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Nonlinear Stability of Traveling Wavefronts for Delayed Reaction-diffusion Equation with Nonlocal Diffusion

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Abstract. In this paper, we consider a class of nonlocal dispersal equation with nonlocal time-delayed reaction. We prove that all noncritical wavefronts are globally exponentially stable by the weighted energy method and comparison principle. However, for the critical wavefronts, we prove that they are globally asymptotically stable.

1. Introduction

The theory of traveling wave solutions of reaction-diffusion equations has attracted much attention due to its significant nature in biology, chemistry, epidemiology and physics (see, [1, 2, 5, 7, 8, 14, 17, 26, 31]). Among the basic problems in the theory of traveling wave solutions, the global stability of traveling wave solutions is an extremely important one. In this paper, we are interested in the stability of traveling waves for the following class of nonlocal diffusion equation with nonlocal time-delayed response term

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= d\int_{\mathbb{R}} J(y)(u(x-y,t)-u(x,t))\,\mathrm{d}y \\ &\quad + f(u(x,t), k*u(x,t-\tau)), \quad t\geq 0,\ x\in\mathbb{R},\end{aligned} \tag{1.1}$$

with the initial data

$$u(x,s) = u_0(x,s), \quad s\in[-\tau,0],\ x\in\mathbb{R}, \tag{1.2}$$

where

$$k*u(x,t-\tau) = \int_{\mathbb{R}} k(y)u(x-y,t-\tau)\,\mathrm{d}y.$$

In [32], Yu and Yuan investigated the existence of traveling wavefronts of equation (1.1) by using upper-lower solutions method and Schauder's theorem. Furthermore, they obtained the asymptotic behavior of traveling waves with the help of Ikehara's theorem by constructing a Laplace transform representation of a solution. For such a local or nonlocal

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diffusion model, the most attractive object is its traveling wavefronts. And the existence, nonexistence, uniqueness, asymptotic behavior, propagation speed and stability of traveling wavefronts are widely studied, such as [3, 4, 9, 11–13, 15, 30] and the references cited therein.

The equation (1.1) includes lots of evolution equations for the single species model. For example, by taking the function $f(u(x,t), k * u(x,t-\tau)) = -du(x,t) + b(u(x,t-\tau))$, equation (1.1) reduces to the following nonlocal diffusion equation with delay

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(y)[u(x-y,t) - u(x,t)]\,\mathrm{d}y - du(x,t) + b(u(x,t-\tau)), \quad x \in \mathbb{R},\ t > 0. \tag{1.3}$$

Here, $u(x,t)$ denotes the total mature population of the species (with the age greater than the maturation age $\tau > 0$) at time t and position x, the nonlinear function $b(u)$ is the birth rate of the mature population, $d > 0$ is the death rate. In [24], Pan, Li and Lin obtained the existence and asymptotic behavior of noncritical traveling wavefronts by constructing proper upper-lower solutions. Moreover, the asymptotic stability with phase shift as well as the uniqueness up to translation of noncritical traveling wavefronts are proved by applying the idea of squeezing technique.

If we take the diffusion kernel $J(x) = \delta(x) + \delta''(x)$ (where δ is the Dirac function) and reaction term $f(u(x,t), k * u(x,t-\tau)) = -d(u(x,t)) + \varepsilon \int_{\mathbb{R}} f(y) b(u(x-y,t-\tau))\,\mathrm{d}y$, then (1.1) reduces to the following local reaction-diffusion equations with time-delayed term

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} - d(u(x,t)) + \varepsilon \int_{\mathbb{R}} f(y) b(u(x-y,t-\tau))\,\mathrm{d}y, \quad x \in \mathbb{R},\ t > 0,$$

where D, ε and the delay τ are positive constant. Nonlinear functions $b(u)$ and $d(u)$ denote the birth and death rates of the population respectively. In [19], Mei, Ou and Zhao proved that all noncritical traveling wavefronts are globally exponentially stable, and critical wavefronts are globally algebraically stable by the combination of weighted energy method and the Green function technique. We can study more works of the weighted energy method by referring to [7, 8, 16–22, 28, 29] and references therein for more details. In [23], Mei and Wang extended this result to the general nonlocal Fisher-Kpp equations in n-dimensional space by the weighted energy method combining Fourier transform.

In the present paper, we make the following assumptions throughout this paper.

(H1) There exist $u_- = 0$ and $u_+ > 0$ such that $f(0,0) = f(u_+,u_+) = 0$, $f \in C^2([0,u_+]^2, \mathbb{R})$, $f(u,u) > 0$ for all $u \in (0,u_+)$;

(H2) $\partial_1 f(0,0) \le 0$ and $\partial_1 f(u_+,u_+) + \partial_2 f(u_+,u_+) < 0$;

(H3) $\partial_2 f(u,v) \ge 0$, $\partial_{11} f(u,v) < 0$ and $\partial_{ij} f(u,v) \le 0$ $(i,j = 1,2)$ for all $(u,v) \in [0,u_+]^2$;

(H4) $J \ge 0$, $J(x) = J(-x)$, $\int_{\mathbb{R}} J(x)\,\mathrm{d}x = 1$, and $\int_{\mathbb{R}} J(x) e^{-\lambda x}\,\mathrm{d}x < +\infty$, $\forall\, \lambda \ge 0$;

(H5) $k \geq 0$, $k(x) = k(-x)$, $\int_{\mathbb{R}} k(x)\,\mathrm{d}x = 1$, and $\int_{\mathbb{R}} k(x)e^{-\lambda x}\,\mathrm{d}x < +\infty$, $\forall\, \lambda \geq 0$.

From (H1) and (H3), it can be verified that both $u_- = 0$ and $u_+ > 0$ are constant equilibria of (1.1). Furthermore, we can see that u_- is unstable equilibria and u_+ is stable equilibria for the spatially homogeneous equation associated with equation (1.1). In the biological environment, the kernel functions $J(x)$ and $k(x)$ can be chosen in the form of $J(x) = k(x) = \frac{1}{\sqrt{4\pi\alpha}}e^{-x^2/(4\alpha)}$. It is not difficult to see the functions $J(x)$ and $k(x)$ satisfy assumption (H4) and (H5). Moreover, the response term $f(u, k * u)$ can be chosen in the form of

$$f(u, k * u) = -d(u(x,t)) + \int_{\mathbb{R}} k(x)b(u(x-y, t-\tau))\,\mathrm{d}y,$$

where $d(u) = -\delta u^2$ is the death rate, and $b(u)$ can be chosen in the so-called Nicholson's birth rate function $b(u) = pue^{-au^q}$ ($p > 0$, $a > 0$, $q > 0$). It is not hard to see that function f satisfies assumptions (H1)–(H3).

A traveling wavefront of (1.1) is special solution in the form of $u(x,t) = \phi(x+ct)$ with $\phi(\pm\infty) = u_\pm$, where c is the wave speed. The main purpose of this paper is to investigate the global stability of traveling wavefronts $\phi(x+ct)$ of (1.1), including the case of critical traveling wavefront $\phi(x+c^*t)$. Regarding the monotone traveling wave problems for some scalar reaction-diffusion equations with delay, lots of investigations has been done concerning the stability of traveling wavefronts by using the spectral analysis method, the squeezing technique, the weighted energy method. The first work on the stability was given by Schaaf [26] through a spectral analysis. The stability of traveling waves for local equations of bistable (i.e., equation has two stable equilibriums) was obtain by Smith and Zhao [27] by the method of the upper-lower solutions and squeezing technique. In the monostable case (i.e., one equilibrium is stable and the other is unstable), the study of the stability of traveling waves is not the same as the bistable case and the main difficulty is caused by the unstable equilibrium. The first study of this case was obtained by Mei et al. [22] by using weighted L^2-energy method.

