On the Solution Existence of Convex Quadratic Programming Problems in Hilbert Spaces

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Abstract. We provide solution existence results for the convex quadratic programming problems in Hilbert spaces, which the constraint set is defined by finitely many convex quadratic inequalities. In order to obtain our results, we shall use either the properties of the Legendre form or the properties of the finite-rank operator. The existence results are established without requesting neither coercivity of the objective function nor compactness of the constraint set.

1. Introduction

Solution existence for convex quadratic programming problems (convex QP problems for brevity) is an interesting question in optimization theory. This question in both the finite-dimensional and infinite-dimensional setting has been studied extensively by several authors. For example, B. G. Belousov [3,4], Z. Q. Lou [15], A. Auslender [2], Z. Dostál [8], G. M. Lee et al. [13], D. S. Kim et al. [11], J. Semple [17], I. E. Schochetman et al. [16], J. F. Bonnans [5] and K. C. Sivakumar et al. [19].

It is some well-known conditions which guarantee the solution existence of convex QP problems. For example, if constraint set of the convex QP problem is nonempty and bounded, objective function is weakly lower semicontinuous and bounded from below on constraint set, then the existence of a solution follows by the compactness argument [12, Theorem 7.3.4]. The coercivity of objective function is also one of the most useful assumptions which guarantee that there is a solution to convex QP problem [18, Theorem 6.2.4]. However, it is to see that a solution of convex QP problem may exist in more general cases (see, for example, [3, 4, 15]).

The purpose of this paper is to extend the results of [3, 11, 15] on solution existence for convex QP problems in Euclidean spaces to Hilbert spaces. By using either the Legendre property of quadratic forms or the finite rank property of operators corresponding

to quadratic forms, we give existence results of the solution for the problems without requesting neither coercivity of the objective function nor compactness of the constraint set. The approach that we have taken here is quite similar to the finite-dimensional setting which can be found in [3,11,15]. As concerning with the Hilbert space setting, our results are completely new.

The idea of using the Legendre property of the quadratic form in objective function in proving solution existence of quadratic programming problems with linear constraints in Hilbert space is due to Bonnans and Shapiro in [5, Theorem 3.128]. We would like to stress that the notion of the Legendre form, which has origin in the Calculus of Variations, is crucial for the just cited solution existence theorem in [5]. In Section 3, we construct an example to show that the conclusion of that theorem fails if the assumption on the Legendre property of the quadratic form is omitted.

The remainder of the paper is structured as follows. In Section 2, we recall some notations and results that will be useful in the sequel. In Section 3, we establish the existence of solutions for convex quadratic programming problems in Hilbert spaces. Finally, in Section 4, we summarize the paper and give an open question.

2. Preliminaries

In this section we recall some basic definitions and facts that are used in the sequel (see [5,6,9,10] for more details).

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. A sequence $\{x^k\}$ in $\mathcal{H}$ is said to converge weakly to x_0, the notation $x^k \rightharpoonup x$, if $\langle a, x^k \rangle \to \langle a, x_0 \rangle$ for each a in $\mathcal{H}$. A sequence $\{x^k\}$ in $\mathcal{H}$ is said to converge strongly to x_0, the notation $x^k \to x$, if $\|x^k - x_0\| \to 0$.

In this paper, we will only consider the continuous quadratic forms in the following form

$$Q(x) = \langle x, Tx \rangle,$$

where $T \colon \mathcal{H} \to \mathcal{H}$ is a continuous linear self-adjoint operator.

Definition 2.1. (See, for instance, [9]) A quadratic form $Q(x)$ is said to be

 (i) nonnegative if $Q(x) \geq 0$ for all $x \in \mathcal{H}$;

 (ii) positive if $Q(x) > 0$ for all $x \in \mathcal{H} \setminus \{0\}$.

The operator $T \colon \mathcal{H} \to \mathcal{H}$ is said to be positive semidefinite (positive definite) if the quadratic form $Q(x) = \langle x, Tx \rangle$ is nonnegative (positive, respectively).

The following properties will be useful in the sequel.

Proposition 2.2. (See, e.g., [5, Proposition 3.71]) *A quadratic form $Q(\cdot)$ on a Hilbert space $\mathcal{H}$ is convex on $\mathcal{H}$ if and only if it is nonnegative.*

Proposition 2.3. (See, e.g., [10, Proposition 3, p. 269]) *A nonnegative continuous quadratic form on a Hilbert space is weakly lower semicontinuous.*

Definition 2.4. (See, for instance, [9, p. 551]) A quadratic form $Q(x)$ on the Hilbert space $\mathcal{H}$ is said to be a Legendre form if

  (i) it is weakly lower semicontinuous, and

  (ii) $x^k \to x_0$ whenever $x^k \rightharpoonup x_0$ and $Q(x^k) \to Q(x_0)$.

It is clear that in the case where $\mathcal{H}$ is of finite dimension, any quadratic form $Q(x)$ on $\mathcal{H}$ is a Legendre form.

Example 2.5. Let ℓ^2 denote the Hilbert space of all square summable real sequences. Define $T\colon \ell^2 \to \ell^2$ by $Tx = (0, x_2, x_3, \ldots, x_n, \ldots)$, where $x = (x_1, x_2, x_3, \ldots, x_n, \ldots) \in \ell^2$. Since $\langle x, Tx \rangle \geq 0$, $\langle x, Tx \rangle$ is convex. It is clear that $\langle x, Tx \rangle = \|x\|^2 - x_1^2 \geq 0$ for all $x \in \ell^2$. By [5, Proposition 3.79], $\langle x, Tx \rangle$ is a Legendre form.

Example 2.6. Let $L_2[0, 1]$ denote the real Hilbert space of all square integrable functions on $[0, 1]$. Let $T\colon L_2[0, 1] \to L_2[0, 1]$ be defined by

$$Tx(t) = tx(t).$$

It is easy to check that T is a continuous linear self-adjoint and positive semidefinite on $L_2[0, 1]$. The quadratic form associated with T given by $\langle x, Tx \rangle = \int_0^1 tx^2(t)\, dt$ is not a Legendre form (see [7, Example 2.2]).

Let us recall the definition of the finite-rank operator.

Definition 2.7. (See, [6, p. 182]) An operator $T\colon \mathcal{H} \to \mathcal{H}$ is called a finite-rank operator if its range is of finite dimension.

Recall that an operator T on a Hilbert space $\mathcal{H}$ is called a compact operator if, for every bounded sequence $\{x_n\}$ in $\mathcal{H}$, the sequence $\{Tx_n\}$ contains a convergent subsequence (see, [6, p. 180]).

Remark 2.8. Any bounded, finite-rank operator T on $\mathcal{H}$ is a compact operator with closed range [6, p. 182], and any compact operator with closed range is a finite-rank operator [1, p. 215].

