

## On Zariski Decomposition with and without Support

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**Abstract.** We study Zariski decomposition with support in a negative definite cycle, a variation of Zariski decomposition introduced by Miyaoka [4]: given a negative definite cycle  $G$ , any  $\mathbb{Q}$ -divisor  $D$  decomposes into the sum of a  $G$ -nef and a rigid  $\mathbb{Q}$ -divisor. We prove that such a decomposition actually exists for an arbitrary  $\mathbb{Q}$ -divisor. Moreover, we show that, under the hypothesis that  $D$  is pseudo-effective, we can drop the assumption of  $G$  being negative definite, and obtain decompositions of  $D$  with respect to arbitrary cycles. Our methods are inspired by a work of Bauer [1], in which he gives a simpler proof of Zariski's original result [5], and by adapting his proof to other cases, we are able to provide an alternative approach to this circle of ideas.

### 1. Introduction

Given a  $\mathbb{Q}$ -divisor  $D$ , a *Zariski decomposition of  $D$*  is a decomposition

$$D = P + N,$$

where  $P$  and  $N$  are  $\mathbb{Q}$ -divisors (called *nef part* and *negative part* respectively) such that:

- (i)  $P$  is nef;
- (ii)  $N$  is effective;
- (iii)  $N$  is either zero or it has negative definite intersection matrix;
- (iv)  $P.C = 0$  for every irreducible component  $C$  of  $N$ .

In 1962, Zariski [5] proved existence and uniqueness of Zariski decomposition of effective  $\mathbb{Q}$ -divisors.

**Theorem 1.1** (Zariski decomposition of effective  $\mathbb{Q}$ -divisors, [5]). *Let  $D$  be an effective  $\mathbb{Q}$ -divisor. Then  $D$  admits a unique Zariski decomposition  $D = P + N$ . Moreover,  $P$  is effective.*

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The original context in which this result was born is the Riemann-Roch problem, i.e., the problem of computing the dimension of a linear system; in particular, Zariski wanted to study the asymptotic behavior of the linear system  $|nD|$  as  $n$  grows. His answer made strong use of the decomposition that bears his name: he proved that the order of growth of  $\dim |nD|$  is determined by the positive part  $P$  of  $D$ ; more precisely, it is the “the self-intersection number” of  $P$ .

In 1979, Fujita [3] extended Zariski’s result to pseudo-effective  $\mathbb{Q}$ -divisors, although the nef part  $P$  is not necessarily effective anymore. In 2008, Bauer [1] provided a simpler proof of Zariski’s original result: the key idea is that the nef part  $P$  of the Zariski decomposition is the largest nef  $\mathbb{Q}$ -subdivisor of  $D$ , and thus he makes use of a maximality argument relative to the nef part of the given  $\mathbb{Q}$ -divisor, rather than on the sophisticated procedure Zariski used to build the negative one; more precisely, he proves the existence and uniqueness of Zariski decomposition by looking at the system of linear inequalities that the nef part has to satisfy. In the same year, Miyaoka [4] introduced the concept of Zariski decomposition with support in a negative definite cycle, in a context which was far from the one where the original problem was born: given a  $\mathbb{Q}$ -divisor  $D$  and a negative definite cycle  $G = \sum_{i=1}^m G_i$ , a *Zariski decomposition of  $D$  with support in  $G$*  is a decomposition

$$D = P_G + N_G$$

where  $P_G$  and  $N_G$  are  $\mathbb{Q}$ -divisors (called  *$G$ -nef part* and *negative part* respectively) such that:

- (a)  $P_G$  is  $G$ -nef (namely,  $P_G \cdot G_i \geq 0$ ,  $\forall i = 1, 2, \dots, m$ );
- (b)  $N_G$  is effective;
- (c)  $N_G$  is supported on a subset of  $G$ , i.e.,  $N_G = \sum_i \nu_i G_i$ ,  $\nu_i \geq 0$ ;
- (d)  $P_G \cdot C = 0$  for every irreducible component  $C$  of  $N_G$ .

