$ and $\beta = 1/2$ on D. Set $T(u) = \varphi'(u)$ and $\gamma = h/2$, from Lemma 2.6, we know that there exists a mapping $W\colon E \to E$ such that W is compact, locally Lipschitzian continuous, and

$$\left\|\varphi'(u) - W(u)\right\| \le \frac{h}{2(1+\|u\|)}, \quad \forall\, u \in E.$$

From the definition of ϕ, we know that ϕ is Lipschitzian continuous. For every $u \in D$, using (2.16), we know that

$$\begin{aligned}\|\phi(u)\| &= \frac{\|Lu + W(u)\|}{\|I'(u)\|^2} \le \frac{\|Lu + \varphi'(u)\|}{\|I'(u)\|^2} + \frac{\|\varphi'(u) - W(u)\|}{\|I'(u)\|^2} \\ &\le \frac{1}{\|I'(u)\|} + \frac{h}{2(1+\|u\|)\,\|I'(u)\|^2} \le \frac{2}{\|I'(u)\|}\end{aligned}$$

and

$$\begin{aligned}(I'(u), \phi(u)) &= \left(I'(u), \frac{Lu + \varphi'(u) - \varphi'(u) + W(u)}{\|I'(u)\|^2}\right) = 1 - \left(I'(u), \frac{\varphi'(u) - W(u)}{\|I'(u)\|^2}\right) \\ &\ge 1 - \frac{1}{2}\frac{h}{(1+\|u\|)\,\|I'(u)\|} \ge 1 - \frac{1}{2} = \frac{1}{2}.\end{aligned}$$

Thus we complete the proof. □

We have completed the proof of Lemma 2.5 via Lemmas 2.6 and 2.7. Next, we will give a generalized critical point theorem under the condition (C) weaker than (PS) condition.

Theorem 2.8. *Let E be a real Hilbert space with $E = E_1 \oplus E_2$. Suppose $I \in C^1(E, \mathbb{R})$ with $I(z) = \frac{1}{2}(Lz, z) + \varphi(z)$ satisfying the condition* (C) *and*

- (I1) *L is a linear, bounded and self-adjoint operator,*
- (I2) *φ' is compact,*
- (I3) *$B(v) = P_2 B_1^{-1} \exp(vL) B_2 \colon E_2 \to E_2$ is invertible for any $v \in [0, +\infty)$, where $P_2 \colon E \to E_2$ is the projective operator, $B_k \colon E \to E$ ($k = 1, 2$) is linear, bounded and invertible.*
- (I4) *there exists a constant $\kappa > 0$ such that*
  - (i) *$S = \{B_1 z \mid z \in E_1, \|z\| = \varrho\}$ and $I|_S \ge \kappa$,*
  - (ii) *$Q = \{B_2(se + z) \mid 0 \le s \le r, \|z\| \le M, z \in E_2\}$ and $I|_{\partial Q} \le 0$, where $e = (p^+, q^+) \in E_1$, $e \ne \mathbf{0}$, $r > \frac{\varrho}{\|B_1^{-1} B_2 e\|}$, $M > \varrho$ and ∂Q refers to the boundary of Q relative to $\{B_2(se + z) \mid s \in \mathbb{R}, z \in E_2\}$, $\varrho > 0$ is a certain constant.*

Then I possesses a critical value $c = \inf_{h \in \Gamma} \sup_{z \in Q} I(h(1, z)) \ge \kappa$, where Γ is defined as

$$\Gamma := \{h \in C([0,1] \times E, E) \mid h \text{ satisfies } (\Gamma_1)\text{–}(\Gamma_3)\},$$

where

- (Γ_1) *$h(0, z) = z$, $z \in Q$,*
- (Γ_2) *$h(t, z) = z$, $z \in \partial Q$,*
- (Γ_3) *$h(t, z) = \exp(\theta(t, z)L)z + K(t, z)$, where $\theta \in C([0,1] \times E, [0, +\infty))$ transforms bounded sets into bounded sets and $K \colon [0,1] \times E \to E$ is compact.*

Proof. The idea comes from [11].

Paper [8] shows that (I3) and (I4) imply

$$h(1,Q) \cap S \neq \varnothing, \quad \forall\, h \in \Gamma. \tag{2.17}$$

By (2.17) and (i) of (I4), we have that $c \geq \kappa$. Book [11, p. 33] shows that (I2) and (I4) imply $c < +\infty$.

Next, we claim that Γ is an invariant set under $\eta(t,\cdot)$, where $\eta(t,\cdot)\colon E \to E$ is the mapping in Theorem 2.3. Because the (PS) condition is replaced by the (C) condition, we must first show that $\eta \in \Gamma$. In fact, η satisfies (Γ_1) and (Γ_3) via Theorem 2.3(1) and Lemma 2.5. From the choice of $\bar{\varepsilon}$, Theorem 2.3(2), the condition (ii) of (I4) and the fact $c \geq \kappa > 0$, we know that (Γ_2) holds. If $h \in \Gamma$, [11, p. 33] shows that $\eta(t,h(t,u)) \in \Gamma$.