In this paper, we will prove that all noncritical traveling wavefronts are globally exponentially stability by using combination of the comparison principle and the weighted energy method. But for the critical wavefront, the expecting optimal convergence rate $O(t^{-1/2})$ is unable to obtain at this moment mainly because of the effect of the nonlocal diffusion, which has the essential difference with the classical Laplacian operator. Here, we not only obtain the L^2-estimate for $v(\xi,t)$ through the L^1-energy estimate, but also obtain the estimate for $\frac{\partial v(\xi,t)}{\partial t}$ and $\frac{\partial v_\xi(\xi,t)}{\partial t}$. Fortunately, we finally obtain the asymptotic stability of the critical traveling wavefront. But the optimal convergence rate result to the critical traveling wavefront with the effect of the nonlocal diffusion is still an open question.

The rest of this paper is organized as follows. In section 2, we introduce some necessary notations and give the existence of traveling wavefronts of (1.1). Furthermore, we present the proof of global existence and uniqueness with respect to the Cauchy problem (1.1) and (1.2). In section 3, we present the stability of traveling wavefronts and establish some energy estimates in weighted L^1 space and L^2 space, further prove the global exponential stability of the noncritical traveling wavefronts and global asymptotical stability of the critical traveling wavefronts. In section 4, we apply our main stability result to nonlocal Nicholson's blowflies equation and local population model with age-structure.

2. Global existence and uniqueness

In this section, we will mainly concentrate on proving the global existence and uniqueness of a solution to (1.1) and (1.2).

First, we introduce some necessary notations throughout this paper. $C > 0$ denotes a generic constant and C_i $(i = 1, 2, \dots)$ represents a specific constant. Let I be an interval, typically $I = \mathbb{R}$ and $L^p(I)$ is the Lebesgue space of the integrable functions defined on I. $W^{k,p}(I)$ $(k \geq 0,\ p \geq 1)$ is the Sobolev space which function $f(x)$ is defined on I and its weak derivatives $\frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)$ $(i = 1, 2, \dots, k)$ also belong to $L^p(I)$. Further, $L^p_w(I)$ denotes the weighted L^p space for a weighted function $w(x) > 0$,

$$L^p_w(I) = \left\{ f(x) \;\middle|\; \|f\|_{L^p_w(I)} = \left(\int_I w(x)\, |f(x)|^p \,\mathrm{d}x \right)^{1/p} < +\infty \right\}.$$

$W^{k,p}_w(I)$ is the weighted Sobolev space

$$W^{k,p}_w(I) = \left\{ f(x) \;\middle|\; \|f\|_{W^{k,p}_w(I)} = \left(\sum_{i=0}^{k} \int_I w(x) \left| \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x) \right|^p \mathrm{d}x \right)^{1/p} < +\infty \right\}.$$

A traveling wave solution of (1.1) connecting with u_- and u_+ is a solution $u(x, t) = \phi(\xi)$, $\xi = x + ct$, satisfying the following equation

$$c\phi'(\xi) = d \int_{\mathbb{R}} J(x) \left[\phi(\xi - x) - \phi(x)\right] \mathrm{d}x + f\left(\phi(\xi), \int_{\mathbb{R}} k(y)\phi(\xi - y - c\tau)\, \mathrm{d}y \right), \tag{2.1}$$

with asymptotic boundary conditions

$$\phi(\pm\infty) = u_\pm.$$

To obtain the existence of traveling wave solutions, we consider the following function

$$\Delta(\lambda, c) = d \int_{\mathbb{R}} J(x)(e^{-\lambda x} - 1)\, \mathrm{d}x - c\lambda + \partial_1 f(0, 0) + \partial_2 f(0, 0) \int_{\mathbb{R}} k(y) e^{-\lambda(y + c\tau)}\, \mathrm{d}y.$$

Lemma 2.1. *Under the conditions* (H1)–(H5)*, there exist* $\lambda^* > 0$ *and* $c^* > 0$ *such that*

$$\Delta(\lambda^*, c^*) = 0, \qquad \frac{\partial \Delta(\lambda, c)}{\partial \lambda}\bigg|_{(\lambda^*, c^*)} = 0.$$

Furthermore,

- *if* $0 < c < c^*$*, we have* $\Delta(\lambda, c) > 0$ *for all* $\lambda > 0$*;*
- *if* $c > c^*$*, the equation* $\Delta(\lambda, c) = 0$ *has two positive real roots* $\lambda_i = \lambda_i(c)$ $(i = 1, 2)$ *with* $0 < \lambda_1 < \lambda^* < \lambda_2 < +\infty$*, and*

$$\Delta(\lambda, c) \begin{cases} < 0, & \lambda \in (\lambda_1, \lambda_2), \\ > 0, & \lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty). \end{cases}$$

The existence of traveling wavefronts of (1.1) is guaranteed by the following Theorem 2.2. In [32], Yu and Yuan proved the existence of traveling wavefronts of (1.1) by using the upper-lower solutions and Schauder's fixed point theorem.

Theorem 2.2 (Existence of Traveling Wavefronts)**.** *Assume that* $c \geq c^*$ *holds, then* (1.1) *admits a nondecreasing positive traveling wavefront* $u(x, t) = \phi(x + ct) = \phi(\xi)$ *satisfying* (2.1) *with* $\phi(\pm\infty) = u_\pm$*.*

Next, we consider the following initial value problem

$$\begin{cases} u_t(x, t) = F[u](x, t), & t > 0,\ x \in \mathbb{R}, \\ u(x, s) = u_0(x, s), & s \in [-\tau, 0],\ x \in \mathbb{R}, \end{cases}$$

where

$$F[u](x, t) = d \int_{\mathbb{R}} J(y)[u(x - y, t) - u(x, t)]\, \mathrm{d}y + f(u(x, t), (k * u)(x, t - \tau)). \tag{2.2}$$

To obtain the stability of traveling waves, we establish the following comparison principle.

Lemma 2.3 (Comparison Principle)**.** *Assume that* u_1 *and* u_2 *are continuous functions on* $\mathbb{R} \times [0, +\infty)$*, such that* $0 \leq u_i \leq u_+$ $(i = 1, 2)$ *on* $\mathbb{R} \times [0, +\infty)$ *and* $u_1 \geq u_2$ *on* $\mathbb{R} \times [-\tau, 0]$*. Furthermore,* u_1 *and* u_2 *satisfy*

$$\frac{\partial u_1(x, t)}{\partial t} - F[u_1](x, t) \geq \frac{\partial u_2(x, t)}{\partial t} - F[u_2](x, t), \tag{2.3}$$

for all $x \in \mathbb{R}$ *and* $t > 0$*. Then* $u_1 \geq u_2$ *on* $\mathbb{R} \times [0, +\infty)$*.*