The so-obtained decomposition has to be thought as a relative version of Zariski decomposition: the nefness of  $P$  is now replaced by the weaker condition of being  $G$ -nef, i.e., nef on the components of  $G$ . In his paper, Miyaoka states the following

**Theorem 1.2** (Zariski decomposition of effective  $\mathbb{Q}$ -divisors with support in a negative definite cycle, Proposition 2.1 of [4]). *Let  $G$  be a negative definite cycle and let  $D$  be an effective  $\mathbb{Q}$ -divisor on  $X$ . Then  $D$  admits a unique Zariski decomposition with support in  $G$ . Moreover,  $P_G$  is effective.*

**Results.** The aim of this paper is give an alternative and simpler approach to questions related to Zariski decomposition (with and without support) for surfaces, by generalizing

Bauer's proof of the existence and uniqueness of Zariski decomposition in the effective case [1]. This is achieved by simply looking at certain systems of inequalities coming from the conjectural properties of the nef part of the decomposition. In all cases, a sufficient condition for the existence of the Zariski decomposition in question turns out to be the existence of a solution to such a system, while uniqueness follows after taking a solution which is maximal in a suitable sense.

Miyaoka's paper refers to [5] for a proof. Instead, we will give an alternative proof which provides a concrete application of Bauer's method [1], distancing itself from the original idea of Zariski. Although the idea is not original, we include the proof because of its instructive nature: in fact, a closer look reveals the points which are the key to the generalizations we study hereinafter, as we observe in Remark 2.7.

Afterwards, we study some generalization of Zariski decomposition with support. A first question we answer concerns the  $\mathbb{Q}$ -divisors which are characterized by the existence of such decomposition, and it is motivated by the analogous question for Zariski decomposition in the sense of Fujita. In fact, Fujita showed that pseudo-effective  $\mathbb{Q}$ -divisors admit Zariski decomposition, and conversely  $\mathbb{Q}$ -divisors admitting Zariski decomposition are necessarily pseudo-effective. In a similar fashion, we investigate which  $\mathbb{Q}$ -divisors admit Zariski decomposition with support in a negative definite cycle: it turns out that every  $\mathbb{Q}$ -divisor admits such a decomposition.

**Theorem 1.3** (Zariski decomposition with support in negative definite cycle for arbitrary  $\mathbb{Q}$ -divisors). *Let  $G = \sum_{i=1}^q G_i$  be a negative definite cycle and let  $D$  be an arbitrary  $\mathbb{Q}$ -divisor on  $X$ . Then  $D$  admits a unique Zariski decomposition with support in  $G$ .*

Another question we address concerns the cycle  $G$  we consider in the Zariski decomposition with support. More precisely, we ask whether we can consider decompositions with support in arbitrary cycles  $G$ , not necessarily negative definite. We consider the case when  $D$  is a pseudo-effective  $\mathbb{Q}$ -divisor: under this assumption, there is a number of results by Fujita [3] that enables us to reduce to Theorem 1.3, and use it iteratively to show the existence and uniqueness of Zariski decomposition with support in arbitrary cycles.

**Theorem 1.4** (Zariski decomposition with support in an arbitrary cycle for pseudo-effective  $\mathbb{Q}$ -divisors). *Let  $G = \sum_{i=1}^m G_i$  be an arbitrary cycle and let  $D$  be a pseudo-effective  $\mathbb{Q}$ -divisor on  $X$ . Then  $D$  admits a unique Zariski decomposition with support in  $G$ .*

The proof also shows that Zariski decomposition in the sense of Fujita [3] is obtained by iterating Zariski decomposition with support, a consideration that might be known to experts. We also notice that for a given cycle  $G$ , it is only the subcycle of curves with negative self-intersection that contributes to the Zariski decomposition with support in  $G$ , and we once again point out the importance of negative definite cycles in the geometry

of surfaces. Nevertheless, not every negative curve in the cycle is needed as shown in examples below.

2. A new proof of Zariski decomposition with support in a negative definite cycle

2.1. Background

Let  $X$  be a surface, i.e., a 2-dimensional nonsingular projective variety over an algebraically closed field  $k$ . A divisor  $D$  is *effective* if  $D = \sum d_i D_i$ ,  $d_i \in \mathbb{Z}$ ,  $D_i$  irreducible, and  $d_i \geq 0 \forall i$ ;  $D$  is *nef* (numerically effective) if for every irreducible curve  $C$  on  $X$  we have  $D.C \geq 0$ ;  $D$  is said to be *pseudo-effective* if  $D.P \geq 0$  for every nef divisor  $P$  on  $X$ .