Using the usual arguments and the above claim, we can prove that c is a critical value of the functional I. The proof can be found in [11, p. 33], so we omit it. $\square$

3. Applications to the Hamiltonian systems

After making change of variables $\varsigma = t/\omega$ with $\omega = T/(2\pi)$, we seek T-periodic solutions of the system (1.1) which correspond to 2π-periodic solutions of the system

$$\begin{cases} \dot{p}(\varsigma) = -\omega H_q'(\omega\varsigma, z), \\ \dot{q}(\varsigma) = \omega H_p'(\omega\varsigma, z). \end{cases}$$

We can hence-force focus our attention on 2π-periodic solutions of the system (1.1).

We introduce some notations and conclusions which are used later:

$$E := W^{\frac{1}{2},2}(S^1,\mathbb{R}^{2n}) = \left\{ z \in L^2(S^1,\mathbb{R}^{2n}) \;\middle|\; \|z\|^2 = \pi \sum_{j\in\mathbb{Z}} |j|\,|a_j|^2 + |a_0|^2 < +\infty \right\},$$

where $S^1 := \mathbb{R}/2\pi\mathbb{Z}$, $z(t) = \sum_{j\in\mathbb{Z}} a_j \exp(\mathrm{i}\,jt)$, $a_j \in \mathbb{C}^{2n}$.

$$\begin{aligned} E^+ &:= \overline{\operatorname{span}}^E \left\{ (\sin jt)e_k - (\cos jt)e_{k+n}, (\cos jt)e_k + (\sin jt)e_{k+n}, j \in \mathbb{N}^*, 1 \leq k \leq n \right\}, \\ E^0 &:= \mathbb{R}^{2n}, \\ E^- &:= \overline{\operatorname{span}}^E \left\{ (\sin jt)e_k + (\cos jt)e_{k+n}, (\cos jt)e_k - (\sin jt)e_{k+n}, j \in \mathbb{N}^*, 1 \leq k \leq n \right\}, \end{aligned}$$

where $\{e_k\}_{1\leq k\leq 2n}$ is the canonical basis in $\mathbb{R}^{2n}$. Set

$$B[z,\zeta] := \int_0^{2\pi} \zeta \cdot (-J\dot{z})\,\mathrm{d}t \quad \text{and} \quad A(z) := \frac{1}{2}B[z,z] = \int_0^{2\pi} p \cdot \dot{q}\,\mathrm{d}t,$$

for $z = (p,q)$, $\zeta \in C^\infty(S^1,\mathbb{R}^{2n})$, both of which can be continuously extended onto E. So B is a bounded bilinear form.

Set $E_1 = E^+$, $E_2 = E^0 \oplus E^-$ and $L_k\colon E_k \to E_k$, $(L_k z, \zeta) = B[z, \zeta]$ $(k = 1, 2)$, where $(\cdot, \cdot)$ denotes the induced inner product. References [10] and [11] indicate the following conclusions. $E = E^+ \oplus E^0 \oplus E^- = E_1 \oplus E_2$, and E^+, E^0 and E^- are orthogonal and B-orthogonal respectively. A is positive on E^+, null on E^0 and negative on E^-. If $z = z^+ + z^0 + z^-$, then $A(z) = \frac{1}{2}(Lz, z) = A(z^+) + A(z^-)$ and $\|z\|^2 = A(z^+) + |z^0|^2 - A(z^-)$, where $Lz := L_1 P_1 z + L_2 P_2 z$ and $P_k\colon E \to E_k$ $(k = 1, 2)$ is the projective operator.

Lemma 3.1. [11, Proposition 6.6] *E can be compactly embedded into $L^s(S^1, \mathbb{R}^{2n})$ $(s \geq 1)$, in particular, there exists a constant $C_s > 0$ such that $\|z\|_{L^s} \leq C_s \|z\|$ holds for $z \in E$.*

Set $I(z) = A(z) - \int_0^{2\pi} H(t, z)\,\mathrm{d}t = \frac{1}{2}(Lz, z) + \varphi(z)$, book [11] tells us that finding 2π-periodic solutions of the system (1.1) is equivalent to finding critical points of the functional $I(z)$ in E. Also, book [11] indicates that $I \in C^1(E, \mathbb{R})$ satisfies (I1) and (I2) in Theorem 2.8, if H satisfies (H1) and (H5).

Choose a fixed

$$e = (p^+, q^+) = (p_1^+, \dots, p_n^+, q_1^+, \dots, q_n^+) \in E^+$$

satisfying $\|e\| = 1$, set $\widehat{E} = \operatorname{span}\{e\} \oplus E_2$ and $W = \{z \in \widehat{E} \mid 1 \leq \|z\| \leq 2 \text{ and } \|z^-\| \leq \|z^+ + z^0\|\}$.

Lemma 3.2. [12] *There exists a constant $\varepsilon_1 > 0$ such that*

$$\operatorname{measure}\{t \in [0, 2\pi] \mid |z(t)| \geq \varepsilon_1\} \geq \varepsilon_1, \quad z \in W.$$

Lemma 3.3. *Functional I satisfies the condition* (C), *if function H satisfies* (H1), (H4) *and* (H5).