It is easy to see that in the infinite-dimensional Hilbert spaces, the quadratic form $\langle x, Ix \rangle$ is a Legendre form while the identity operator I is not compact, the quadratic form $\langle x, 0x \rangle$ is not a Legendre form while the zero operator 0 is compact.

Throughout this paper, we consider the convex QP problems of the form

(CQP)
$$\min \quad f(x) := \frac{1}{2} \langle x, Tx \rangle + \langle c, x \rangle$$
$$\text{s.t.} \quad x \in \mathcal{H} : \; g_i(x) := \frac{1}{2} \langle x, T_i x \rangle + \langle c_i, x \rangle + \alpha_i \leq 0, \; i = 1, 2, \ldots, m,$$

where $T, T_i \colon \mathcal{H} \to \mathcal{H}$ are positive semidefinite, continuous and linear self-adjoint, and $c, c_i \in \mathcal{H}$, $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$.

If for each $i = 1, 2, \ldots, m$, T_i is a zero operator, then we say that (CQP) is a *convex quadratic programming problem under linear constraints* and denote it by (QPL). If T, T_i are zero operators for all $i = 1, 2, \ldots, m$, then (CQP) becomes a *linear programming problem* and denoted by (LP).

It follows from Propositions 2.2 and 2.3 that $f(x), g_i(x), (i = 1, 2, \ldots, m)$ convex and weakly lower semicontinuous.

Let

(C) $\qquad F = \left\{ x \in \mathcal{H} \;\middle|\; g_i(x) = \frac{1}{2} \langle x, T_i x \rangle + \langle c_i, x \rangle + \alpha_i \leq 0 \text{ for all } i = 1, 2, \ldots, m \right\}$

denote the constraint set of (CQP).

Since g_i, $i = 1, 2, \ldots, m$, are continuous and convex, F is closed and convex. Hence, the constraint set F of (CQP) is convex and weakly closed (see, for instance, [5, Theorem 2.23, p. 24]).

The *recession cone* of a nonempty closed convex set $X \subset \mathcal{H}$ plays an important role in our results. Let $X \subset \mathcal{H}$. The recession cone of X is defined (see [5, p. 33]) by $0^+ X = \{ v \in \mathcal{H} \mid \exists x \in X \text{ with } x + tv \in X \; \forall t \geq 0 \}$.

The recession of the constraint set of (CQP) can be described explicitly as follows.

Lemma 2.9. *If F is nonempty, then*

(2.1) $\qquad\qquad 0^+ F = \{ v \in \mathcal{H} \mid T_i v = 0, \langle c_i, v \rangle \leq 0, \forall\, i = 1, 2, \ldots, m \} .$

Proof. This proof is similar to the proof of Lemma 1.1 in [11]. $\qquad\qquad\square$

Remark 2.10. Suppose that $x^k \in F \setminus \{0\}$ for all k, $\left\| x^k \right\| \to \infty$ as $k \to \infty$ and $\left\| x^k \right\|^{-1} x^k$ weakly converges to $\overline{v}$. Then, $\overline{v} \in 0^+ F$ (see, [7, Lemma 3.3]).

3. Optimal solution existence

In this section, we provide existence solution results for (CQP). To prove our main results we need the following lemma.

Lemma 3.1. *Let $f(x) = \frac{1}{2}\langle x, Tx \rangle + \langle c, x \rangle$, where $T \colon \mathcal{H} \to \mathcal{H}$ is a positive semidefinite continuous linear self-adjoint operator, and $c \in \mathcal{H}$. Suppose that f is bounded from below over $\mathcal{H}$ and one of the following conditions holds:*

  (i) *$\langle x, Tx \rangle$ is a Legendre form,*

  (ii) *T is a finite-rank operator.*

Then, there exists $\overline{x} \in \mathcal{H}$ such that $f(\overline{x}) \leq f(x)$ for all $x \in \mathcal{H}$.

Proof. Let $f^* = \inf\{f(x) \mid x \in \mathcal{H}\} > -\infty$. For each k, consider the set $S_k = \{x \in \mathcal{H} \mid f(x) \leq f^* + \frac{1}{k}\}$. By assumption that $f^* > -\infty$, there exists $x^k \in \mathcal{H}$ such that

$$(3.1) \qquad f(x^k) = \frac{1}{2}\left\langle x^k, Tx^k \right\rangle + \left\langle c, x^k \right\rangle \leq f^* + \frac{1}{k}.$$

Thus S_k is nonempty. Since f is convex and continuous, each set S_k is convex and closed. Hence, S_k admits the least norm element (see, for instance, [14, Theorem 1, p. 69]). Without loss of generality, we can assume that x^k in (3.1) is the least norm element in S_k.

Consider the sequence $\{x^k\}$. We shall prove that $\{x^k\}$ is bounded. On the contrary, suppose that $\{x^k\}$ is unbounded. Without loss of generality we may assume that $\|x^k\| \to \infty$ as $k \to \infty$, $\|x^k\| \neq 0$ for all k. Put $v^k := x^k / \|x^k\|$, one has $\|v^k\| = 1$. Then, there exists a subsequence of $\{v^k\}$ which converges weakly to $\overline{v}$. Without loss of generality we can assume that $v^k \rightharpoonup \overline{v}$ as $k \to \infty$.

We will show that

$$(3.2) \qquad T\overline{v} = 0, \quad \langle c, \overline{v} \rangle = 0.$$

Since T is positive semidefinite, by Proposition 2.3, $\langle x, Tx \rangle$ is weakly lower semicontinuous. Dividing both sides of inequality in (3.1) by $\|x^k\|^2$, letting $k \to \infty$, we have

$$0 \leq \frac{1}{2}\langle \overline{v}, T\overline{v} \rangle \leq \frac{1}{2}\liminf_{k \to \infty}\left\langle v^k, Tv^k \right\rangle \leq \frac{1}{2}\limsup_{k \to \infty}\left\langle v^k, Tv^k \right\rangle \leq 0.$$

It follows that

$$(3.3) \qquad \langle \overline{v}, T\overline{v} \rangle = \lim_{k \to \infty}\left\langle v^k, Tv^k \right\rangle = 0.$$

Since T is positive semidefinite, from the above we can deduce

$$(3.4) \qquad T\overline{v} = 0.$$

By positive semidefiniteness of T, from (3.1) it follows that

$$(3.5) \qquad\qquad \left\langle c, x^k \right\rangle \leq f^* + \frac{1}{k}.$$

Dividing both sides of inequality in (3.5) by $\left\| x^k \right\|$, letting $k \to \infty$ we obtain

$$\langle c, \overline{v} \rangle \leq 0.$$

We now show that $\langle c, \overline{v} \rangle = 0$. Indeed, suppose that $\langle c, \overline{v} \rangle < 0$. For fixed k and for all $t > 0$, we have $x^k + t\overline{v} \in \mathcal{H}$ and

$$f(x^k + t\overline{v}) = f(x^k) + \frac{t^2}{2} \langle \overline{v}, T\overline{v} \rangle + t \left\langle Tx^k + c, \overline{v} \right\rangle$$
$$= f(x^k) + t \langle c, \overline{v} \rangle \to -\infty \quad \text{as } t \to +\infty.$$

This contradicts the fact that f is bounded from below over $\mathcal{H}$. Thus, we have

$$(3.6) \qquad\qquad \langle c, \overline{v} \rangle = 0.$$

Combining (3.4) with (3.6) we obtain (3.2).