Given a divisor  $D = \sum_{i=1}^n d_i D_i$ , where  $d_i \in \mathbb{Z}$  and  $D_i$  is an irreducible curve, the matrix

$$\mu_D := \begin{bmatrix} D_1.D_1 & \cdots & D_1.D_n \\ \vdots & \ddots & \vdots \\ D_n.D_1 & \cdots & D_n.D_n \end{bmatrix}$$

is called the *intersection matrix* of  $D$ . The intersection matrix of a divisor is independent of the coefficients of the irreducible components, and therefore the definition above can be extended in a natural way to  $\mathbb{Q}$ -divisors and  $\mathbb{R}$ -divisors.

Every ( $\mathbb{Q}$ - or  $\mathbb{R}$ -) divisor  $D = \sum_{i=1}^n d_i D_i$  induces a quadratic form  $\Phi_D$  by means of its intersection matrix. A finite sum  $G = \sum_{i=1}^m G_i$  of irreducible curves  $G_i \subset X$  is said to be a *negative definite cycle* if the intersection matrix  $\mu_G$  is negative definite, meaning that the quadratic form  $\Phi_G$  induced by  $G$  is negative definite. The components  $G_i$  of a negative definite cycle must be distinct, i.e.,  $G$  is reduced: if not, the matrix  $\mu_G$  would have 2 equal columns, hence the quadratic form  $\Phi_G$  would not be negative definite. A typical example of negative definite cycle is given by the  $f$ -exceptional locus of a surjective morphism of surfaces  $f: X \rightarrow Y$ , i.e., the union of the curves that  $f$  contracts to a point (cf. [4], § 2). For another example, one may consider an effective  $\mathbb{Q}$ -divisor  $D \subset X$  and write its Zariski decomposition  $D = P + N$ ,  $N = \sum_i \nu_i N_i$  (with regard to Theorem 1.1); then  $N_{\text{red}} := \sum_i N_i$  is a negative definite cycle.

2.2. An alternative approach to Theorem 1.2

We recall the following *componentwise ordering*: given

$$\underline{x} = (x_1, \dots, x_n), \underline{y} = (y_1, \dots, y_n) \in \mathbb{R}^n,$$

we put  $\underline{x} \leq \underline{y}$  if and only if  $x_i \leq y_i, \forall i = 1, \dots, n$ ; similarly,  $\underline{x} < \underline{y}$  if and only if  $x_i < y_i, \forall i = 1, \dots, n$ . This ordering naturally carries over to divisors  $\sum d_i D_i$  with specified curves  $D_1, \dots, D_n$ .

We now proceed with proving Theorem 1.2: we include the proof because of its instructive nature, as a closer look reveals the points which are the key to generalize this approach to more general setting (see Remark 2.7).

*Proof of Theorem 1.2.* We start by proving the existence of such a decomposition. Let  $D = \sum_{i=1}^n d_i D_i$ , with  $D_i$  integral curve and  $d_i \in \mathbb{Q}_{>0}$ ,  $\forall i = 1, \dots, n$ . Consider now  $P$  such that  $0 \leq P \leq D$ ,  $P = \sum_{i=1}^n x_i D_i$ ,  $0 \leq x_i \leq d_i$ . We now have that

$$(2.1) \quad \begin{aligned} P \text{ is } G\text{-nef} &\iff P \cdot G_j \geq 0 \quad \forall j = 1, \dots, m \\ &\iff \sum_{i=1}^n x_i (D_i \cdot G_j) \geq 0 \quad \forall j = 1, \dots, m. \end{aligned}$$

*Claim 2.1.* The system of inequalities (2.1) has a maximal solution (with respect to the ordering  $\leq$ ) in the cuboid  $[0, d_1] \times \dots \times [0, d_n] \subset \mathbb{R}^n$ .

Now let  $P$  be an  $\mathbb{R}$ -divisor defined by a maximal solution to the system of inequalities above,  $P = \sum_{i=1}^n x_i D_i$ . Set  $N := D - P$ ; then, conditions (a) and (c) are satisfied by construction (although  $P$  and  $N$  might have real coefficients).