Proof. The idea comes from [9].

Condition (H4) implies that

$$\begin{aligned}
&I(z) - I'(z) \cdot V(z) \\
&= A(z) - \int_0^{2\pi} (-J\dot{z}) \cdot V(z)\,\mathrm{d}t + \int_0^{2\pi} \big(H_z'(t, z) \cdot V(z) - H(t, z)\big)\,\mathrm{d}t \\
&= A(z) - \int_0^{2\pi} \sum_{i=1}^{n} \dot{q}_i \cdot p_i\,\mathrm{d}t + \int_0^{2\pi} \big(H_z'(t, z) \cdot V(z) - H(t, z)\big)\,\mathrm{d}t \\
&= A(z) - \int_0^{2\pi} p \cdot \dot{q}\,\mathrm{d}t + \int_0^{2\pi} \big(H_z'(t, z) \cdot V(z) - H(t, z)\big)\,\mathrm{d}t \\
&= \int_0^{2\pi} \big(H_z'(t, z) \cdot V(z) - H(t, z)\big)\,\mathrm{d}t \\
&\geq c_1 \int_0^{2\pi} |z|^\beta\,\mathrm{d}t - 2\pi c_2.
\end{aligned} \tag{3.1}$$

Let $\{z_m\}$ be a (C) sequence, that is, $\{I(z_m)\}$ is bounded and $(1+\|z_m\|)\,\|I'(z_m)\| \to 0$ as $m \to +\infty$. We first claim that $\{z_m\}$ is bounded. If not, there exists a subsequence $\{z_{m_k}\}$ of sequence $\{z_m\}$ such that $\|z_{m_k}\| \to +\infty$ as $k \to +\infty$. For simplicity of notations, we use sequence $\{z_m\}$ represent subsequence $\{z_{m_k}\}$, so we have $\|I'(z_m)\| \to 0$ as $m \to +\infty$.

Inequality (3.1) implies that

$$\int_0^{2\pi} |z_m|^\beta \, \mathrm{d}t \le d_1, \tag{3.2}$$

where d_1 is a positive constant. For β, λ in (H4) and (H5), set $p = \frac{2\beta+1}{2\lambda-1} > 1$ and $q = \frac{p}{p-1} = \frac{2\beta+1}{2(\beta+1-\lambda)}$, then we have $\lambda - \frac{\beta}{p} = \frac{\lambda+\beta}{2\beta+1}$ and $2q(\lambda - \frac{\beta}{p}) = \frac{\lambda+\beta}{\beta-\lambda+1}$. Hölder's inequality, Lemma 3.1 and (3.2) imply that

$$\begin{aligned}
&\int_0^{2\pi} |z_m|^\lambda \left|z_m^+\right| \mathrm{d}t \\
&= \int_0^{2\pi} |z_m|^{\frac{\beta}{p}} |z_m|^{\lambda-\frac{\beta}{p}} \left|z_m^+\right| \mathrm{d}t \\
&\le \left(\int_0^{2\pi} \left(|z_m|^{\frac{\beta}{p}}\right)^p \mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^{2\pi} |z_m|^{(\lambda-\frac{\beta}{p})q} \left|z_m^+\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \left(\int_0^{2\pi} |z_m|^\beta \, \mathrm{d}t\right)^{\frac{1}{p}} \left(\int_0^{2\pi} \left(|z_m|^{\lambda-\frac{\beta}{p}}\right)^{2q} \mathrm{d}t\right)^{\frac{1}{2q}} \left(\int_0^{2\pi} \left|z_m^+\right|^{2q} \mathrm{d}t\right)^{\frac{1}{2q}} \\
&\le d_1^{\frac{1}{p}} \|z_m\|_{L^{\frac{\beta+\lambda}{\beta-\lambda+1}}}^{\frac{\beta+\lambda}{2\beta+1}} \left\|z_m^+\right\|_{L^{2q}} \\
&\le d_1^{\frac{1}{p}} C \|z_m\|^{\frac{\beta+\lambda}{2\beta+1}} \left\|z_m^+\right\|,
\end{aligned} \tag{3.3}$$

where $C > 0$ is the product of two powers of embedding constant in Lemma 3.1. Lemma 3.1, (3.3) and (H5) imply that

$$\begin{aligned}
\left\|I'(z_m)\right\| \left\|z_m^+\right\| &\ge I'(z_m) \cdot z_m^+ = A'(z_m) \cdot z_m^+ - \int_0^{2\pi} H_z'(t, z_m) \cdot z_m^+ \, \mathrm{d}t \\
&= (Lz_m, z_m^+) - \int_0^{2\pi} H_z'(t, z_m) \cdot z_m^+ \, \mathrm{d}t \\
&\ge 2\left\|z_m^+\right\|^2 - \int_0^{2\pi} \left|H_z'(t, z_m)\right| \cdot \left|z_m^+\right| \mathrm{d}t \\
&\ge 2\left\|z_m^+\right\|^2 - \int_0^{2\pi} (c_2 |z_m|^\lambda + c_2) \left|z_m^+\right| \mathrm{d}t \\
&\ge 2\left\|z_m^+\right\|^2 - c_2 d_1^{\frac{1}{p}} C \|z_m\|^{\frac{\beta+\lambda}{2\beta+1}} \left\|z_m^+\right\| - c_2 C_1 \left\|z_m^+\right\|,
\end{aligned} \tag{3.4}$$

where C_1 is the embedding constant in Lemma 3.1. (3.4) implies that

$$\left\|z_m^+\right\| \le \left\|I'(z_m)\right\| + c_2 d_1^{\frac{1}{p}} C \|z_m\|^{\frac{\beta+\lambda}{2\beta+1}} + c_2 C_1. \tag{3.5}$$