Consider the case where $\langle x, Tx \rangle$ is a Legendre form. Since $\langle x, Tx \rangle$ is a Legendre form and $v^k \rightharpoonup \overline{v}$ as $k \to \infty$, by (3.3), v^k converges to $\overline{v}$. Hence we have $\overline{v} \neq 0$ and $\|\overline{v}\| = 1$. Let $y^k(t) := x^k - t\overline{v}$, $t \in \mathbb{R}$, by (3.2), we have

$$f(y^k(t)) = f(x^k - t\overline{v})$$
$$= \frac{1}{2} \left\langle x^k, Tx^k \right\rangle + \left\langle c, x^k \right\rangle + \frac{t^2}{2} \langle \overline{v}, T\overline{v} \rangle - t \left\langle x^k, T\overline{v} \right\rangle - t \langle c, \overline{v} \rangle$$
$$= f(x^k) \leq f^* + \frac{1}{k}.$$

This shows that $y^k(t) \in S_k$ for any real number t. On the other hand, we have

$$(3.7) \qquad \left\| y^k(t) \right\|^2 = \left\| x^k - t\overline{v} \right\|^2 = \left\| x^k \right\|^2 - t \left(2 \left\langle x^k, \overline{v} \right\rangle + t \|\overline{v}\|^2 \right).$$

Since $\langle \overline{v}, \overline{v} \rangle = 1$ and $\left\langle x^k / \left\| x^k \right\|, \overline{v} \right\rangle \to \langle \overline{v}, \overline{v} \rangle$, there exists k_1 such that

$$\left\langle x^k, \overline{v} \right\rangle > 0 \quad \forall k \geq k_1.$$

Therefore, for $k \geq k_1$, by (3.7) there exists $\gamma > 0$ such that

$$(3.8) \qquad\qquad \left\| x^k - t\overline{v} \right\|^2 < \left\| x^k \right\|^2 \quad \forall t \in (0, \gamma).$$

From (3.8) it follows that $\left\| x^k - t\overline{v} \right\| < \left\| x^k \right\|$. This contradicts the fact that x^k is the least norm element in S_k. Thus $\left\{ x^k \right\}$ is bounded.

Consider the case where T is a finite-rank operator. Let $\mathcal{L} = \mathcal{H} \oplus \mathbb{R}$, where $\oplus$ denotes the direct sum of Hilbert spaces, and let $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ and $\|\cdot\|_{\mathcal{L}}$ stand for the scalar product and the norm on $\mathcal{L}$, respectively. Let $A \colon \mathcal{H} \to \mathcal{H} \oplus \mathbb{R}$ be defined by $Ax = (Tx, \langle c, x \rangle)$. Since T is a finite-rank operator, it is compact with closed range (see Remark 2.8). Hence A is compact with closed range. For each k, consider the linear system

$$(3.9) \qquad\qquad\qquad Ax = Ax^k.$$

Since A is compact with closed range, there exist the continuous pseudoinverse A^+ of A and a solution $\overline{x}^k$ to (3.9) such that $\overline{x}^k = A^+ Ax^k$ (see, for instance, [14, p. 163]). Therefore, there exists $\rho > 0$, depending only on A, such that $\left\| \overline{x}^k \right\| \leq \rho \left(\left\| Ax^k \right\|_{\mathcal{L}} \right)$. This gives $\left\| \overline{x}^k \right\| \leq \rho \left(\left\| Tx^k \right\| + \left| \langle c, x^k \rangle \right| \right)$. Since $A\overline{x}^k = Ax^k$, we can check that

$$f(\overline{x}^k) = f(x^k) \leq f^* + \frac{1}{k}.$$

Since x^k is the least norm element in S_k, we have

$$\left\| x^k \right\| \leq \left\| \overline{x}^k \right\| \leq \rho \left(\left\| Qx^k \right\| + \left| \langle c, x^k \rangle \right| \right) \quad \forall k.$$

Dividing both sides of this inequality by $\left\| x^k \right\|$, letting $k \to \infty$ and by the compactness of T, one has

$$1 \leq \rho \left(\left\| T\overline{v} \right\| + \left| \langle c, \overline{v} \rangle \right| \right).$$

This contradicts the fact that $T\overline{v} = 0$ and $\langle c, \overline{v} \rangle = 0$. Thus we have shown that $\left\{ x^k \right\}$ is bounded.

The sequence $\left\{ x^k \right\}$ is bounded and hence it has a weakly convergent subsequence. Without loss of generality, we may assume that $x^k \rightharpoonup \overline{x}$ as $k \to \infty$. Since T is positive semidefinite, $\langle x, Tx \rangle$ is weakly lower semicontinuous. Hence, one has

$$\frac{1}{2} \langle \overline{x}, T\overline{x} \rangle \leq \liminf_{k \to \infty} \frac{1}{2} \left\langle x^k, Tx^k \right\rangle.$$

Therefore, by (3.1),

$$f(\overline{x}) = \frac{1}{2} \langle \overline{x}, T\overline{x} \rangle + \langle c, \overline{x} \rangle \leq \liminf_{k \to \infty} \left(\frac{1}{2} \left\langle x^k, Tx^k \right\rangle + \left\langle c, x^k \right\rangle \right)$$

$$\leq \liminf_{k \to \infty} \left(f^* + \frac{1}{k} \right) = f^*.$$

It follows that there exists $\overline{x} \in \mathcal{H}$ such that $f(\overline{x}) \leq f(x)$ for all $x \in \mathcal{H}$. $\qquad\square$

We will now prove the existence results of (CQP) by using the Legendre property of quadratic forms. In the following theorem we extend the result in [3, 15] to Hilbert spaces.