*Claim 2.2.* Properties (b) and (d) hold as well.

This ends the proof of the existence of such a decomposition with real coefficients. A closer look reveals that

*Claim 2.3.* The decomposition actually takes place at the level of  $\mathbb{Q}$ -divisors, i.e.,  $P, N$  as above are  $\mathbb{Q}$ -divisors.

Assume now that we are given a decomposition  $D = P + N$ .

*Claim 2.4.*  $P$  is a maximal  $G$ -nef subdivisor of  $D$ .

Now we are left to prove uniqueness. We show that a maximal  $G$ -nef  $\mathbb{Q}$ -subdivisor of  $D$  is in fact unique.

*Claim 2.5.* If  $P' = \sum_{i=1}^n x'_i D_i$  and  $P'' = \sum_{i=1}^n x''_i D_i$  are  $G$ -nef  $\mathbb{Q}$ -subdivisors of  $D$ , then so is  $P = \max(P', P'') := \sum_{i=1}^n x_i D_i$ , where  $x_i := \max(x'_i, x''_i)$ .

It follows that  $P$  is the maximal  $G$ -nef subdivisor of  $D$ ; this concludes the proof of the theorem. □

*Remark 2.6.* We remark that given an effective  $\mathbb{Q}$ -divisor  $D$  on  $X$ , its Zariski decomposition  $D = P + N$  in the sense of Theorem 1.1 coincides with the Zariski decomposition of  $D$  with support in  $N_{\text{red}}$ .

2.3. Technical lemmata

We now pay our debts, by proving the claims mentioned in the proof above.

*Proof of Claim 2.1.* Indeed, the subset  $K$  of  $\mathbb{R}^n$  described by these inequalities is the intersection of finitely many half-spaces; notice that we always have a solution (the vector  $\underline{x} = \underline{0}$ ). Consider the family of hyperplanes

$$H_t := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = t \sum_{i=1}^n d_i \right\};$$

then, there is a maximal  $t$  such that  $H_t$  intersects  $K$ . □

*Proof of Claim 2.2.* Notice that in case  $D = P$ , conditions (b) and (d) hold, and thus we can assume that  $N$  is nonzero.

(b) Write  $N = \sum_i \nu_i D_i$ . For a fixed  $i$ , consider the intersection numbers  $D_i \cdot G_j$ ,  $j = 1, \dots, m$ ; if  $D_i \cdot G_j \geq 0 \forall j = 1, \dots, m$ , then  $P + \varepsilon D_i$  is  $G$ -nef (for a suitable  $\varepsilon > 0$ ), and this contradicts the maximality of  $P$ . Thus there must exist a  $j = j(i)$  such that  $D_i \cdot G_j < 0$ , thus  $D_i = G_j$  and  $D_i^2 = G_j^2 < 0$ . Since this holds for every  $i$ , we get  $N = \sum_i \nu_i G_i$ , after possibly rearranging indexes.

(d) By (b),  $N = \sum_i \nu_i G_i$ . If  $P \cdot G_i > 0$ , with  $G_i \subseteq \text{supp}(N)$ , then  $P + \varepsilon G_i \leq D$  and  $P + \varepsilon G_i$  is  $G$ -nef, for small enough  $\varepsilon > 0$ , contradicting the maximality of  $P$ . Then  $P \cdot G_i = 0$ , because  $P$  is  $G$ -nef, and  $P \cdot N = 0$ . □

*Proof of Claim 2.3.* Assume we are given a decomposition  $D = P + N$  with real coefficients; then the negative part  $N = \sum_i \nu_i G_i$  has negative definite matrix because  $\Phi_N$  is the restriction of  $\Phi_G$  to the subspace  $V$  of  $\mathbb{R}^m$  defined by

$$V := \{ \underline{x} \in \mathbb{R}^m \mid x_i = 0 \forall i : \nu_i = 0 \}.$$

Now, notice that  $P = \sum_{i=1}^n x_i D_i$  is such that

1.  $0 = P \cdot D_j = \sum_{i=1}^n x_i (D_i \cdot D_j)$ , for all  $j$  corresponding to  $D_j \subset \text{supp}(N)$ , by orthogonality of  $P$  and  $N$ ;
2.  $x_i = d_i$ , for all  $i$  such that  $D_i \not\subseteq \text{supp}(N)$ .