Proof. Let μ be such that $\mu-\partial_1 f(0,0)-\partial_2(0,0)e^{-\mu\tau}>0$. Since $v(x,t)=u_2(x,t)-u_1(x,t)$ is continuous and bounded, $v(t)=\sup_{x\in\mathbb{R}} v(x,t)$ is continuous on $[0,+\infty)$. Suppose the assertion is not true. Then there exists $t_0>0$ such that $v(t_0)>0$. It is no loss of generality to assume that

$$v(t_0)e^{-\mu t_0}>v(s)e^{-\mu s}, \quad \text{for all } s\in[-\tau,t_0).$$

Let $\{x_j\}_{j=1}^{\infty}$ be a sequence on $\mathbb{R}$ such that $v(x_j,t_0)>0$ for all $j\geq 1$ and

$$\lim_{j\to+\infty} v(x_j,t_0)=v(t_0).$$

Let $\{t_j\}_{j=1}^{\infty}$ be a sequence in $[0,t_0)$ such that

$$e^{-\mu t_j}v(x_j,t_j)=\max_{t\in[0,t_0]}\left\{e^{-\mu t}v(x_j,t)\right\}.$$

Since $v(t)e^{-\mu t}<v(t_0)e^{-\mu t_0}$ for all $t\in[0,t_0)$, we have $\lim_{j\to+\infty} t_j=t_0$ and

$$\lim_{j\to+\infty} v(x_j,t_j)=v(t_0).$$

Furthermore, for each $j\geq 1$, we obtain

$$0\leq \underline{D}_t\left\{e^{-\mu t}v(x_j,t)\right\}\big|_{t=t_j^-}=e^{-\mu t_j}\left\{\underline{D}_t v(x_j,t_j)-\mu v(x_j,t_j)\right\},$$

where $\underline{D}_t u(x,t)=\liminf_{h\to 0_+}\frac{u(x,t)-u(x,t-h)}{h}$. Thus, we obtain

$$\underline{D}_t(u_2-u_1)(x_j,t_j)=\underline{D}_t v(x_j,t_j)\geq \mu v(x_j,t_j).$$

It follows from (2.3) and assumption (H3) that

$$\begin{aligned}
0 &\geq \underline{D}_t v(x_j,t_j)-d[J*v-v](x_j,t_j)+f(u_1(x_j,t_j),k*u_1(x_j,t_j-\tau))\\
&\quad -f(u_2(x_j,t_j),k*u_2(x_j,t_j-\tau))\\
&\geq (\mu+d)v(x_j,t_j)-dJ*v(x_j,t_j)-\partial_1 f(0,0)v(x_j,t_j)-\partial_2 f(0,0)h*v(x_j,t_j-\tau)\\
&\geq [\mu+d-\partial_1 f(0,0)]v(x_j,t_j)-dv(t_j)-\partial_2 f(0,0)v(t_j-\tau).
\end{aligned}$$

Letting $j\to+\infty$, we obtain

$$\begin{aligned}
0 &\geq [\mu+d-\partial_1 f(0,0)]v(t_0)-dv(t_0)-\partial_2 f(0,0)v(t_0-\tau)\\
&\geq [\mu-\partial_1 f(0,0)-\partial_2 f(0,0)e^{-\mu\tau}]v(t_0).
\end{aligned}$$

Noticing that $\mu-\partial_1 f(0,0)-\partial_2 f(0,0)e^{-\mu\tau}>0$, we obtain $v(t_0)\leq 0$, which contradicts $v(t_0)>0$. Therefore, $v(x,t)=u_2(x,t)-u_1(x,t)\leq 0$ for all $x\in\mathbb{R}$ and $t>0$. This completes the proof. $\square$

Next, we present the global existence and uniqueness result of solution $u(x,t)$ to the Cauchy problem (1.1) and (1.2).

Theorem 2.4 (Global Existence and Uniqueness)**.** *Assume that the initial data satisfy* $0 \le u_0(x,s) \le u_+$ *for all* $(x,s) \in \mathbb{R} \times [-\tau, 0]$*. For a given traveling wave solution* $\phi(x+ct)$ *of* (1.1) *satisfying* (2.1)*, if the initial perturbation data* $u_0(\cdot,s) - \phi(\cdot + cs)$ *is in* $C([-\tau,0], H^1(\mathbb{R}))$*, then there exists a unique global solution* $u(x,t)$ *of the Cauchy problem* (1.1) *and* (1.2) *such that* $0 \le u(x,t) \le u_+$ *for all* $(x,t) \in \mathbb{R} \times [0,+\infty)$ *and* $u(\cdot,t) - \phi(\cdot + ct) \in C([0,+\infty), H^1(\mathbb{R}))$.

Let $v(x,t) = u(x,t) - \phi(x+ct)$, where $\phi(x+ct)$ is a given traveling wave solution of (2.1). Then the Cauchy problem (1.1) and (1.2) can be rewritten as

$$\begin{cases} v_t(x,t) = d[J * v - v](x,t) + Q_0(x,t), & (x,t) \in \mathbb{R} \times [0,+\infty), \\ v(x,s) = v_0(x,s), & (x,s) \in \mathbb{R} \times [-\tau, 0], \end{cases} \tag{2.4}$$

where $v_0(x,x) = u_0(x,s) - \phi(x+cs)$ and

$$Q_0(x,t) = f(u(x,t), k * u(x,t-\tau)) - f(\phi(x+ct), k * \phi(x+ct-c\tau)).$$

In order to prove Theorem 2.4, we give the following two results on the local existence, uniqueness, extension of solutions and boundedness of solutions of (2.4).

Lemma 2.5 (Local Existence and Uniqueness)**.** *For* $v_0 \in C([-\tau,0], H^1(\mathbb{R}))$*, there exists* $t_0 > 0$ *such that* (2.4) *has a unique solution* $v(\cdot,t) \in C([0,t_0), H^1(\mathbb{R}))$*. Furthermore, if* $[0,T)$ *is its maximal interval of existence and* $v(\cdot,t) \in C([0,T), H^1(\mathbb{R}))$*, then either* $T = +\infty$ *or the solution blows up in finite time, in the sense that* $T < +\infty$ *and* $\lim_{t \to T^-} \|v(\cdot,t)\|_{H^1(\mathbb{R})} = +\infty$.

It can be proved by using the standard iteration method (see [6, 25]). Thus the proof is omitted here.

Lemma 2.6 (Boundedness)**.** *If* $v(\cdot,t)$ *is a solution in* $C([0,T), H^1(\mathbb{R}))$ *for* $0 < T < +\infty$*, then there exists a positive constant* C_0*, independent of* T*, such that*

$$\|v(\cdot,t)\|_{H^1(\mathbb{R})} \le C_0 \left(\|v_0(\cdot,0)\|^2_{H^1(\mathbb{R})} + \int_{-\tau}^0 \|v_0(\cdot,s)\|^2_{H^1(\mathbb{R})} \, \mathrm{d}s \right) e^{4\mu t} \tag{2.5}$$

for all $t \in [0,T)$*, where* $\mu = \max\{|\partial_i f(u,v)|, |\partial_{ij} f(u,v)|, u,v \in [0,u_+], i,j = 1,2\}$.

Proof. From the assumption (H3) and the mean-value theorem, we obtain

$$|Q_0(x,t)| \le \mu |v(x,t)| + \mu |k * v(x,t-\tau)|.$$

Multiplying (2.4) by $v(x,t)$ and integrating it over $\mathbb{R}\times[0,t]$ with respect to x and t, we obtain

(2.6)
$$\|v(\cdot,t)\|^2_{L^2(\mathbb{R})} \le \|v_0(\cdot,0)\|^2_{L^2(\mathbb{R})} + 2d\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)(v(x-y,s)-v(x,s))v(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s \\ + 2\mu\int_0^t \|v(\cdot,s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + 2\mu\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)v(x-y,s-\tau)v(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s.$$