Theorem 3.2 (Frank-Wolfe-type Theorem 1). *Consider convex quadratic programming problem* (CQP). *Assume that* $\langle x, Tx \rangle$ *is a Legendre form and objective function* f *is bounded from below over nonempty set* F. *Then,* (CQP) *has a solution.*

Proof. Let $f^* = \inf \{ f(x) \mid x \in F \} > -\infty$. For each k, consider the set $S_k = \{ x \in F \mid f(x) \leq f^* + \frac{1}{k} \}$. By assumption that $f^* > -\infty$, there exists $x^k \in \mathcal{H}$ such that

$$(3.10) \qquad f(x^k) = \frac{1}{2} \left\langle x^k, Tx^k \right\rangle + \left\langle c, x^k \right\rangle \leq f^* + \frac{1}{k},$$

$$(3.11) \qquad g_i(x^k) = \frac{1}{2} \left\langle x^k, T_i x^k \right\rangle + \left\langle c_i, x^k \right\rangle + \alpha_i \leq 0, \quad i = 1, 2, \ldots, m.$$

Thus S_k is nonempty. Since f, g_i are convex and continuous, each set S_k is convex and closed. Hence, S_k admits the least norm element (see, for instance, [14, Theorem 1, p. 69]). Without loss of generality, we can assume that x^k in (3.10) and (3.11) is the least norm element in S_k.

Let us consider the sequence $\{ x^k \}$. If $\{ x^k \}$ is bounded, then it has a weakly convergent subsequence. Without loss of generality, we may assume that $x^k \rightharpoonup \overline{x}$ as $k \to \infty$. From (3.11), it follows that $x^k \in F$. By the weakly closedness of F, we have $\overline{x} \in F$. Since $\langle x, Tx \rangle$ is a Legendre form, it is weakly lower semicontinuous. Consequently,

$$\frac{1}{2} \langle \overline{x}, T\overline{x} \rangle \leq \liminf_{k \to \infty} \frac{1}{2} \left\langle x^k, Tx^k \right\rangle.$$

From this, by (3.10),

$$f(\overline{x}) = \frac{1}{2} \langle \overline{x}, T\overline{x} \rangle + \langle c, \overline{x} \rangle \leq \liminf_{k \to \infty} \left(\frac{1}{2} \left\langle x^k, Tx^k \right\rangle + \left\langle c, x^k \right\rangle \right)$$

$$\leq \liminf_{k \to \infty} \left(f^* + \frac{1}{k} \right) = f^*.$$

Thus, $\overline{x}$ is a solution of (CQP).

Consider the case where $\{ x^k \}$ is unbounded. Without loss of generality we may assume that $\| x^k \| \to \infty$ as $k \to \infty$, $\| x^k \| \neq 0$ for all k. Put $v^k := x^k / \| x^k \|$, one has $\| v^k \| = 1$. Since $\| v^k \|$ is bounded, it has a weakly convergent subsequence. Without loss of generality we can assume that $v^k \rightharpoonup \overline{v}$ as $k \to \infty$. By Remark 2.10, we have $\overline{v} \in 0^+ F$. By repeating similar arguments as in the proof of Lemma 3.1 we can deduce

$$(3.12) \qquad\qquad \overline{v} \neq 0 \quad \text{and} \quad T\overline{v} = 0, \ \langle c, \overline{v} \rangle = 0.$$

We now consider three distinguish subcases:

Subcase 1: $\langle c_i, \overline{v} \rangle = 0$ for all $i = 1, 2, \ldots, m$. Then,

$$T\overline{v} = 0, \quad \langle c, \overline{v} \rangle = 0, \quad T_i \overline{v} = 0, \quad \forall \, i = 1, 2, \ldots, m.$$

Let $y^k(t) := x^k - t\overline{v}$, $t > 0$. It is easy to check that $y^k(t) := x^k - t\overline{v} \in F$ for all $t > 0$. Since $T\overline{v} = 0$, $\langle c, \overline{v} \rangle = 0$, we have

$$f(y^k(t)) = f(x^k - t\overline{v})$$
$$= \frac{1}{2}\left\langle x^k, Tx^k \right\rangle + \left\langle c, x^k \right\rangle + \frac{t^2}{2}\left\langle \overline{v}, T\overline{v} \right\rangle - t\left\langle x^k, T\overline{v} \right\rangle - t\left\langle c, \overline{v} \right\rangle$$
$$= f(x^k) \le f^* + \frac{1}{k}.$$

Thus, $y^k(t) \in S_k$ for all $t > 0$. On the other hand, we have

$$(3.13) \qquad \left\| y^k(t) \right\|^2 = \left\| x^k - t\overline{v} \right\|^2 = \left\| x^k \right\|^2 - t\left(2\left\langle x^k, \overline{v} \right\rangle + t\|\overline{v}\|^2 \right).$$

Since $\langle \overline{v}, \overline{v} \rangle = 1$ and $\left\langle x^k/\|x^k\|, \overline{v} \right\rangle \to \langle \overline{v}, \overline{v} \rangle$, there exists k_1 such that

$$\left\langle x^k, \overline{v} \right\rangle > 0 \quad \forall k \ge k_1.$$

Therefore, for $k \ge k_1$, by (3.13) there exists $\gamma > 0$ such that

$$(3.14) \qquad \left\| x^k - t\overline{v} \right\|^2 < \left\| x^k \right\|^2 \quad \forall t \in (0, \gamma).$$

From (3.14) it follows that $\left\| x^k - t\overline{v} \right\| < \left\| x^k \right\|$. This contradicts the fact that x^k is the least norm element in S_k, implying that Subcase 1 can never occur.

Subcase 2: $\langle c_i, \overline{v} \rangle < 0$ for all $i = 1, 2, \ldots, m$.

Consider quadratic programming problem

$$\min\left\{ f(x) := \frac{1}{2}\langle x, Tx \rangle + \langle c, x \rangle \;\Big|\; x \in \mathcal{H} \right\}.$$

If $f(x)$ is bounded from below over Hilbert space $\mathcal{H}$ then, by Lemma 3.1, there exists $\overline{x} \in \mathcal{H}$ such that $f(\overline{x}) \le f^*$. If $f(x)$ is unbounded from below over Hilbert space $\mathcal{H}$ then, there exists $\widehat{x} \in \mathcal{H}$ such that $f(\widehat{x}) \le f^*$. Hence, in both cases there exists $x^* \in \mathcal{H}$ such that $f(x^*) \le f^*$.

Let $t^* = \max\left\{ -g_i(x^*)/\langle c_i, \overline{v} \rangle, i = 1, 2, \ldots, m \right\}$. It is easy to check that $x^* + t\overline{v} \in F$ for all $t \ge t^*$. Then, we have

$$f(x^* + t^*\overline{v}) = \frac{1}{2}\langle x^*, Tx^* \rangle + \langle c, x^* \rangle \le f^*.$$

This shows that $x^* + t^*\overline{v}$ is a solution of problem (CQP).