Since $0 < \frac{\beta+\lambda}{2\beta+1} < 1$ and $\|I'(z_m)\| \to 0$ as $m \to +\infty$, (3.5) implies that

$$\frac{\|z_m^+\|}{\|z_m\|} \to 0, \quad \text{as } m \to +\infty. \tag{3.6}$$

Similarly for z_m^-, we can obtain that

$$\frac{\|z_m^-\|}{\|z_m\|} \to 0, \quad \text{as } m \to +\infty. \tag{3.7}$$

Since E^0 is finite-dimensional, there exists $d_2 > 0$ such that

$$\|u\| \le d_2 \|u\|_{L^2}, \quad \text{for all } u \in E^0. \tag{3.8}$$

(3.8), Hölder's inequality, (3.2) and Lemma 3.1 imply that

$$\begin{aligned} \frac{1}{d_2^2}\left\|z_m^0\right\|^2 &\le \int_0^{2\pi} \left|z_m^0\right|^2 \mathrm{d}t \le \int_0^{2\pi} |z_m|^2 \mathrm{d}t = \int_0^{2\pi} |z_m|^{\frac{\beta}{\beta+1}} |z_m|^{\frac{\beta+2}{\beta+1}} \mathrm{d}t \\ &\le \left(\int_0^{2\pi} |z_m|^\beta \mathrm{d}t\right)^{\frac{1}{\beta+1}} \left(\int_0^{2\pi} |z_m|^{\frac{\beta+2}{\beta}} \mathrm{d}t\right)^{\frac{\beta}{\beta+1}} \le d_1^{\frac{1}{\beta+1}} C_{\frac{\beta+2}{\beta}} \|z_m\|^{\frac{\beta+2}{\beta+1}}, \end{aligned} \tag{3.9}$$

where $C_{\frac{\beta+2}{\beta}}$ is embedding constant in Lemma 3.1. (3.9) implies that

$$\frac{\left\|z_m^0\right\|}{\|z_m\|} \to 0, \quad \text{as } m \to +\infty. \tag{3.10}$$

Hence, (3.6), (3.7) and (3.10) imply that

$$1 = \frac{\|z_m^+\|^2 + \left\|z_m^0\right\|^2 + \|z_m^-\|^2}{\|z_m\|^2} \to 0 \quad \text{as } m \to +\infty,$$

which is a contradiction. Hence $\{z_m\}$ must be bounded.

Now we show that $\{z_m\}$ has a convergent subsequence. We may suppose that $z_m \rightharpoonup z$ in E as $m \to +\infty$. Since $2\|z_m^+ - z^+\|^2 = (I'(z_m) - I'(z)) \cdot (z_m^+ - z^+) + \int_0^{2\pi}(H_z'(t, z_m) - H_z'(t, z)) \cdot (z_m^+ - z^+)\,\mathrm{d}t$, which implies that $z_m^+ \to z^+$ in E as $m \to +\infty$. Similarly, $z_m^- \to z^-$ in E as $m \to +\infty$. Furthermore, the fact that E^0 has finite dimension implies that $z_m^0 \to z^0$ in E as $m \to +\infty$. Thus $\{z_m\}$ has a convergent subsequence. □

Set $M = \max\{\sigma_1 + \tau_1, \ldots, \sigma_n + \tau_n\}$, then there exist positive constants $\mu_i \ge \sigma_i$, $\nu_i \ge \tau_i$, $x_i \ge 1$ $(i = 1, 2, \ldots, n)$ such that $\mu_i = x_i\sigma_i$, $\nu_i = x_i\tau_i$ and $\mu_i + \nu_i = M$ $(i = 1, 2, \ldots, n)$. Define operator $B_1 \colon E \to E$ as $B_1(p_1, \ldots, p_n, q_1, \ldots, q_n) = (\varrho^{\nu_1 - 1}p_1, \ldots, \varrho^{\nu_n - 1}p_n, \varrho^{\mu_1 - 1}q_n, \ldots, \varrho^{\mu_n - 1}q_n)$, where $(p_1, \ldots, p_n, q_1, \ldots, q_n) \in E$, and constant ϱ is determined in Lemma 3.4. Then B_1 is linear, bounded, invertible. Set $S = \{B_1 z \mid \|z\| = \varrho \text{ and } z \in E_1\}$.