Clearly, we have

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)v(x-y,s-\tau)v(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s\\
&\le \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)[v^2(x-y,s-\tau)+v^2(x,s)]\,\mathrm{d}y\mathrm{d}x\mathrm{d}s\\
&= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)v^2(x-y,s-\tau)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)v^2(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s\\
&= \int_{-\tau}^{t-\tau}\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)v^2(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s + \int_0^t\int_{\mathbb{R}} v^2(x,s)\,\mathrm{d}x\mathrm{d}s\\
&\le 2\int_0^t\int_{\mathbb{R}} v^2(x,s)\,\mathrm{d}x\mathrm{d}s + \int_{-\tau}^0\int_{\mathbb{R}} v^2(x,s)\,\mathrm{d}x\mathrm{d}s\\
&= 2\int_0^t \|v(\cdot,s)\|\,2_{L^2(\mathbb{R})}\,\mathrm{d}s + \int_{-\tau}^0 \|v_0(\cdot,s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&2\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)[v(x-y,s)-v(x,s)]v(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s\\
&= 2\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)v(x-y,s)v(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s - 2\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)v^2(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s\\
&\le \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)[v^2(x-y,s)+v^2(x,s)]\,\mathrm{d}y\mathrm{d}x\mathrm{d}s - 2\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)v^2(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s\\
&= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)v^2(x-y,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)v^2(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s\\
&= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)v^2(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s - \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} J(y)v^2(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s\\
&= 0.
\end{aligned} \tag{2.8}$$

Substituting (2.7) and (2.8) into (2.6), we obtain

$$\|v(\cdot,t)\|^2_{L^2(\mathbb{R})} \le 4\mu\int_0^t \|v(\cdot,s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + \left[\int_{-\tau}^0 \|v_0(\cdot,s)\|^2_{L^2(\mathbb{R})} + \|v_0(\cdot,0)\|^2_{L^2(\mathbb{R})}\right]. \tag{2.9}$$

Applying Gronwall's inequality to (2.9), we obtain

$$\|v(\cdot,t)\|^2_{L^2(\mathbb{R})} \le \left[\int_{-\tau}^0 \|v_0(\cdot,s)\|^2_{L^2(\mathbb{R})} + \|v_0(\cdot,0)\|^2_{L^2(\mathbb{R})}\right] e^{4\mu t} \tag{2.10}$$

for $t \in [0, T)$.

Let us differentiate (2.4) with respect to x, then we have

$$(v_x)_t(x,t) = d[J * v_x - v_x] + Q_{0x}(x,t), \quad (x,t) \in \mathbb{R} \times [0, +\infty).$$

By the similar method above, we obtain

$$\|v_x(\cdot,t)\|^2_{L^2(\mathbb{R})} \le C_1 \left[\int_{-\tau}^{0} \|v_{0x}(\cdot,s)\|^2_{L^2(\mathbb{R})} + \|v_{0x}(\cdot,0)\|^2_{L^2(\mathbb{R})} \right] e^{4\mu t} \tag{2.11}$$

for $t \in [0, T)$, where C_1 is a positive constant, independent of T.

Combining (2.10) and (2.11), we obtain (2.5) immediately. This completes the proof.

Finally, Theorem 2.4 follows immediately from the Lemmas 2.5 and 2.6. □

3. Stability of traveling wavefronts

In this section, we mainly concentrate on proving the stability for all noncritical traveling wavefront to (1.1) with an exponential convergence rate, and present the proof of global asymptotic stability for the critical traveling wavefronts. Due to the difficulty caused by the unstable equilibrium, we first establish a weighted L^1-estimate by selecting a suitable weight function. Then using the L^1-estimate, we further obtain the desired L^2-energy estimate. Now, we present the main result of this paper.

Theorem 3.1 (Stability). *Assume that* (H1)–(H5) *hold. For a given traveling wavefront* $\phi(x+ct)$ *of* (2.1) *with* $c \ge c^*$ *and* $\phi(\pm\infty) = u_\pm$*, if the initial data satisfies*

$$0 = u_- \le u_0(x,s) \le u_+, \quad (x,s) \in \mathbb{R} \times [-\tau, 0],$$

and the initial perturbation $u_0(\cdot,s) - \phi(\cdot + cs) \in C^1([-\tau,0], H^1_w(\mathbb{R}) \cap H^1(\mathbb{R}))$*, where the weighted function* $w(x)$ *is defined as*

$$w(x) = \begin{cases} e^{-\lambda^*(x-x_0)} & x \le x_0, \\ 1 & x > x_0, \end{cases}$$

where x_0 *is a very large constant, then the solution* $u(x,t)$ *of* (1.1) *and* (1.2) *satisfies*

$$0 = u_- \le u(x,t) \le u_+, \quad \forall\,(x,t) \in \mathbb{R}_+ \times \mathbb{R},$$
$$u(t,\cdot) - \phi(\cdot + ct) \in C^1([0,+\infty), H^1_w(\mathbb{R}) \cap H^1(\mathbb{R})).$$

Furthermore, we have

(i) *when* $c > c^*$*, the solution* $u(x,t)$ *converges to the noncritical traveling wavefront* $\phi(x+ct)$ *exponentially*

$$\sup_{x \in \mathbb{R}} |u(x,t) - \phi(x+ct)| \le Ce^{-\mu t}, \quad t > 0,$$

for a small positive constant $\mu > 0$*;*

(ii) *when $c = c^*$, the solution $u(x,t)$ converges to the critical traveling wavefront $\phi(x+c^*t)$ time-asymptotically*

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|u(x,t)-\phi(x+c^*t)|=0.$$

Remark 3.2. Theorem 3.1 not only shows the convergence rate μ to the noncritical traveling wavefronts, but also tells us how the time delay τ effects the convergence rate μ from the proof of Theorem 3.1. The effect of the time delay τ will make the decay rate μ of the solution slow down. That is, μ becomes the smallest 0 as $\tau \to +\infty$ and μ tends to biggest as $\tau \to 0$. For more details, we can refer to the proofs of Lemmas 3.3 and 3.4.

Let $c \ge c^*$ and initial data $u_0(x,s)$ be such that $0 = u_- \le u_0(x,s) \le u_+$ for $(x,s) \in \mathbb{R}\times[-\tau,0]$, and define

$$\begin{cases}\widetilde{U}_0^+(x,s)=\max\{u_0(x,s),\phi(x+cs)\}, \\ \widetilde{U}_0^-(x,s)=\min\{u_0(x,s),\phi(x+cs)\},\end{cases} \qquad \forall\,(x,s)\in\mathbb{R}\times[-\tau,0],$$

which implies

$$0=u_-\le\widetilde{U}_0^-(x,s)\le u_0(x,s)\le\widetilde{U}_0^+(x,s)\le u_+,$$
$$0=u_-\le\widetilde{U}_0^-(x,s)\le \phi(x+cs)\le\widetilde{U}_0^+(x,s)\le u_+,$$

for $(x,s)\in\mathbb{R}\times[-\tau,0]$. Clearly, the initial data $\widetilde{U}_0^\pm(x,s)$ are piecewise continuous and don't have a good enough regularity, which may also cause the absence of regularity for the corresponding solutions. In order to overcome such a shortcoming and establish the energy estimate, instead of these initial data, we choose smooth functions $U_0^\pm(x,s)$ as the new initial data and $U_0^\pm(x,s)$ satisfy

$$0=u_-\le U_0^-(x,s)\le\widetilde{U}_0^-(x,s)\le u_0(x,s)\le\widetilde{U}_0^+(x,s)\le U_0^+(x,s)\le u_+,$$

for $(x,s)\in\mathbb{R}\times[-\tau,0]$.

Let $U^\pm(x,t)$ be the corresponding solution of (1.1) with the initial data $U_0^\pm(x,s)$, that is

$$\begin{cases}\dfrac{\partial U^\pm}{\partial t}=D[J*U^\pm+f(U^\pm,k*U^\pm(x,t-\tau)), \\ U^\pm(x,s)=U_0^\pm(x,s), \quad (x,s)\in\mathbb{R}\times[-\tau,0].\end{cases}$$

By the comparison principle in Lemma 2.3, we have

$$u_-\le U^-(x,t)\le u(x,t)\le U^+(x,t)\le u_+, \tag{3.1}$$

$$u_-\le U^-(x,t)\le \phi(t+ct)\le U^+(x,t)\le u_+, \tag{3.2}$$

for $(x,t)\in\mathbb{R}_+\times\mathbb{R}$.