Subcase 3: There exist i and j such that $\langle c_i, \overline{v} \rangle = 0$ and $\langle c_j, \overline{v} \rangle < 0$. Let $I = \{1, 2, \ldots, m\}$, $I_1 = \{i \in I \mid \langle c_i, \overline{v} \rangle = 0\}$ and $F_1 = \{x \in \mathcal{H} \mid g_i(x) \le 0, i \in I_1\}$. Consider the problem

$$\min\left\{ f(x) := \frac{1}{2}\langle x, Tx \rangle + \langle c, x \rangle \;\Big|\; x \in F_1 \right\}.$$

It is clear that x_k in F_1 for all k. In view of Subcase 1 above, there exists $x_1^* \in F_1$ such that $f(x_1^*) \leq f^*$. Put $t_1^* = \max\{-g_j(x_1^*)/\langle c_j, \overline{v}\rangle, j \in I \setminus I_1\}$. It is easy to check that $x_1^* + t\overline{v} \in F$ for all $t \geq \max\{0, t_1^*\}$. Then, we have

$$f(x_1^* + t^*\overline{v}) = \frac{1}{2}\langle x_1^*, Tx_1^*\rangle + \langle c, x_1^*\rangle \leq f^*.$$

It follows that $x_1^* + t_1^*\overline{v}$ is a solution of (CQP). The proof is complete. $\square$

The referee asked for an example or evidences to see that in Hilbert spaces, the boundedness of a (CQP) generally does not imply the attainment as in Euclidean space $\mathbb{R}^n$. We now construct such an example.

Example 3.3. Let ℓ^2 denote the Hilbert space of all square summable real sequence, $\ell^2 = \{x = (x_1, x_2, \ldots, x_n, \ldots) \mid \sum_{n=1}^{\infty} x_n^2 < \infty, x_n \in \mathbb{R}, n = 1, 2, \ldots\}$. The scalar product and the norm in ℓ^2 are defined, respectively, by

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n, \quad \|x\| = \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}.$$

For each $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2$, let us define $T \colon \ell^2 \to \ell^2$ by

$$Tx = \left(x_1, \frac{x_2}{2^2}, \ldots, \frac{x_n}{n^n}, \ldots\right).$$

It is easily seen that T is a positive semidefinite continuous linear self-adjoint operator and $\|T\| = 1$.

The quadratic form associated with T given by

$$Q(x) = \langle x, Tx\rangle = \sum_{n=1}^{\infty} \frac{x_n^2}{n^n}.$$

We claim that $Q(x) = \langle x, Tx\rangle = \sum_{n=1}^{\infty} x_n^2/n^n$ is not a Legendre form. Indeed, let $\{e^k\}$ be a sequence in ℓ^2, where $e^k = (e_1^k, e_2^k, \ldots, e_n^k, \ldots)$ such that $e_n^k = 1$ for $n = k$, and $e_n^k = 0$ for $n \neq k$. It is easy to check that e^k converges weakly to 0 in ℓ^2 but not strongly. We also have

$$\left\langle e^k, Te^k\right\rangle = \frac{1}{k^k} \to 0 = \langle 0, T0\rangle \quad \text{as } k \to \infty.$$

Hence, $Q(x) = \langle x, Tx\rangle$ is not a Legendre form.

Since $Q(x) = \langle x, Tx\rangle = \sum_{n=1}^{\infty} x_n^2/n^n \geq 0$ for all $x \in \ell^2$, by Propositions 2.2 and 2.3, $\langle x, Tx\rangle$ is convex and weakly lower semicontinuous.

We now consider the programming problem (CQP)

$$(3.15) \qquad \min \quad f(x) = \frac{1}{2}\langle x, Tx\rangle \quad \text{subject to} \quad x \in \ell^2 : \langle c_1, x\rangle + \alpha_1 \leq 0.$$

where

$$\langle x, Tx \rangle = \sum_{n=1}^{\infty} \frac{x_n^2}{n^n}, \quad c_1 = \left(-1, -\frac{1}{2}, \ldots, -\frac{1}{n}, \ldots \right) \in \ell^2, \quad \alpha_1 = 1.$$

Let

$$F = \left\{ x \in \ell^2 \mid \langle c_1, x \rangle + \alpha_1 \le 0 \right\} = \left\{ x \in \ell^2 \;\Big|\; -\sum_{n=1}^{\infty} \frac{x_n}{n} + 1 \le 0 \right\}.$$

The set F is nonempty. Indeed, for any positive integer k, let $x^k = ke^k \in \ell^2$, where e^k is the above mentioned vector. It is easy to check that $\langle c_1, x^k \rangle + 1 = -1 + 1 = 0$. Hence $x^k \in F$ for all k.

Since $\langle x, Tx \rangle = \sum_{n=1}^{\infty} x_n^2/n^n$, $\langle x, Tx \rangle = 0$ if and only if $x = 0$. It is easily seen that $0 \notin F$, we have

(3.16)
$$f(x) = \frac{1}{2} \langle x, Tx \rangle = \frac{1}{2} \sum_{n=1}^{\infty} \frac{x_n^2}{n^n} > 0 \quad \text{for all } x \in F.$$

On the other hand,

$$f(x^k) = \frac{1}{2} \left\langle x^k, Tx^k \right\rangle = \frac{1}{2} \frac{1}{k^{k-2}} \to 0 \quad \text{as } k \to \infty.$$

This, together with (3.16), shows that the infimum of f over F is 0. Since $0 \notin F$, the inequality (3.16) shows that the problem (3.15) has no solution.

The above example shows that the conclusion of Theorem 3.2 and the conclusion Theorem 3.128 in [5] fail if the assumption on the Legendre property of the quadratic form is omitted.

Remark 3.4. In finite-dimensional setting, the proof of Theorem 3.2 can be found in [4] and [15]. Belousov and Klatte [4, p. 45] showed that there exists a nonconvex quadratic program in $\mathbb{R}^3$ with two convex quadratic constraints whose objective function is bounded from below over a nonempty constraint set, which has no solution. Thus, even in the case of three-dimensional quadratic programs, the positive semidefiniteness of T cannot be dropped from the assumptions of Theorem 3.2.

In order to apply Theorem 3.2, we have to find out whether the objective function $f(x)$ of (CQP) is bounded from below over F, or not. This is a rather difficult task. In the following theorem we extend the result in [11] to Hilbert spaces which gives a necessary condition and sufficient conditions for existence solution of (CQP).

Theorem 3.5 (Eaves-type Theorem). *Consider problem* (CQP) *where F is nonempty and $\langle x, Tx \rangle$ is a Legendre form. The following statements are valid:*

(a) *If* (CQP) *has a solution, then*

(3.17)
$$(v \in 0^+ F, Tv = 0) \implies \langle c, v \rangle \ge 0.$$

(b) *Problem* (CQP) *has a solution if one of the following conditions holds:*

(3.18) $$c = 0,$$

(3.19) $$\left(v \in (0^+F) \setminus \{0\}, Tv = 0\right) \implies \langle c, v \rangle > 0,$$

(3.20) $$\left(v \in (0^+F), Tv = 0\right) \implies \left(\langle c, v \rangle \geq 0, \langle c_i, v \rangle = 0, \forall i \in I_1\right),$$

where $I = \{1, 2, \ldots, m\}$ *and* $I_1 = \{i \in I \mid T_i \neq 0\}$.