Lemma 3.4. *There exist constants $\varrho > 0$ and $\kappa > 0$ such that $I|_S \ge \kappa$, if H satisfies* (H1), (H2) *and* (H5).

Proof. The idea comes from [12]. (H5) implies that there exist constants $c_5 > 0$ and $c_6 > 0$ such that

$$H(t,z) \le c_5 + c_6 |z|^{\lambda+1}, \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n}. \tag{3.11}$$

For arbitrary $\varepsilon > 0$ and using (H2), there exists a constant $\delta_\varepsilon > 0$ such that

$$H(t,z) \le \varepsilon \sum_{i=1}^{n} \left(|p_i|^{1+\frac{\sigma_i}{\tau_i}} + |q_i|^{1+\frac{\tau_i}{\sigma_i}} \right), \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n} \text{ with } |z| \le \delta_\varepsilon. \tag{3.12}$$

Choosing $M_\varepsilon > \max\left\{2c_5\delta_\varepsilon^{-\lambda-1}, 2c_6\right\}$, and using (3.11), (3.12) and (1.6), we see

$$\begin{aligned} H(t,z) &\le \varepsilon \sum_{i=1}^{n} \left(|p_i|^{1+\frac{\sigma_i}{\tau_i}} + |q_i|^{1+\frac{\tau_i}{\sigma_i}} \right) + M_\varepsilon |z|^{\lambda+1} \\ &\le \varepsilon \sum_{i=1}^{n} \left(|p_i|^{1+\frac{\sigma_i}{\tau_i}} + |q_i|^{1+\frac{\tau_i}{\sigma_i}} \right) \\ &\quad + M_\varepsilon (2n)^{\lambda+1} \sum_{i=1}^{n} \left(|p_i|^{\lambda+1} + |q_i|^{\lambda+1} \right), \quad (t,z) \in \mathbb{R} \times \mathbb{R}^{2n}. \end{aligned} \tag{3.13}$$

For $z = (\varrho^{\nu_1-1}p_1, \ldots, \varrho^{\nu_n-1}p_n, \varrho^{\mu_1-1}q_1, \ldots, \varrho^{\mu_n-1}q_n) \in S$, note that $\frac{\mu_i}{\nu_i} = \frac{\sigma_i}{\tau_i}$ $(i = 1, 2, \ldots, n)$, (3.13) and Lemma 3.1 imply that

$$\begin{aligned} I(z) &= \int_0^{2\pi} (\varrho^{\nu_1-1}p_1, \ldots, \varrho^{\nu_n-1}p_n) \cdot (\varrho^{\mu_1-1}\dot{q}_1, \ldots, \varrho^{\mu_n-1}\dot{q}_n)\,\mathrm{d}t - \int_0^{2\pi} H(t,z)\,\mathrm{d}t \\ &\ge \sum_{i=1}^{n} \int_0^{2\pi} \varrho^{\mu_i+\nu_i-2} p_i \cdot \dot{q}_i \,\mathrm{d}t \\ &\quad - \varepsilon \sum_{i=1}^{n} \int_0^{2\pi} \left(\varrho^{(\nu_i-1)(1+\frac{\mu_i}{\nu_i})} |p_i|^{1+\frac{\mu_i}{\nu_i}} + \varrho^{(\mu_i-1)(1+\frac{\nu_i}{\mu_i})} |q_i|^{1+\frac{\nu_i}{\mu_i}} \right) \mathrm{d}t \\ &\quad - M_\varepsilon (2n)^{\frac{\lambda+1}{2}} \sum_{i=1}^{n} \int_0^{2\pi} \left(\varrho^{(\lambda+1)(\nu_i-1)} |p_i|^{1+\lambda} + \varrho^{(\lambda+1)(\mu_i-1)} |q_i|^{1+\lambda} \right) \mathrm{d}t \\ &\ge \varrho^{M-2} \int_0^{2\pi} p \cdot \dot{q}\,\mathrm{d}t \\ &\quad - \varepsilon \sum_{i=1}^{n} C(\mu_i,\nu_i) \left[\varrho^{(\nu_i-1)(1+\frac{\mu_i}{\nu_i})} \|(p_i,\mathbf{0})\|^{1+\frac{\mu_i}{\nu_i}} + \varrho^{(\mu_i-1)(1+\frac{\nu_i}{\mu_i})} \|(\mathbf{0},q_i)\|^{1+\frac{\nu_i}{\mu_i}} \right] \\ &\quad - M_\varepsilon (2n)^{\frac{\lambda+1}{2}} C_{\lambda+1} \sum_{i=1}^{n} \left(\varrho^{(\lambda+1)(\nu_i-1)} \|(p_i,\mathbf{0})\|^{1+\lambda} + \varrho^{(\lambda+1)(\mu_i-1)} \|(\mathbf{0},q_i)\|^{1+\lambda} \right) \\ &\ge \varrho^M - \varepsilon \sum_{i=1}^{n} C(\mu_i,\nu_i)(\varrho^M + \varrho^M) - M_\varepsilon 2^{\frac{\lambda+1}{2}} C_{\lambda+1} \sum_{i=1}^{n} \left(\varrho^{(\lambda+1)\nu_i} + \varrho^{(\lambda+1)\mu_i} \right) \\ &= \varrho^M \left(1 - 2\varepsilon \sum_{i=1}^{n} C(\mu_i,\nu_i) - M_\varepsilon (2n)^{\frac{\lambda+1}{2}} C_{\lambda+1} \sum_{i=1}^{n} \left(\varrho^{(\lambda+1)\nu_i - M} + \varrho^{(\lambda+1)\mu_i - M} \right) \right) \end{aligned} \tag{3.14}$$

where $C(\mu_i,\nu_i) > 0$ and $C_{\lambda+1} > 0$ are embedding numbers.