Next, we will complete the proof of Theorem 3.1 in three steps.

Step 1. The convergence of $U^+(x,t)$ to $\phi(x+ct)$. For any given $c \geq c^*$, let $\xi = x+ct$ and

$$v(\xi,t) := U^+(t,x) - \phi(x+ct), \quad v_0(\xi,s) := U_0^+(s,x) - \phi(x+cs).$$

It follows from (3.1) and (3.2) that

$$v(\xi,t) \geq 0, \quad v_0(\xi,s) \geq 0,$$

and

$$v_t + cv_\xi = d[J * v - v] + f(v+\phi, k*(v_\tau+\phi_\tau)) - f(\phi, k*\phi_\tau), \tag{3.3}$$

where $v_\tau = v(\xi - c\tau, t-\tau)$, $\phi_\tau = \phi(\xi - c\tau)$. Furthermore, (3.3) can be rewritten as

$$v_t + cv_\xi = d[J * v - v] + \partial_1 f(0,0)v + \partial_2 f(0,0)k * v_\tau + Q, \tag{3.4}$$

where

$$Q = Q(\xi,t) = f(v+\phi, k*(v_\tau+\phi_\tau)) - f(\phi, k*\phi_\tau) - \partial_1 f(0,0)v - \partial_2 f(0,0)k*v_\tau.$$

By the assumptions (H1)–(H3) and mean-value theorem, we can obtain $Q(\xi,t) \leq -mv^2$ for the nonnegativity of k, ϕ and v, where m is a positive constant.

Lemma 3.3. *It holds that*

$$\|v(t)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^t e^{-\mu_1(t-s)}\|v(s)\|_{L^1_{w_1}(\mathbb{R})}\,\mathrm{d}s + \int_0^t e^{-\mu_1(t-s)}\|v(s)\|^2_{L^2_{w_1}(\mathbb{R})}\,\mathrm{d}s \leq Ce^{-\mu_1 t}$$

for $c > c^*$, and

$$\|v(t)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^t \|v(s)\|^2_{L^2_{w_1}(\mathbb{R})}\,\mathrm{d}s \leq C \quad \textit{for } c = c^*,$$

where $w_1(\xi) = e^{-\lambda^*(\xi - x_0)}$ (x_0 is a large constant) and μ_1 is the small constant.

Proof. Using the similar arguments as proof of Theorem 2.4 and noticing that initial data v_0 is a smooth function, we have

$$v(\cdot,t) \in C([0,+\infty), H^1_{w_1}(\mathbb{R}) \cap H^1(\mathbb{R})). \tag{3.5}$$

Multiplying (3.4) by $w_1(\xi)e^{\mu_1 t}$, where $\mu_1 > 0$ is a small constant, we obtain

$$\frac{\partial}{\partial t}(w_1 e^{\mu_1 t} v) + e^{\mu_1 t}\frac{\partial}{\partial \xi}(cw_1 v) = w_1 e^{\mu_1 t}\left[d(J*v - v) + c\frac{w_1'}{w_1}v + \mu_1 v + \partial_1 f(0,0)v\right] + \partial_2 f(0,0)w_1 e^{\mu_1 t}v_\tau + w_1 e^{\mu_1 t}Q.$$

Integrating the above equation over $\mathbb{R}\times[0,t]$ with respect to ξ and t, we have

$$
\begin{aligned}
&e^{\mu_1 t}\int_{\mathbb{R}} w_1(\xi)v(\xi,t)\,\mathrm{d}\xi\\
&= \|v_0(0)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^t\int_{\mathbb{R}} w_1(\xi)e^{\mu_1 s}Q(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&\quad + \int_0^t\int_{\mathbb{R}} w_\xi e^{\mu_1 s}\left[d(J*v-v)+c\frac{w_1'}{w_1}v+\mu_1 v+\partial_1 f(0,0)v+\partial_2 f(0,0)k*v_\tau\right]\mathrm{d}\xi\mathrm{d}s. \qquad (3.6)
\end{aligned}
$$

Here, we use (3.5) to ensure that

$$
\int_0^t\int_{\mathbb{R}} e^{\mu_1 s}\frac{\partial}{\partial\xi}\left[cw_1(\xi)v(\xi,s)\right]\mathrm{d}\xi\mathrm{d}s = 0.
$$

Because of $Q(\xi,s)\le -mv^2$ and $w_1(\xi)e^{\mu_1 t}\ge 0$, we have

$$
\int_0^t\int_{\mathbb{R}} e^{\mu_1 s}w_1(\xi)Q(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \le -m\int_0^t e^{\mu_1 s}\|v(s)\|^2_{L^2_{w_1}(\mathbb{R})}\,\mathrm{d}s. \qquad (3.7)
$$

By changing variable $y\to y$, $\xi-y-c\tau\to\xi$, $s-\tau\to s$ and using the fact

$$
\int_{\mathbb{R}} k(y)\frac{w_1(\xi+y+c\tau)}{w_1(\xi)}\,\mathrm{d}y = \int_{\mathbb{R}} k(y)e^{-\lambda^*(y+c\tau)}\,\mathrm{d}y := k_0,
$$

we obtain

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}} w_1(\xi)e^{\mu_1 s}\int_{\mathbb{R}} k(y)v(\xi-y-c\tau)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\
&= \int_{-\tau}^{t-\tau}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{\mu_1(s+\tau)}w_1(\xi+y+c\tau)k(y)v(\xi,s)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s\\
&= \int_{-\tau}^{t-\tau}\int_{\mathbb{R}} w_1(\xi)e^{\mu_1(s+\tau)}v(\xi,s)\left[\int_{\mathbb{R}} k(y)\frac{w_1(\xi+y+c\tau)}{w_1(\xi)}\,\mathrm{d}y\right]\mathrm{d}\xi\mathrm{d}s\\
&= \int_{-\tau}^{t-\tau}\int_{\mathbb{R}} w_1(\xi)e^{\mu_1(s+\tau)}v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\int_{\mathbb{R}} k(y)e^{-\lambda^*(y+c\tau)}\,\mathrm{d}y\\
&\le e^{\mu_1\tau}\int_{\mathbb{R}} k(y)e^{-\lambda^*(y+c\tau)}\,\mathrm{d}y\int_0^t\int_{\mathbb{R}} w_1(\xi)e^{\mu_1 s}v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\
&\quad + e^{\mu_1\tau}\int_{\mathbb{R}} k(y)e^{-\lambda^*(y+c\tau)}\,\mathrm{d}y\int_{-\tau}^0\int_{\mathbb{R}} w_1(\xi)e^{mu_1 s}v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\\
&= k_0 e^{\mu_1\tau}\left[\int_0^t e^{\mu_1 s}\|v(s)\|_{L^1_{w_1}(\mathbb{R})}\,\mathrm{d}s + \int_{-\tau}^0 e^{\mu_1 s}\|v_0(s)\|_{L^1_{w_1}(\mathbb{R})}\,\mathrm{d}s\right].
\end{aligned}
$$