Hence, possibly after rearranging, this can be written into matricial form as

$$\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \underline{d} \end{bmatrix} \in \mathbb{Q}^n,$$

where  $A = \mu_N$  is negative definite,  $0$  has zero entries,  $I$  is the identity matrix, and  $\underline{d}$  is the vector of the  $d_i$ 's, everything with the appropriate dimension, in accordance with condition 1 and 2 above. Since the matrix on the left-hand side has integer coefficients and is invertible, we get that the vector  $(x_1, \dots, x_n)$  actually lies in  $\mathbb{Q}^n$ , proving the existence part of the theorem.  $\square$

*Proof of Claim 2.4.* Indeed, given a  $G$ -nef divisor  $P'$ ,

$$P \leq P' \leq D = P + N,$$

we have that  $P' = P + \sum_i y_i G_i$  ( $y_i \geq 0$ ).  $G$ -nefness of  $P'$  and orthogonality between  $P$  and  $N$  imply that

$$0 \leq P'.G_j = \sum_i y_i (G_i.G_j) \quad \forall j.$$

Then, by multiplying by  $y_j$ , we get  $\sum_i y_i y_j (G_i.G_j) \geq 0 \quad \forall j$ , and by summing over all  $j$ , we get

$$0 \leq \sum_{i,j} y_i y_j (G_i.G_j) = \left( \sum_i y_i G_i \right)^2 = \Phi_G(\underline{y}).$$

Since  $\Phi_G$  is negative definite, it can only be  $\Phi_G(\underline{y}) = 0$ , and this happens if and only if  $\underline{y} = \underline{0}$ , yielding  $P' = P$  and thus the maximality of  $P$ .  $\square$

*Proof of Claim 2.5.* Showing that  $P$  is  $G$ -nef is equivalent to showing that it is  $G_i$ -nef for every  $i = 1, \dots, m$ . Notice that we can write  $P$  as the sum of a nef and an effective  $\mathbb{Q}$ -divisor, for instance

$$P = P' + E, \quad E := P - P' = \sum_k e_k D_k.$$

If  $G_i \not\subseteq \text{supp}(E)$ , then  $P.G_i \geq 0$ ; otherwise,  $G_i = D_j$  for some  $j$ . If  $x'_j \geq x''_j$ , we get

$$\begin{aligned} E.G_i &= E.D_j = \sum_{k=1}^n \{ \max(x'_k, x''_k) - x'_k \} (D_k.D_j) \\ &= \sum_{k \neq j} \{ \max(x'_k, x''_k) - x'_k \} (D_k.D_j) \geq 0, \end{aligned}$$

and thus

$$P.G_i = \underbrace{E.G_i}_{\geq 0} + \underbrace{P'.G_i}_{\geq 0} \geq 0,$$

i.e.,  $P$  is  $G_i$ -nef. If  $x'_j < x''_j$  instead, we just consider the analogous decomposition of  $P$  as

$$P = P'' + F, \quad F := P - P'' = \sum_k f_k D_k,$$

and the  $G_i$ -nefness of  $P$  follows as above.  $\square$

*Remark 2.7.* Notice that the proofs of Claim 2.3, Claim 2.4 and Claim 2.5 do not rely on  $P$  being effective, a fact we will make use of in what follows.

### 3. Improvements to Zariski decomposition with support

#### 3.1. Decomposition of arbitrary $\mathbb{Q}$ -divisors

We now would like to generalize Theorem 1.2; we will first try to relax the assumption on  $D$  being effective. The motivation lies in the fact that Zariski decomposition characterizes pseudo-effectivity. Fujita [3] showed that if  $D$  is pseudo-effective, then we do have a Zariski decomposition; it is straightforward to see that the converse also holds true. Therefore, we formulate the following

**Question 3.1.** Which  $\mathbb{Q}$ -divisors on a surface are characterized by the existence of a Zariski decomposition with support in  $G$ , for some fixed negative definite cycle  $G$ ?