Proof. (a) Suppose that (CQP) has a solution $\bar{x}$. To obtain (3.17), let $v \in 0^+F$ be such that $Tv = 0$. Since $\bar{x} + v \in F$ and $Tv = 0$, we have

$$0 \leq f(\bar{x} + v) - f(\bar{x})$$
$$= \left[\frac{1}{2} \langle \bar{x} + v, T(\bar{x} + v) \rangle + \langle c, \bar{x} + v \rangle\right] - \left[\frac{1}{2} \langle \bar{x}, T\bar{x} \rangle + \langle c, v \rangle\right]$$
$$= \langle c, v \rangle.$$

We have thus proved that $\langle c, v \rangle \geq 0$ for any $v \in 0^+F$ satisfying $Tv = 0$.

(b) Suppose that $c = 0$. To prove that (CQP) has a solution, by Theorem 3.2, it suffices to verify that f is bounded from below over F. Since $\langle x, Tx \rangle \geq 0$ for every $x \in \mathcal{H}$, we get

$$f(x) = \frac{1}{2} \langle x, Tx \rangle + \langle c, x \rangle \geq 0 \quad \forall x \in \mathcal{H},$$

which shows that f admits 0 as a lower bound over F.

Next, suppose that (3.19) holds. We shall prove that f is coercive on F. On the contrary, suppose that f is noncoercive on F. Then, one can find some $a \in \mathbb{R}$ and a sequence $\{x^k\} \subset F$ with $\|x^k\| \to \infty$ as $k \to \infty$ and

(3.21) $$f(x^k) = \frac{1}{2} \left\langle x^k, Tx^k \right\rangle + \left\langle c, x^k \right\rangle \leq a \quad \forall k.$$

We may assume that $\|x^k\| \neq 0$ for all k, $v^k := \|x^k\|^{-1} x^k \rightharpoonup v$. By Remark 2.10, we have $v \in 0^+F$.

Multiplying the inequality $\frac{1}{2} \langle x^k, Tx^k \rangle + \langle c, x^k \rangle \leq a$ in (3.21) by $\|x^k\|^{-2}$ and passing to the limit as $k \to \infty$, we obtain

(3.22) $$\langle v, Tv \rangle \leq \liminf_{k \to \infty} \left\langle v^k, Tv^k \right\rangle \leq \limsup_{k \to \infty} \left\langle v^k, Tv^k \right\rangle \leq 0.$$

Combining this with positive semidefiniteness of T, we can deduce that

(3.23) $$\lim_{k \to \infty} \left\langle v^k, Tv^k \right\rangle = \langle v, Tv \rangle = 0,$$

(3.24) $$Tv = 0.$$

Since $\langle x, Tx \rangle$ is a Legendre form and $v^k \rightharpoonup v$ as $k \to \infty$, by (3.23), v^k converges to v. Hence, we have $v \neq 0$.

As $\langle x^k, Tx^k \rangle \geq 0$, from (3.21) it follows that

$$\langle c, x^k \rangle \leq a. \tag{3.25}$$

Multiplying the inequality $\langle c, x^k \rangle \leq a$ by $\|x^k\|^{-1}$ and letting $k \to \infty$, we get

$$\langle c, v \rangle \leq 0. \tag{3.26}$$

This contradicts (3.19). Thus, f is coercive on F. By [18, Theorem 6.2.4], (CQP) has a solution.

Finally, suppose that (3.20) holds. Then, all the assumptions of Theorem 3.3 in [7] hold. By [7, Theorem 3.3], (CQP) has a solution. This completes the proof. □

It is worthy stressing that condition (3.17) is necessary but not sufficient for the solution existence of (CQP). We now construct such an example.

Example 3.6. Let $\mathcal{H} = \ell^2$, $Tx = (0, 0, x_3, \ldots, x_n, \ldots)$, $T_1x = (0, x_2, 0, 0, \ldots)$ for $x = (x_1, x_2, x_3, \ldots, x_n, \ldots) \in \ell^2$, and let $c = (0, -1, 0, 0, \ldots), c_1 = (1, 0, 0, \ldots) \in \ell^2$, $\alpha_1 = -1$. Then, (CQP) becomes

$$\min \quad f(x) = \frac{1}{2}(0x_1^2 + 0x_2^2 + x_3^2 + x_4^2 + \cdots) - x_2 \quad \text{subject to } x \in F, \tag{3.27}$$

where $F = \left\{ x = (x_1, x_2, \ldots) \in \ell^2 \mid \frac{1}{2}x_2^2 + x_1 - 1 \leq 0 \right\}$.

Since $\langle x, Tx \rangle \geq 0$ for all $x \in F$, $f(x)$ is convex. It is easy to check that $\langle x, Tx \rangle = \|x\|^2 - (x_1^2 + x_2^2)$, and so the quadratic form $\langle x, Tx \rangle$ is the sum of an elliptic quadratic form and a quadratic form of finite rank. From [5, Proposition 3.79] it follows that $\langle x, Tx \rangle$ is a Legendre form. By (2.1), we have

$$0^+ F = \{v \in F \mid T_1v = 0, \langle c_1, v \rangle \leq 0\} = \left\{(v_1, 0, v_3, v_4, \ldots) \in \ell^2 \mid v_1 \leq 0\right\}.$$

Note that if $v = (v_1, 0, v_3, v_4, \ldots) \in 0^+F$ and $Tv = 0$ then, $\langle c, v \rangle = -v_2 = 0$. Thus condition (3.17) is satisfied. Since $x^k := (-\frac{1}{2}k^2, k, 0, \ldots) \in F$ for each integer $k \geq 1$ and $f(x^k) = -k$, we infer that f is unbounded from below over F. Hence, (3.27) has no solution.

Remark 3.7. Note that if either $T_i = 0$ for all $i = 1, 2, \ldots, m$ or $c_i = 0$ for all $i \in I_1$, then (3.17) is a necessary and sufficient condition for the solution existence of (CQP), provided that $F \neq \emptyset$.

The following example shows that each of the conditions (3.18), (3.19), (3.20) is sufficient but not necessary for the solution existence of (CQP).

Example 3.8. Consider the programming problem

$$\min \quad f(x) = \frac{1}{2}\langle x, Tx \rangle + \langle c, x \rangle,$$

(3.28)

$$\text{s.t.} \quad x = (x_1, x_2, \ldots) \in \ell^2 : \frac{1}{2}\langle x, T_i x \rangle + \langle c_i, x \rangle + \alpha_i \leq 0, \quad i = 1, 2.$$

where $Tx = (0, x_2, x_3, x_4, \ldots)$, $c = (0, 1, 0, 0, \ldots)$, $T_1 x = (0, x_2, x_3, \ldots)$, $c_1 = (0, 0, 0, \ldots)$, $\alpha_1 = 0$, $T_2 x = (0, 0, x_3, 0, 0, \ldots)$, $c_2 = (1, 0, 0, \ldots)$, $\alpha_2 = 0$.