By the similar method above, we obtain

$$\begin{aligned}
& d\int_0^t\int_{\mathbb{R}} w_1(\xi)e^{\mu_1 s}\int_{\mathbb{R}} J(x)v(\xi-x,s)\,\mathrm{d}x\mathrm{d}\xi\mathrm{d}s \\
&= d\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} w_1(\xi+x)e^{\mu_1 s}J(x)v(\xi,s)\,\mathrm{d}x\mathrm{d}\xi\mathrm{d}s \\
&= d\int_0^t\int_{\mathbb{R}} w_1(\xi)e^{\mu_1 s}v(\xi,s)\left[\int_{\mathbb{R}} J(x)\frac{w_1(\xi+x)}{w_1(\xi)}\,\mathrm{d}x\right]\mathrm{d}\xi\mathrm{d}s \\
&= d\int_{\mathbb{R}} J(x)e^{-\lambda^* x}\,\mathrm{d}x\int_0^t e^{\mu_1 s}\int_{\mathbb{R}} w_1(\xi)v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&= d\int_{\mathbb{R}} J(x)e^{-\lambda^* x}\,\mathrm{d}x\int_0^t e^{\mu_1 s}\|v(s)\|_{L^1_{w_1}(\mathbb{R})}\,\mathrm{d}s.
\end{aligned}$$

So we obtain

$$\begin{aligned}
& \int_0^t\int_{\mathbb{R}} w_1(\xi)e^{\mu_1 s}\left[d(J*v-v)+c\frac{w_1'}{w_1}v+\mu_1 v+\partial_1 f(0,0)v+\partial_2 f(0,0)k*v_\tau\right]\mathrm{d}\xi\mathrm{d}s \\
&\le \int_0^t\int_{\mathbb{R}} w_1(\xi)e^{\mu_1 s}v(\xi,s) \\
&\qquad\times\left[d\int_{\mathbb{R}} J(x)(e^{-\lambda^* x}-1)\,\mathrm{d}x-c\lambda^*+\mu_1+\partial_1 f(0,0)+\partial_2 f(0,0)k_0e^{\mu_1\tau}\right]\mathrm{d}\xi\mathrm{d}s \\
&\quad+\partial_2 f(0,0)k_0e^{\mu_1\tau}\int_{-\tau}^0 e^{\mu_1 s}\|v_0(s)\|_{L^1_{w_1}(\mathbb{R})}\,\mathrm{d}s \\
&= A(\lambda^*,c,\mu_1)\int_0^t e^{\mu_1 s}\|v(s)\|_{L^1_{w_1}(\mathbb{R})}\,\mathrm{d}s+\partial_2 f(0,0)k_0e^{\mu_1\tau}\int_{-\tau}^0 e^{\mu_1 s}\|v_0(s)\|_{L^1_{w_1}(\mathbb{R})}\,\mathrm{d}s,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
A(\lambda^*,c,\mu_1) &= d\int_{\mathbb{R}} J(x)(e^{-\lambda^* x}-1)\,\mathrm{d}x-c\lambda^*+\mu_1+\partial_1 f(0,0)+\partial_2 f(0,0)k_0e^{\mu_1\tau} \\
&= \Delta(\lambda^*,c)+\mu_1+\partial_2 f(0,0)k_0(e^{\mu_1\tau}-1).
\end{aligned}$$

Substituting (3.7) and (3.8) to (3.6), we have

$$\begin{aligned}
& e^{\mu_1 t}\|v(t)\|_{L^1_{w_1}(\mathbb{R})}-A(\lambda^*,c,\mu_1)\int_0^t e^{\mu_1 s}\|v(s)\|_{L^1_{w_1}(\mathbb{R})}\,\mathrm{d}s \\
&\quad+m\int_0^t e^{\mu_1 s}\|v(s)\|^2_{L^2_{w_1}(\mathbb{R})}\,\mathrm{d}s \\
&\le \|v_0(0)\|_{L^1_{w_1}(\mathbb{R})}+\partial_2 f(0,0)k_0e^{\mu_1\tau}\int_{-\tau}^0 e^{\mu_1 s}\|v_0(s)\|_{L^1_{w_1}(\mathbb{R})}\,\mathrm{d}s.
\end{aligned}$$

Furthermore, when $c>c^*$, due to $\Delta(\lambda^*,c)<0$ and $\lim_{\mu_1\to 0}\partial_2 f(0,0)k_0(e^{\mu_1\tau}-1)=0$, we can choose a small $\mu_1>0$ such that

$$A(\lambda^*,c,\mu_1)<0.$$

When $c = c^*$, because of $\Delta(\lambda^*, c^*) = 0$, we can take only $\mu_1 = 0$ such that

$$A(\lambda^*, c^*, \mu_1) = 0.$$

Thus, we obtain

$$\|v(t)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^t e^{-\mu_1(t-s)} \|v(s)\|_{L^1_{w_1}(\mathbb{R})} \, \mathrm{d}s + \int_0^t e^{-\mu_1(t-s)} \|v(s)\|^2_{L^2_{w_1}(\mathbb{R})} \, \mathrm{d}s \le C e^{-\mu_1 t}$$

for $c > c^*$, and

$$\|v(t)\|_{L^1_{w_1}(\mathbb{R})} + \int_0^t \|v(s)\|^2_{L^2_{w_1}(\mathbb{R})} \, \mathrm{d}s \le C \quad \text{for } c = c^*.$$

This completes the proof. □

Lemma 3.4. *For $c \ge c^*$, it holds that*

$$\|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|v(s)\|^2_{L^2(\mathbb{R})} \, \mathrm{d}s \le C.$$

Proof. Since $w_1(\xi) = e^{-\lambda^*(\xi - x_0)} \ge 1$ for $\xi \in (-\infty, x_0]$, it follows from Lemma 3.3 that

$$\int_{-\infty}^{x_0} v(\xi,t) \, \mathrm{d}\xi + \int_0^{t-\tau} \int_{-\infty}^{x_0} v^2(\xi,s) \, \mathrm{d}\xi \mathrm{d}s \le C, \quad \forall\, t \ge 0, \quad \text{for } c \ge c^*,$$

and in particular by taking $t = +\infty$, we obtain

$$\int_0^{+\infty} \int_{-\infty}^{x_0} v^2(\xi,s) \, \mathrm{d}\xi \mathrm{d}s \le C. \tag{3.9}$$

(3.3) can be rewritten as

$$v_t + c v_\xi = d[J * v - v] + \partial_1 f(\phi, k * \phi_\tau) v + \partial_2 f(\phi, k * \phi_\tau) k * v_\tau + Q_1, \tag{3.10}$$

where

$$\begin{aligned} Q_1 = Q_1(\xi,t) = {} & f(v + \phi, k * (v_\tau + \phi_\tau)) - f(\phi, k * \phi_\tau) \\ & - \partial_1 f(\phi, k * \phi_\tau) v - \partial_2 f(\phi, k * \phi_\tau) k * v_\tau. \end{aligned}$$

By using the mean value theorem and (H3), we can obtain $Q_1(\xi,t) \le 0$ for the nonnegativity of k and ϕ, v.

Multiplying (3.10) by $v(\xi,t)$ and integrating it over $\mathbb{R} \times [0,t]$ with respect to ξ and t, then we have

$$\begin{aligned} \|v(t)\|^2_{L^2(\mathbb{R})} \le {} & \|v_0(0)\|^2_{L^2(\mathbb{R})} + 2 \int_0^t \int_{\mathbb{R}} d(J * v - v) v \, \mathrm{d}\xi \mathrm{d}s \\ & + 2 \int_0^t \int_{\mathbb{R}} \partial_1 f(\phi, k * \phi_\tau) v^2 \, \mathrm{d}\xi \mathrm{d}s + 2 \int_0^t \int_{\mathbb{R}} \partial_2 f(\phi, k * \phi_\tau) k * v_\tau v \, \mathrm{d}\xi \mathrm{d}s. \end{aligned} \tag{3.11}$$

Using the Cauchy inequality $|ab| \le \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2$ for $\eta > 0$, which will be specified later, we obtain

$$\begin{aligned}
&2\int_0^t\int_{\mathbb{R}} \partial_2 f(\phi, k*\phi_\tau)\int_{\mathbb{R}} k(y)v(\xi-y-c\tau, s-\tau)v(\xi,s)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\
&\le \frac{1}{\eta}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)\partial_2^2 f(\phi, k*\phi_\tau)v^2(\xi-y-c\tau, s-\tau)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\
&\quad + \eta\int_0^t\int_{\mathbb{R}} v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s.
\end{aligned}$$