The answer, which lies in Theorem 1.3, is that every  $\mathbb{Q}$ -divisor admits such a decomposition, as we will prove shortly. The proof of Theorem 1.2 shows that Theorem 1.3 would follow from

**Lemma 3.2.** *Let  $D$  be a  $\mathbb{Q}$ -divisor, and let  $G = \sum_{i=1}^q G_i$  be a negative definite cycle. Then there exists a subdivisor  $P \leq D$ , possibly with real coefficients, such that  $P$  is  $G$ -nef.*

*Proof.* If  $D$  is  $G$ -nef, we put  $P := D$ ; otherwise,  $D$  is negative on some of the  $G_i$ 's. Set

$$P := D - \sum_{i=1}^q x_i G_i,$$

where  $\underline{x} = (x_1, \dots, x_q) \in \mathbb{R}_{\geq 0}^q$ . Then  $P$  is  $G$ -nef if and only if

$$P.G_j \geq 0, \forall j = 1, \dots, q \iff \sum_{i=1}^q x_i (G_i.G_j) \leq D.G_j, \forall j = 1, \dots, q$$

and the last condition is equivalent to the matrix inequality

$$(3.1) \quad \begin{bmatrix} G_1.G_1 & \cdots & G_1.G_q \\ \vdots & \ddots & \vdots \\ G_q.G_1 & \cdots & G_q.G_q \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} \leq \begin{bmatrix} D.G_1 \\ \vdots \\ D.G_q \end{bmatrix}.$$

The inequality (3.1) is equivalent to the following homogeneous one

$$(3.2) \quad \begin{bmatrix} -G_1.G_1 & \cdots & -G_1.G_q & D.G_1 \\ \vdots & \ddots & \vdots & \vdots \\ -G_q.G_1 & \cdots & -G_q.G_q & D.G_q \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_q \\ 1 \end{bmatrix} \geq 0;$$



if we denote the matrix in (3.2) by  $M$ , (3.2) has a solution if and only if there exists a solution to the system

$$(3.3) \quad \begin{cases} t \neq 0, \\ M \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_q \\ t \end{bmatrix} \geq 0. \end{cases}$$

Moreover, all principal minors of  $M$  are positive: for, notice that  $M$  is built out of  $-\mu_G$ , which is positive definite since  $\mu_G$  is negative definite; the claim now clearly follows. Finally, we get the result by applying

**Fact 3.3.** [2] Let  $A$  be an  $n \times n$  real matrix. If all the principal minors are positive, then the system

$$(3.4) \quad \begin{cases} \underline{x} \geq 0 \\ \underline{x} \neq 0 \\ A\underline{x} > 0 \end{cases}$$

has a solution.

□

*Proof of Theorem 1.3.* The lemma ensures the existence of a  $G$ -nef subdivisor of  $D$  (possibly with real coefficients); now we can choose the solution  $\underline{x}$  to be minimal with respect to the ordering  $\leq$  defined in Section 2. This leads to a maximal subdivisor  $P$  of  $D$ , with respect to the property of being  $G$ -nef, and we can now apply the same argument in Theorem 1.2 to conclude, by virtue of Remark 2.7. □

### 3.2. The case of arbitrary support

So far, we can decompose any  $\mathbb{Q}$ -divisor with respect to a negative definite cycle  $G$ ; we now would like to weaken the hypothesis on  $G$  being negative definite. Thus, we formulate the following

**Question 3.4.** Does there exist a Zariski decomposition with support in an arbitrary cycle?

We might have to assume  $D$  satisfies some additional property, as the following example shows.

**Example 3.5.** Let  $X = \mathbb{P}_k^2$ ,  $D = -H$  and  $G = H$ ,  $H$  being the hyperplane divisor. Then,  $-H.H = -1$ , and thus  $-H$  is not  $H$ -nef, but also  $H^2 > 0$ , meaning that  $H$  is not a negative definite cycle. Hence, there is no Zariski decomposition of  $-H$  with support in  $H$ .

A nice setup seems to be assuming  $D$  to be pseudo-effective. In fact, pseudo-effective  $\mathbb{Q}$ -divisors have the property to detect negative definite cycles by means of the intersection form, as the following result points out.