Since $\langle x, Tx \rangle = \sum_{n=1}^{\infty} x_n^2 \geq 0$ for all $x \in \ell^2$, $\langle x, Tx \rangle$ is convex.

It is easy to check that $\langle x, Tx \rangle = \|x\|^2 - x_1^2$, and so the quadratic form $\langle x, Tx \rangle$ is the sum of an elliptic quadratic form and a quadratic form of finite rank. By [5, Proposition 3.79], $\langle x, Tx \rangle$ is a Legendre form. From Lemma 2.9 it follows that

$$0^+ F = F = \left\{ v = (v_1, 0, 0, \ldots) \in \ell^2 \mid v_1 \leq 0 \right\},$$

$$\left\{ v \in 0^+ F \mid Tv = 0 \right\} = \left\{ v = (v_1, 0, 0, \ldots) \in \ell^2 \mid v_1 \leq 0 \right\}.$$

Since $f(x) = 0$ on F, the solution set of (3.28) coincides with F.

We see that problem (3.28) has a solution while $c = (0, 1, 0, 0, \ldots) \neq 0$, $\langle c, \overline{v} \rangle = 0$ and $\langle c_2, \overline{v} \rangle \neq 0$, where $\overline{v} := (-1, 0, \ldots, 0, \ldots)$. Thus, one of the conditions (3.18), (3.19), (3.20) is sufficient but not necessary for the solution existence of (CQP).

In the remainder of this section we provide an existence result of the solution for (CQP) under the assumption that all the operators corresponding to quadratic forms are finite-rank operators. Note that this assumption is very restrictive but by using this assumption we can investigate the solution existence for a class of (CQP) problems, where the quadratic form in the objective function is not a Legendre form. The next statement may be seen as a complement to Theorem 3.2.

Theorem 3.9 (Frank-Wolfe type Theorem 2)**.** *Consider the problem* (CQP), *where T and T_i ($i = 1, 2, \ldots, m$) are positive semidefinite and finite-rank operators. Assume that the objective function f is bounded from below over the nonempty F. Then,* (CQP) *has a solution.*

Proof. We shall prove this theorem by induction on the number m of quadratic functions that define the constraint set F in (CQP).

For $m = 1$: For each k, consider the set $S_k = \left\{ x \in F \mid f(x) \leq f^* + \frac{1}{k} \right\}$. Since $f^* > -\infty$, there exists $\{x^k\} \subset \mathcal{H}$ such that

(3.29)

$$f(x^k) = \frac{1}{2}\left\langle x^k, Tx^k \right\rangle + \left\langle c, x^k \right\rangle \leq f^* + \frac{1}{k},$$

$$g_1(x^k) = \frac{1}{2}\left\langle x^k, T_1 x^k \right\rangle + \left\langle c_1, x^k \right\rangle + \alpha_1 \leq 0.$$

Suppose that x^k is the least norm element in S_k. If $\{x^k\}$ is bounded then $\{x^k\}$ has a weakly convergent subsequence. Without loss of generality, we may assume that x^k converges weakly to $\overline{x}$. As T and T_1 are finite-rank operators, they are compact. By [10, Theorem 1, p. 261], $f(x)$ and $g_1(x)$ are weakly continuous. Hence, taking the limits in (3.29) as $x^k \rightharpoonup \overline{x}$ we see that $\overline{x}$ is a solution of (CQP).

Consider the case where $\{x^k\}$ is unbounded. Put $v^k := x^k / \|x^k\|$, one has $\|v^k\| = 1$. Without loss of generality, we may assume that $v^k := x^k / \|x^k\| \rightharpoonup \overline{v}$ as $k \to \infty$. Since T and T_1 are positive semidefinite, by an argument analogous to those in the proof of Theorem 3.2 we can deduce

$$T\overline{v} = 0, \quad \langle c, \overline{v} \rangle = 0, \quad T_1\overline{v} = 0, \quad \langle c_1, \overline{v} \rangle \leq 0.$$

We now claim that $\langle c_1, \overline{v} \rangle < 0$. Suppose, contrary to our claim, that $\langle c_1, \overline{v} \rangle = 0$. Then,

$$T\overline{v} = 0, \quad \langle c, \overline{v} \rangle = 0, \quad T_1\overline{v} = 0, \quad \langle c_1, \overline{v} \rangle = 0.$$

Let $\mathcal{L}_1 = \mathcal{H} \oplus \mathcal{H} \oplus \mathbb{R}^2$, where $\oplus$ denotes the direct sum of Hilbert spaces, and let $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ and $\|\cdot\|_{\mathcal{L}}$ stand for the scalar product and the norm on $\mathcal{L}$, respectively.

Let $A \colon \mathcal{H} \to \mathcal{L}_1$ be defined by

$$Ax = (Tx, T_1x, \langle c, x \rangle, \langle c_1, x \rangle).$$

Since T and T_1 are finite-rank operators, so is A. For each k, consider the linear system

$$(3.30) \qquad\qquad\qquad\qquad Ax = Ax^k.$$

Since A is a finite-rank operator, A is a compact operator with closed range (see Remark 2.8). Hence, there exist the continuous pseudoinverse A^+ of A and a solution $\overline{x}^k$ to (3.30) such that $\overline{x}^k = A^+ A x^k$ (see, for instance, [14, p. 163]). Therefore, there exists $\rho > 0$, depending on A, such that

$$\left\| \overline{x}^k \right\| \leq \rho \left(\left\| A x^k \right\|_{\mathcal{L}} \right).$$

This gives

$$\left\| \overline{x}^k \right\| \leq \rho \left(\left\| T x^k \right\| + \left\| T_1 x^k \right\| + \left| \left\langle c, x^k \right\rangle \right| + \left| \left\langle c_1, x^k \right\rangle \right| \right).$$

By (3.30), $A\overline{x}^k = Ax^k$, we can check that

$$f(\overline{x}^k) = f(x^k) \leq f^* + \frac{1}{k}.$$

Since x^k is the least norm element in S_k, we have

$$\left\| x^k \right\| \leq \left\| \overline{x}^k \right\| \leq \rho \left(\left\| T x^k \right\| + \left\| T_1 x^k \right\| + \left| \left\langle c, x^k \right\rangle \right| + \left| \left\langle c_1, x^k \right\rangle \right| \right) \quad \forall k.$$

By assumption that T and T_1 are finite-rank operators, T and T_1 are compact with closed range (see Remark 2.8). Dividing both sides this inequality by $\|x^k\|$, letting $k \to \infty$ and by the compactness of T and T_1, one has

$$1 \leq \rho \left(\|T\overline{v}\| + \|T_1\overline{v}\| + |\langle c, \overline{v}\rangle| + |\langle c_1, \overline{v}\rangle| \right).$$

This contradicts the fact that $T\overline{v} = 0$, $\langle c, \overline{v}\rangle = 0$, $T_1\overline{v} = 0$, $\langle c_1, \overline{v}\rangle = 0$. Thus $\langle c_1, \overline{v}\rangle < 0$. The claim is proved.