By changing variables $y \to y$, $\xi - y - c\tau \to \xi$, $s - \tau \to s$, we have

$$\begin{aligned}
&\frac{1}{\eta}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)\partial_2^2 f(\phi, k*\phi_\tau)v^2(\xi-y-c\tau, s-\tau)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\
&= \frac{1}{\eta}\int_{-\tau}^{t-\tau}\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)\partial_2^2 f\left(\phi(\xi+y+c\tau), \int_{\mathbb{R}} k(z)\phi(\phi+y-z)\,\mathrm{d}z\right)v^2(\xi,s)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\
&= \frac{1}{\eta}\int_{-\tau}^{0}\int_{\mathbb{R}}\int_{\mathbb{R}} k(y)\partial_2^2 f\left(\phi(\xi+y+c\tau), \int_{\mathbb{R}} k(z)\phi(\phi+y-z)\,\mathrm{d}z\right)v^2(\xi,s)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\
&\quad + \frac{1}{\eta}\int_0^{t-\tau}\left(\int_{-\infty}^{x_0} + \int_{x_0}^{+\infty}\right)k(y) \\
&\qquad\qquad \times \partial_2^2 f\left(\phi(\xi+y+c\tau), \int_{\mathbb{R}} k(z)\phi(\phi+y-z)\,\mathrm{d}z\right)v^2(\xi,s)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\
&\le \frac{1}{\eta}\partial_2^2 f(0,0)\,\|v_0(0)\|^2_{L^2(\mathbb{R})} + \frac{1}{\eta}\partial_2^2 f(0,0)\int_0^{+\infty}\int_{-\infty}^{x_0} v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&\quad + \frac{1}{\eta}\int_0^t\int_{x_0}^{+\infty}\left[\int_{\mathbb{R}} k(y)\partial_2^2 f\left(\phi(\xi+y+c\tau), \int_{\mathbb{R}} k(z)\phi(\xi+y-z)\,\mathrm{d}z\right)\mathrm{d}y\right]v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&\le C + \frac{1}{\eta}\int_0^t\int_{x_0}^{+\infty}\left[\int_{\mathbb{R}} k(y)\partial_2^2 f\left(\phi(\xi+y+c\tau), \int_{\mathbb{R}} k(z)\phi(\xi+y-z)\,\mathrm{d}z\right)\mathrm{d}y\right]v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s,
\end{aligned} \tag{3.12}$$

where we have used $\partial_2 f(u,v) \ge 0$ and $\partial_{ij} f(u,v) \le 0$ for all $(u,v) \in [0,u_+]^2$ and (3.9). Similarly, we obtain

$$\begin{aligned}
&2\int_0^t\int_{\mathbb{R}} \partial_1 f(\phi, k*\phi_\tau)v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&= 2\int_0^t\int_{-\infty}^{x_0} \partial_1 f(\phi, k*\phi_\tau)v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s + 2\int_0^t\int_{x_0}^{+\infty} \partial_1 f(\phi, k*\phi_\tau)v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\
&\le 2\int_0^t\int_{x_0}^{+\infty} \partial_1 f(\phi, k*\phi_\tau)v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s,
\end{aligned} \tag{3.13}$$

and

$$\eta\int_0^t\int_{\mathbb{R}} v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \le C + \eta\int_0^t\int_{x_0}^{+\infty} v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s, \tag{3.14}$$

where we use the fact that $\partial_1 f(u,v) \le 0$ and (3.9). Substituting (3.12)–(3.14) to (3.11) and noticing (2.8), we obtain

$$\|v(t)\|^2_{L^2(\mathbb{R})} - \int_0^t\int_{x_0}^{+\infty} C_2(\xi)v^2(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \le C,$$

where

$$C_2(\xi) = \eta + \frac{1}{\eta}\int_{\mathbb{R}} k(y)\partial_2^2 f\left(\phi(\xi+y+c\tau), \int_{\mathbb{R}} k(z)\phi(\xi+y-z)\,\mathrm{d}z\right)\mathrm{d}y$$
$$+ 2\partial_1 f(\phi, k*\phi_\tau)$$

for $\xi \in [x_0, +\infty)$. If we let $\xi \to +\infty$, we have

$$\begin{aligned} C_2(+\infty) &= \eta + \frac{1}{\eta}\int_{\mathbb{R}} k(y)\partial_2^2 f(u_+, u_+)\,\mathrm{d}y + 2\partial_1 f(u_+, u_+) \\ &= \eta + \partial_2^2 f(u_+, u_+) + 2\partial_1 f(u_+, u_+) \\ &= \frac{1}{\eta}\left[\eta^2 + 2\eta\partial_1 f(u_+, u_+) + \partial_2^2 f(u_+, u_+)\right]. \end{aligned}$$

Noting $\partial_1 f(u_+, u_+) < 0$ and $\partial_1 f(u_+, u_+) + \partial_2 f(u_+, u_+) < 0$, and using the properties of quadratic function, we can choose a suitable $\eta > 0$ such that $C_2(+\infty) < 0$. Furthermore, we can choose x_0 large enough such that

$$C_2(\xi) < \frac{1}{2}C_2(+\infty) < 0, \quad \xi \in [x_0, +\infty).$$

Thus, we have

$$\|v(t)\|^2_{L^2(\mathbb{R})} - \frac{1}{2}C_2(+\infty)\int_0^t\int_{x_0}^{+\infty} v^2(\xi, s)\,\mathrm{d}\xi\mathrm{d}s \le C.$$

In particular, we have

$$\int_0^t\int_{x_0}^{+\infty} v^2(\xi, s)\,\mathrm{d}\xi\mathrm{d}s \le C. \tag{3.15}$$

Combining (3.9) and (3.15), we have $\int_0^t \|v(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \le C$. Thus, we can immediately obtain

$$\|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|v(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \le C,$$

and complete this proof. □

To obtain the stability of traveling waves, we need to get the estimate for $v_\xi(\xi, t)$. Let us differentiate (3.3) with respect to ξ and multiply the resulting equation by $v_\xi(\xi, t)$ and then integrate it over $\mathbb{R} \times [0, t]$ with respect to ξ and t. By the similar method of Lemma 3.4, we can obtain the following estimate for $v_\xi(\xi, t)$. The detail of the proof is omitted.

Lemma 3.5. *For any $c \ge c^*$, it holds that*

$$\|v_\xi(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|v_\xi(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s \le C. \tag{3.16}$$

Now, based on lemmas above, we obtain the exponential stability for noncritical traveling waves in $(-\infty, x_0]$.