Note that $\lambda > \max\left\{\frac{\sigma_1}{\tau_1},\dots,\frac{\sigma_n}{\tau_n},\frac{\tau_1}{\sigma_1},\dots,\frac{\tau_n}{\sigma_n}\right\}$, so we have that $(\lambda+1)\mu_i - M > (\frac{\tau_i}{\sigma_i}+1)\mu_i - M = 0$ and $(\lambda+1)\nu_i - M > (\frac{\sigma_i}{\tau_i}+1)\nu_i - M = 0$ $(i=1,2,\dots,n)$. If we choose $\varepsilon,\varrho\in(0,1)$ satisfying that

$$2\varepsilon\sum_{i=1}^{n} C(\mu_i,\nu_i) < \frac{1}{3}, \quad M_\varepsilon(2n)^{\lambda+1}C_{\lambda+1}\sum_{i=1}^{n}\left(\varrho^{(\lambda+1)\nu_i-M}+\varrho^{(\lambda+1)\mu_i-M}\right) < \frac{1}{3},$$

then (3.14) implies that $I|_S \ge \kappa = \varrho^M/3 > 0$. □

Define operator $B_2\colon E\to E$ as

$$B_2(p_1,\dots,p_n,q_1,\dots,q_n) = (r^{\nu_1-1}p_1,\dots,r^{\nu_n-1}p_n,r^{\mu_1-1}q_1,\dots,r^{\mu_n-1}q_n),$$

where $(p_1,\dots,p_n,q_1,\dots,q_n)\in E$, and constant $r>0$ is determined in following Lemma 3.5. Then B_2 is linear, bounded, invertible. For $s\in\mathbb{R}$, $z^\pm = (p_1^\pm,\dots,p_n^\pm,q_1^\pm,\dots,q_n^\pm)\in E^\pm$ and $z^0 = (p_1^0,\dots,p_n^0,q_1^0,\dots,q_n^0)\in E^0$, define

$$\begin{aligned} f(s,e,z,z^0) = {} & s(r^{\nu_1-1}p_1^+,\dots,r^{\nu_n-1}p_n^+,r^{\mu_1-1}q_1^+,\dots,r^{\mu_n-1}q_n^+) \\ & + (r^{\nu_1-1}p_1^-,\dots,r^{\nu_n-1}p_n^-,r^{\mu_1-1}q_1^-,\dots,r^{\mu_n-1}q_n^-) \\ & + (r^{\nu_1-1}p_1^0,\dots,r^{\nu_n-1}p_n^0,r^{\mu_1-1}q_1^0,\dots,r^{\mu_n-1}q_n^0). \end{aligned}$$

Set $Q = \left\{f(s,e,z^-,z^0)\mid 0\le s\le r, \left\|z^-+z^0\right\|\le r\right\}$, ∂Q refers to the boundary of Q relative to $\left\{f(s,e,z^-,z^0)\mid s\in\mathbb{R}, z^-\in E^-, z^0\in E^0\right\}$.

Lemma 3.5. *There exists a constant* $r > \frac{\varrho}{\left\|B_1^{-1}B_2e\right\|}$ *such that* $I|_{\partial Q}\le 0$, *if* H *satisfies* (H1) *and* (H3).

Proof. The idea comes from [12].

Set $\overline{m} = \min_{1\le i\le n}\left\{\left(\frac{\varepsilon_1}{\sqrt{2}}\right)^{1+\frac{\sigma_i}{\tau_i}}, \left(\frac{\varepsilon_1}{\sqrt{2}}\right)^{1+\frac{\tau_i}{\sigma_i}}\right\}$ and $A_1 = \frac{\sqrt{2n}}{\varepsilon_1\overline{m}}$, where ε_1 is as in Lemma 3.2. Condition (H3) implies that there exists a constant $A_2 > \sqrt{2n}$ such that

$$H(t,z)\ge A_1\sum_{i=1}^{n}\left(|p_i|^{1+\frac{\sigma_i}{\tau_i}}+|q_i|^{1+\frac{\tau_i}{\sigma_i}}\right), \quad (t,z)\in\mathbb{R}\times\mathbb{R}^{2n} \text{ with } |z|\ge A_2. \tag{3.15}$$