Consider quadratic programming problem

$$\min \left\{ f(x) := \frac{1}{2} \langle x, Tx \rangle + \langle c, x \rangle \ \Big| \ x \in \mathcal{H} \right\}.$$

If $f(x)$ is bounded from below over Hilbert space $\mathcal{H}$ then, by Lemma 3.1, there exists $\overline{x} \in \mathcal{H}$ such that $f(\overline{x}) \leq f^*$. If $f(x)$ is unbounded from below over $\mathcal{H}$ then, it is clear that there exists $\widehat{x} \in \mathcal{H}$ such that $f(\widehat{x}) \leq f^*$. Hence, in both cases there exists $x^* \in \mathcal{H}$ such that $f(x^*) \leq f^*$.

By $\langle c_1, \overline{v}\rangle < 0$, we can check that $x^* + t\overline{v} \in F$ for all $t > 0$ large enough. Choose $t^* > 0$ so that $x^* + t^*\overline{v} \in F$ and, by (3.29), we have

$$f(x^* + t^*\overline{v}) = \frac{1}{2} \langle x^*, Tx^* \rangle + \langle c, x^* \rangle \leq f^*.$$

This shows that $x^* + t^*\overline{v}$ is a solution of problem (CQP). This completes the proof for the case $m = 1$.

Suppose that the assertion is shown for all constraint sets F defined by $m-1$ quadratic functions, and let now F defined by m quadratic functions. Let $f^* = \inf \{f(x) \mid x \in F\} > -\infty$. We define $S_k = \left\{ x \in F \mid f(x) \leq f^* + \frac{1}{k} \right\}$. Since $f^* > -\infty$, S_k is nonempty, convex and closed. Hence S_k admits the least norm element (see, for instance, [14, Theorem 1, p. 69]). Let x^k be the least norm element in S_k, we have

$$f(x^k) = \frac{1}{2} \left\langle x^k, Tx^k \right\rangle + \left\langle c, x^k \right\rangle \leq f^* + \frac{1}{k},$$
$$g_i(x^k) = \frac{1}{2} \left\langle x^k, T_i x^k \right\rangle + \left\langle c_i, x^k \right\rangle + \alpha_i \leq 0, \quad i = 1, 2, \ldots, m.$$

If $\{x^k\}$ is bounded then $\{x^k\}$ has a weakly convergent subsequence. Without loss of generality, we may assume that x^k converges weakly to $\overline{x}$. Then, it is easy to check that $\overline{x}$ is a solution of (CQP).

Consider the case where $\{x^k\}$ is unbounded. Let $v^k := x^k / \|x^k\|$, one has $\|v^k\| = 1$. Without loss of generality, we may assume that $v^k := x^k / \|x^k\| \rightharpoonup \overline{v}$ as $k \to \infty$. Since T and T_i are positive semidefinite, by similar argument as in the case $m = 1$, we can deduce that

$$T\overline{v} = 0, \quad \langle c, \overline{v}\rangle = 0, \quad T_i\overline{v} = 0, \quad \langle c_i, \overline{v}\rangle \leq 0, \quad i = 1, 2, \ldots, m,$$

and there exists $j \in \{1, 2, \ldots, m\}$ such that $\langle c_j, \overline{v} \rangle < 0$. Without loss of generality, we may assume that $\langle c_m, \overline{v} \rangle < 0$.

Consider the problem

$$(P_1) \qquad\qquad \min \left\{ f(x) := \frac{1}{2} \langle x, Tx \rangle + \langle c, x \rangle \;\middle|\; x \in F_1 \right\}$$

where $F_1 := \{x \in \mathcal{H} : g_i(x) \leq 0, i = 1, 2, \ldots, m-1\}$.

For this problem, either f is bounded from below over F_1 or not. If f is bounded from below over F_1 then by the assumption of induction, the problem (P_1) has a solution. Hence in both cases, there exists $\overline{x} \in F_1$ such that $f(\overline{x}) \leq f^*$.

Consider the vector $x(t) := \overline{x} + t\overline{v}$, $t \geq 0$. We have, by $\langle c, \overline{v} \rangle = 0$,

$$f(x(t)) = f(\overline{x}) + t \langle c, \overline{v} \rangle = f(\overline{x}) \leq f^*, \quad \forall\, t > 0.$$

Since

$$g_m(x(t)) = g_m(\overline{x}) + t \langle c_m, \overline{v} \rangle,$$

and $\langle c_m, \overline{v} \rangle < 0$, we can choose $t^* > 0$ so that $x(t^*) \in F$ and $f(x(t^*)) \leq f^*$. This proves that $x(t^*)$ is a solution of (CQP).

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark 3.10. Note in Example 3.3 that T is a compact operator whose range is not closed, so T is not a finite-rank operator. Therefore, Theorem 3.9 does not hold true if the assumption that T, T_i are finite-rank operators is replaced by the assumption that T, T_i are compact, $i = 1, 2, \ldots, m$.

In the case where $T = 0$ and $T_i = 0$ $(i = 1, 2, \ldots, m)$, we obtain the following solution existence of the linear programming problem (LP) in Hilbert spaces.

Corollary 3.11. *Consider the linear programming problem (LP) (i.e., (CQP), where $T = 0$ and $T_i = 0$ for all $i = 1, 2, \ldots, m$). Suppose that $f(x)$ is bounded from below over nonempty F. Then, problem (CQP) has a solution.*

Proof. It is easy to see that the zero operator is a finite-rank operator. The assertion is immediate from Theorem 3.9. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 3.12. If $\mathcal{H}$ is of finite dimension then, any continuous operator T on $\mathcal{H}$ is a finite-rank operator and $\langle x, Tx \rangle$ is a Legendre form. Therefore, in the finite-dimensional setting Theorem 3.2 and Theorem 3.9 are identical.

4. Conclusions

In this paper we consider convex quadratic programming problems in Hilbert spaces and propose conditions for the solution existence of convex quadratic programming problems whose constraint set is defined by finitely many convex quadratic inequalities. Our results extend some previous existence results for convex quadratic programming problems in finite-dimensional setting.

In connection with Theorem 3.5, Examples 3.6 and 3.8 the following question seems to be interesting: Is there any verifiable necessary and sufficient condition (stronger than (3.17), but weaker than (3.20)) for the solution existence of (CQP)?

Acknowledgments

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