**Lemma 3.6** (Lemma 1.10 of [3]). *Let  $\{C_i\}_{i=1,\dots,q}$  be a family of integral curves such that the matrix*

$$\begin{bmatrix} C_1.C_1 & \cdots & C_1.C_r \\ \vdots & \ddots & \vdots \\ C_r.C_1 & \cdots & C_r.C_r \end{bmatrix}$$

*is negative definite for some  $r < q$ . If  $D$  is a pseudo-effective  $\mathbb{Q}$ -divisor such that  $D.C_i \leq 0$  for every  $i = 1, \dots, q$  and  $D.C_i < 0$  for  $i = r + 1, \dots, q$ , then the matrix*

$$\begin{bmatrix} C_1.C_1 & \cdots & C_1.C_r & C_1.C_{r+1} & \cdots & C_1.C_q \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_r.C_1 & \cdots & C_r.C_r & C_r.C_{r+1} & \cdots & C_r.C_q \\ C_{r+1}.C_1 & \cdots & C_{r+1}.C_r & C_{r+1}.C_{r+1} & \cdots & C_{r+1}.C_q \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ C_q.C_1 & \cdots & C_q.C_r & C_q.C_{r+1} & \cdots & C_q.C_q \end{bmatrix}$$

*is also negative definite.*

More precisely, suppose  $G = \sum G_i$  is an arbitrary cycle, and let  $D$  be a pseudo-effective  $\mathbb{Q}$ -divisor. Then, the cycle of curves on which  $D$  is negative,

$$G' := \sum_{D.G_i < 0} G_i,$$

is a negative definite cycle (apply Lemma 3.6 to the case  $r = 0$ ); this puts us in the situation of Theorem 1.3. We will show that, under the assumption that  $D$  is pseudo-effective, we do have Zariski decomposition with support in an arbitrary cycle  $G$ . The main idea is to fix the non- $G$ -nefness of  $D$  step by step, by iterating Theorem 1.3. The key to the iteration process lies within the following

**Lemma 3.7** (Lemma 1.8 of [3]). *Let  $\{C_i\}_{i=1,\dots,q}$  be a family of distinct integral curves, and let  $E := \sum_{i=1}^q a_i C_i$  be a  $\mathbb{Q}$ -divisor. If  $D$  is a pseudo-effective  $\mathbb{Q}$ -divisor such that  $(D - E).C_i \leq 0 \forall i = 1, \dots, q$ , then  $D - E$  is pseudo-effective.*

*Proof of Theorem 1.4.* If  $D$  is  $G$ -nef, then we are done; otherwise, define

$$G^{(1)} := \sum_{D.G_i < 0} G_i;$$

by Lemma 3.6,  $G^{(1)}$  is a negative definite cycle. Hence, we can write the Zariski decomposition of  $D$  with support in  $G^{(1)}$ :

$$D = P_{G^{(1)}} + N_{G^{(1)}}.$$

If  $P_{G^{(1)}}$  is  $G$ -nef, then we are done; otherwise, we notice that  $P_{G^{(1)}}$  is pseudo-effective by Lemma 3.7. Now, define

$$G^{(2)} := G^{(1)} + \sum_{P_{G^{(1)}}.G_i < 0} G_i;$$

again by Lemma 3.6,  $G^{(2)}$  is a negative definite cycle. Write the Zariski decomposition with support in  $G^{(2)}$

$$P_{G^{(1)}} = P_{G^{(2)}} + N_{G^{(2)}},$$

and obtain the decomposition

$$D = P_{G^{(2)}} + (N_{G^{(1)}} + N_{G^{(2)}}).$$

If  $P_{G^{(2)}}$  is  $G$ -nef, then we are done; otherwise we repeat this process, which must come to an end since  $G$  is a finite sum of integral curves. □

The iterative approach to Theorem 1.4 shows that Zariski decomposition in the sense of Fujita [3] is obtained by iterating Zariski decomposition with support: if  $D$  is pseudo-effective, let  $G$  be the cycle of curves which are negative on  $D$  (their number is bounded by the Picard number, thanks to Lemma 3.6). Then, we apply repeatedly Theorem 1.4 until we get the desired decomposition.