Fix $r\ge\max\left\{\frac{A_2}{\varepsilon_1}+1, \frac{\varrho}{\left\|B_1^{-1}B_2e\right\|}\right\}$, for any $z = f(s,e,z^-,z^0)\in\partial Q$, we have

$$\begin{aligned} A(z) &= \int_0^{2\pi}\sum_{i=1}^{n}\left(r^{\mu_i+\nu_i-2}sp_i^+\cdot s\dot{q}_i^+ + r^{\mu_i+\nu_i-2}p_i^-\cdot\dot{q}_i^-\right)\mathrm{d}t \\ &= r^{M-2}\int_0^{2\pi}\left(sp^+\cdot s\dot{q}^+ + p^-\cdot\dot{q}^-\right)\mathrm{d}t \\ &= r^{M-2}\left[A(s(p^+,q^+)) + A((p^-,q^-))\right] \\ &= r^M\left\|\frac{s}{r}(p^+,q^+)\right\|^2 - r^M\left\|\frac{1}{r}(p^-,q^-)\right\|^2. \end{aligned} \tag{3.16}$$

We will check $I(z) \leq 0$. The process is divided into several cases.

Case 1. If $s = 0$, then (3.16) and (H1) imply that $I(z) \leq 0$.

Case 2. If $s \neq 0$, then $z \in \partial Q$ indicates that either $s = r$ and $\left\|z^- + z^0\right\| \leq r$ or $0 < s \leq r$ and $\left\|z^- + z^0\right\| = r$. Whatever the case is, we have $1 \leq \|\widetilde{z}\| \leq 2$, where

$$\begin{aligned}\widetilde{z} &= (\widetilde{p}_1, \ldots, \widetilde{p}_n, \widetilde{q}_1, \ldots, \widetilde{q}_n) \\ &= \frac{1}{r}(sp_1^+ + p_1^- + p_1^0, \ldots, sp_n^+ + p_n^- + p_n^0, sq_1^+ + q_1^- + q_1^0, \ldots, sq_n^+ + q_n^- + q_n^0).\end{aligned}$$

Next, we will consider two subcases below.

Subcase 1. If $\left\|(sp^+ + p^0, sq^+ + q^0)\right\| < \|(p^-, q^-)\|$, then (3.16) and (H1) imply that $I(z) \leq 0$.

Subcase 2. If $\left\|(sp^+ + p^0, sq^+ + q^0)\right\| \geq \|(p^-, q^-)\|$, set $\Omega_{\widetilde{z}} = \{t \in [0, 2\pi] \mid |\widetilde{z}(t)| \geq \varepsilon_1\}$, then Lemma 3.2 implies that measure$(\Omega_{\widetilde{z}}) \geq \varepsilon_1$. For $t \in \Omega_{\widetilde{z}}$, we have $\left|\frac{\sqrt{2n}}{\varepsilon_1}\widetilde{z}(t)\right| \geq \sqrt{2n}$ and

$$\begin{aligned}|z(t)| &= \big|(r^{\nu_1-1}(sp_1^+(t) + p_1^-(t) + p_1^0), \ldots, r^{\nu_1-1}(sp_n^+(t) + p_n^-(t) + p_n^0), \\ &\qquad r^{\mu_1-1}(sq_1^+(t) + q_1^-(t) + q_1^0), \ldots, r^{\mu_n-1}(sq_n^+(t) + q_n^-(t) + q_n^0))\big| \\ &= r\,|\widetilde{z}(t)| \geq r\varepsilon_1 > A_2.\end{aligned} \tag{3.17}$$

Using (3.17), (3.15), (1.5) and the choice of A_1, we have

$$\begin{aligned}&H(t, z(t)) \\ &\geq A_1 \sum_{i=1}^{n} \left(\left|r^{\nu_i-1}(sp_i^+(t) + p_i^-(t) + p_i^0(t))\right|^{1+\frac{\sigma_i}{\tau_i}} + \left|r^{\mu_i-1}(sq_i^+(t) + q_i^-(t) + q_i^0(t))\right|^{1+\frac{\tau_i}{\sigma_i}}\right) \\ &= A_1 \sum_{i=1}^{n} \left[r^M \left(|\widetilde{p}_i|^{1+\frac{\sigma_i}{\tau_i}} + |\widetilde{q}_i|^{1+\frac{\tau_i}{\sigma_i}}\right)\right] \\ &= A_1 r^M \sum_{i=1}^{n} \left[\left(\frac{\varepsilon_1}{\sqrt{2n}}\right)^{1+\frac{\sigma_i}{\tau_i}} \left|\frac{\sqrt{2n}}{\varepsilon_1}\widetilde{p}_i\right|^{1+\frac{\sigma_i}{\tau_i}} + \left(\frac{\varepsilon_1}{\sqrt{2n}}\right)^{1+\frac{\tau_i}{\sigma_i}} \left|\frac{\sqrt{2n}}{\varepsilon_1}\widetilde{p}_i\right|^{1+\frac{\tau_i}{\sigma_i}}\right] \\ &\geq A_1 r^M \min_{1 \leq i \leq n} \left\{\left(\frac{\varepsilon_1}{\sqrt{2}}\right)^{1+\frac{\sigma_i}{\tau_i}}, \left(\frac{\varepsilon_1}{\sqrt{2}}\right)^{1+\frac{\tau_i}{\sigma_i}}\right\} \sum_{i=1}^{n} \left(\left|\frac{\sqrt{2n}}{\varepsilon_1}\widetilde{p}_i\right|^{1+\frac{\sigma_i}{\tau_i}} + \left|\frac{\sqrt{2n}}{\varepsilon_1}\widetilde{p}_i\right|^{1+\frac{\tau_i}{\sigma_i}}\right) \\ &\geq r^M \frac{\sqrt{2n}}{\varepsilon_1} \cdot \frac{1}{2n} \sum_{i=1}^{n} \left(\left|\frac{\sqrt{2n}}{\varepsilon_1}\widetilde{p}_i\right| + \left|\frac{\sqrt{2n}}{\varepsilon_1}\widetilde{p}_i\right|\right) \\ &\geq r^M \frac{\sqrt{2n}}{\varepsilon_1} \cdot \frac{1}{2n} \left|\frac{\sqrt{2n}}{\varepsilon_1}\widetilde{z}(t)\right| \geq \frac{r^M}{\varepsilon_1}, \quad t \in \Omega_{\widetilde{z}(t)}.\end{aligned} \tag{3.18}$$

So (3.16), (3.18) and (H1) imply that

$$I(z) = A(z) - \int_0^{2\pi} H(t, z)\,\mathrm{d}t \leq r^M - \int_{\Omega_{\widetilde{z}}} H(t, z)\,\mathrm{d}t \leq 0. \qquad \square$$

Lemma 3.6. *If H satisfies* (H1)–(H3) *and* (H5), *then* (I3) *in Theorem* 2.8 *holds for I.*

Proof. As [2, Lemma 2.8] demonstrates, for ϱ and r as in Lemmas 3.4–3.5, $B(v)$ ($v \geq 0$) has an explicit formula, that is,

$$\begin{aligned} B(v)\left((p^-, q^-) + (p^0, q^0)\right) &= P_2 B_1^{-1} \exp(vl) B_2\left((p^-, q^-) + (p^0, q^0)\right) \\ &= \sum_{i=1}^{n} m_i(\varrho, r, s)(p_i^-, q_i^-) + \left(\left(\frac{r}{\varrho}\right)^{\nu_i - 1} p^0, \left(\frac{r}{\varrho}\right)^{\mu_i - 1} q^0\right), \end{aligned}$$

where $(p^-, q^-) \in E^-$, $(p^0, q^0) \in E^0$ and

$$2m_i(\varrho, r, s) = \left[\left(\frac{r}{\varrho}\right)^{\nu_i - 1} + \left(\frac{r}{\varrho}\right)^{\mu_i - 1}\right] \cosh(v) - \left(\frac{r^{\mu_i - 1}}{\varrho^{\nu_i - 1}} + \frac{r^{\nu_i - 1}}{\varrho_i^{\mu_i - 1}}\right) \sinh(v).$$

We note that $\varrho < 1$ and $r > 1$, thus we have

$$\begin{aligned} 2m_i(\varrho, r, s) &= \left[\left(\frac{r}{\varrho}\right)^{\nu_i - 1} + \left(\frac{r}{\varrho}\right)^{\mu_i - 1} - \frac{r^{\mu_i - 1}}{\varrho^{\nu_i - 1}} + \frac{r^{\nu_i - 1}}{\varrho_i^{\mu_i - 1}}\right] \frac{\exp(v)}{2} \\ &\quad + \left[\left(\frac{r}{\varrho}\right)^{\nu_i - 1} + \left(\frac{r}{\varrho}\right)^{\mu_i - 1} + \frac{r^{\mu_i - 1}}{\varrho^{\nu_i - 1}} + \frac{r^{\nu_i - 1}}{\varrho_i^{\mu_i - 1}}\right] \frac{\exp(-v)}{2} \\ &= \left(\frac{r}{\varrho}\right)^{\nu_i - 1} (r^{\mu_i - \nu_i} - 1) \left[\left(\frac{1}{\varrho}\right)^{\mu_i - \nu_i} - 1\right] \frac{\exp(v)}{2} \\ &\quad + \left[\left(\frac{r}{\varrho}\right)^{\nu_i - 1} + \left(\frac{r}{\varrho}\right)^{\mu_i - 1} + \frac{r^{\mu_i - 1}}{\varrho^{\nu_i - 1}} + \frac{r^{\nu_i - 1}}{\varrho_i^{\mu_i - 1}}\right] \frac{\exp(-v)}{2} > 0. \end{aligned}$$

So $\widehat{B}(v)\colon E_2 \to E_2$ is linear, bounded and invertible for $v \geq 0$. □

Finally, we shall give the proof of Theorem 1.3.

Proof of Theorem 1.3. Book [11] and Lemmas 3.3–3.6 imply that $I \in C^1(E, \mathbb{R})$ satisfies all conditions of Theorem 2.8, if H satisfies (H1)–(H5). So there exists a critical point z of I which is a weak solution of the system (1.1) and $I(z) \geq \kappa > 0$. [11, pp. 40–41] indicate that z is a nontrivial classical 2π-periodic solution of the system (1.1). □

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