Notice that this approach indeed coincides with Fujita's: given  $D$  pseudo-effective, he builds a  $\mathbb{Q}$ -divisor  $N_1$  such that  $D.G_i = N_1.G_i$ , for every curve  $G_i$  which is negative on  $D$ . But since the cycle  $\sum G_i$  is negative definite, then  $N_1$  must coincide with the one we built analogously, and similarly at any other step of the iteration.

*Remark 3.8.* In this new setting, we could not make use of Bauer's method. In fact, Bauer's idea was used in an effective setup, in which we did know where to look for curves which are negative on the divisor we started with: in fact, these bad curves are among the components of  $D$  itself, since  $D.C \geq 0$  for every irreducible  $C$  not in the support of  $D$ , and thus there are only finitely many conditions to impose in order to get nefness; moreover, the coefficients of  $P$  are bounded from below and above. However, in the pseudoeffective case, these conditions do not hold in general, for:

1. there may exist some curve which is not in the support of  $D$ , but on which  $D$  is negative;
2. there is no elementary reasoning (e.g., by using linear algebra) that allows us to argue that the number of curves on which  $D$  is negative is finite (Fujita dealt with these nontrivially in [3]);
3. the coefficients of  $P$  are not bounded from below anymore.

### 3.3. More on the support

Now that we have established the existence and uniqueness of Zariski decomposition with support in an arbitrary cycle, it remains to determine how much of the support  $G$  we really need.

**Question 3.9.** To what extent is the support  $G$  really necessary? In other words, does there exist a subdivisor  $G'$  of  $G$  which is enough to realize the Zariski decomposition with support?

First of all, notice that the proof of Theorem 1.4 shows that  $N_G$  has support in the components of  $G$  having negative self-intersection. Secondly, curves with nonnegative self-intersection play no role in the Zariski decomposition with support. In fact, writing

$$G = \sum_i G_i = G^+ + G^-, \quad G^+ := \sum_{G_i^2 \geq 0} G_i, \quad G^- := \sum_{G_i^2 < 0} G_i,$$

we see that, for every pseudo-effective  $D$ ,  $D \cdot G_i \geq 0$  for all components  $G_i \subset G^+$ . This means that the curves in  $G^+$  play no role at any stage of the iteration, and in particular they will not appear as components of the negative part of  $D$ . It follows that, we can restrict to consider cycles whose components have negative self-intersection, once again pointing out the importance of this class of curves in the geometry of surfaces.

**Example 3.10** (Example 3.5 reloaded). Going back to the example  $X = \mathbb{P}^2$ ,  $D = -H$  and  $G = H$ , we see that we were actually asking for the Zariski decomposition of  $D$  with support in  $G^- = 0$ .

However, in general not all the curves of a cycle will be essential for the existence of a decomposition. It is straightforward to come up with examples: we just need a surface whose Picard number is exceeded by the number of negative curves.

**Example 3.11.** Let

$$X : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

be the Fermat quartic surface in  $\mathbb{P}^3$ : it is a K3 surface containing 48 lines, each having self-intersection  $-2$ . Since  $X$  has Picard number 20, the 48 lines cannot form a negative definite cycle. Same for the Schur quartic surface in  $\mathbb{P}^3$

$$Y : x_0^4 + x_0x_1^3 + x_2^4 + x_2x_3^3 = 0,$$

which contains 64 lines of self-intersection  $-2$  and has Picard number 20.

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### References

- [1] T. Bauer, *A simple proof for the existence of Zariski decompositions on surfaces*, J. Algebraic Geom. **18** (2009), no. 4, 789–793.  
<http://dx.doi.org/10.1090/s1056-3911-08-00509-2>
- [2] M. Fiedler and V. Pták, *Some generalizations of positive definiteness and monotonicity*, Numer. Math. **9** (1966), no. 2, 163–172 . <http://dx.doi.org/10.1007/bf02166034>
- [3] T. Fujita, *On Zariski problem*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), no. 3, 106–110 . <http://dx.doi.org/10.3792/pjaa.55.106>
- [4] Y. Miyaoka, *The orbibundle Miyaoka-Yau-Sakai inequality and an effective Bogomolov-McQuillan theorem*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 403–417. <http://dx.doi.org/10.2977/prims/1210167331>
- [5] O. Zariski, *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. of Math. (2) **76** (1962), no. 3, 560–615.  
<http://dx.doi.org/10.2307/1970376>

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