Lemma 3.6. *For any $c > c^*$, it holds that*

$$\|v(t)\|_{L^\infty(-\infty,x_0]} \le Ce^{-\mu_1 t/3}, \quad t \ge 0.$$

Proof. Let $I = (-\infty, x_0]$. Then we have

$$\|v(t)\|^2_{L^2(I)} = \int_{-\infty}^{x_0} |v(\xi,t)|^2 \, \mathrm{d}\xi \le \|v(t)\|_{L^\infty(I)} \|v(t)\|_{L^1(I)}, \tag{3.17}$$

and

$$\begin{aligned} v^2(\xi,t) &= \int_{-\infty}^{x_0} \partial_\xi(v^2)\,\mathrm{d}\xi = 2\int_{-\infty}^{x_0} v(\xi,t)v_\xi(\xi,t)\,\mathrm{d}\xi \\ &\le 2\left(\int_{-\infty}^{x_0} v^2(\xi,t)\,\mathrm{d}\xi\right)^{1/2}\left(\int_{-\infty}^{x_0} v_\xi^2(\xi,t)\,\mathrm{d}\xi\right)^{1/2} \\ &= 2\|v(t)\|_{L^2(I)}\|v_\xi\|_{L^2(I)}, \end{aligned}$$

which implies

$$\|v(t)\|^2_{L^\infty(I)} \le 2\|v(t)\|_{L^2(I)}\|v_\xi\|_{L^2(I)}. \tag{3.18}$$

Combining (3.17) and (3.18), we obtain

$$\|v(t)\|_{L^\infty(I)} \le \sqrt[3]{4}\,\|v(t)\|^{1/3}_{L^1(I)}\|v_\xi(t)\|^{2/3}_{L^2(I)}. \tag{3.19}$$

In view of $\|v_\xi(t)\|_{L^2(I)} \le C$ from (3.16), $w_1(\xi) = e^{-\lambda^*(\xi - x_0)} \ge 1$ for $x \in I = (-\infty, x_0]$, and Lemma 3.3, we obtain

$$\|v(t)\|_{L^1(I)} \le \|v(t)\|_{L^1_{w_1}(I)} \le Ce^{-\mu_1 t}, \quad t \ge 0. \tag{3.20}$$

Thus, combining (3.19) and (3.20), we immediately get

$$\|v(t)\|_{L^\infty(I)} \le Ce^{-\mu_1 t/3}, \quad t \ge 0.$$

and compete the proof. □

Now we are going to prove the exponential stability for noncritical traveling waves in $[x_0, +\infty)$.

Lemma 3.7. *For any $c > c^*$, it holds that*

$$\|v(t)\|_{L^\infty[x_0,+\infty)} \le Ce^{-\mu_1 t/3}, \quad t \ge 0.$$

Proof. Multiplying (3.10) by $e^{\mu_1 t}$ and integrating it with respect to (ξ,t) over $\mathbb{R}\times[0,t]$, and noticing that $Q_1(\xi,t)\le 0$, we obtain

$$\begin{aligned} & e^{\mu_1 t}\|v(t)\|_{L^1(\mathbb{R})}-\int_0^t e^{\mu_1 s}\int_{\mathbb{R}}\partial_1 f(\phi,k*\phi_\tau)v\,\mathrm{d}\xi\mathrm{d}s \\ & -\int_0^t e^{\mu_1 s}\int_{\mathbb{R}}\partial_2 f(\phi,k*\phi_\tau)k*v_\tau\,\mathrm{d}\xi\mathrm{d}s \\ \le{}& \|v_0(0)\|_{L^1(\mathbb{R})}+d\int_0^t e^{\mu_1 s}\int_{\mathbb{R}}(J*v-v)\,\mathrm{d}\xi\mathrm{d}s. \end{aligned} \tag{3.21}$$

By changing variables $y\to y$, $s-\tau\to s$ and $\xi-y-c\tau\to\xi$, we have

$$\begin{aligned} & \int_0^t e^{\mu_1 s}\int_{\mathbb{R}}\partial_2 f(\phi,k*\phi_\tau)k*v_\tau\,\mathrm{d}\xi\mathrm{d}s \\ ={}& \int_0^t e^{\mu_1 s}\int_{\mathbb{R}}\int_{\mathbb{R}}k(y)\partial_2 f\left(\phi(\xi),\int_{\mathbb{R}}k(z)\phi(\xi-z-c\tau)\,\mathrm{d}z\right)v(\xi-y-c\tau,s-\tau)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\ ={}& \int_{-\tau}^{t-\tau} e^{\mu_1(s+\tau)}\int_{\mathbb{R}}\int_{\mathbb{R}}k(y)\partial_2 f\left(\phi(\xi+y+c\tau),\int_{\mathbb{R}}k(z)\phi(\xi+y-z)\,\mathrm{d}z\right)v(\xi,s)\,\mathrm{d}y\mathrm{d}\xi\mathrm{d}s \\ \le{}& \int_0^t e^{\mu_1 s}\int_{\mathbb{R}}v(\xi,s)\left[e^{\mu_1\tau}\int_{\mathbb{R}}k(y)\partial_2 f\left(\phi(\xi+y+c\tau),\int_{\mathbb{R}}k(z)\phi(\xi+y-z)\,\mathrm{d}z\right)\mathrm{d}y\right]\mathrm{d}\xi\mathrm{d}s \\ & +\int_{-\tau}^0 e^{\mu_1 s}\int_{\mathbb{R}}v(\xi,s)\left[e^{\mu_1\tau}\int_{\mathbb{R}}k(y)\partial_2 f\left(\phi(\xi+y+c\tau),\int_{\mathbb{R}}k(z)\phi(\xi+y-z)\,\mathrm{d}z\right)\mathrm{d}y\right]\mathrm{d}\xi\mathrm{d}s. \end{aligned}$$

Let

$$\begin{aligned} C_3(\xi)={}&\partial_1 f(\phi,k*\phi_\tau) \\ & +e^{\mu_1\tau}\int_{\mathbb{R}}k(y)\partial_2 f\left(\phi(\xi+y+c\tau),\int_{\mathbb{R}}k(z)\phi(\xi+y-z)\,\mathrm{d}z\right)\mathrm{d}y. \end{aligned}$$

As $\xi\to+\infty$, we obtain

$$C_3(+\infty)=\partial_1 f(u_+,u_+)+e^{\mu_1\tau}\partial_2 f(u_+,u_+).$$

Due to the fact that $\partial_1 f(u_+,u_+)+\partial_2 f(u_+,u_+)<0$, we can choose μ_1 small enough such that $C_3(+\infty)<0$. Thus, we can choose x_0 large enough such that $C_3(\xi)\le\frac{1}{2}C_3(+\infty)<0$ for $\xi\in[x_0,+\infty)$.

$$\begin{aligned} & \int_0^t e^{\mu_1 s}\int_{\mathbb{R}}C_3(\xi)v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \\ ={}& \int_0^t e^{\mu_1 s}\left(\int_{-\infty}^{x_0}+\int_{x_0}^{+\infty}\right)C_3(\xi)\,\mathrm{d}\xi\mathrm{d}s \\ \le{}& \int_0^t e^{\mu_1 s}\int_{-\infty}^{x_0}C_3(\xi)v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s+\frac{1}{2}C_3(+\infty)\int_0^t e^{\mu_1 s}\int_{x_0}^{+\infty}v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s. \end{aligned}$$

From Lemma 3.3 and $w_1(\xi)\ge 1$ in $(-\infty,x_0]$, we obtain

$$e^{\mu_1 t}\int_{-\infty}^{x_0}v(\xi,t)\,\mathrm{d}\xi\le C \quad\text{and}\quad \int_0^t e^{\mu_1 s}\int_{-\infty}^{x_0}v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\le C$$

which, together with the boundedness of $C_3(\xi)$, implies that

$$\left|\int_0^t e^{\mu_1 s}\int_{-\infty}^{x_0} C_3(\xi)v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s\right| \le C.$$

Furthermore, splitting each integral on the inequality (3.21) into two parts due to $\mathbb{R} = (-\infty, x_0]\cup[x_0,+\infty)$, we get

$$e^{\mu_1 t}\|v(t)\|_{L^1[x_0,+\infty)} - \frac{1}{2}C_3(+\infty)\int_0^t e^{\mu_1 s}\int_{x_0}^{+\infty} v(\xi,s)\,\mathrm{d}\xi\mathrm{d}s \le C.$$

In particular, noticing $C_3(+\infty) < 0$, we have

$$\|v(t)\|_{L^1[x_0,+\infty)} \le Ce^{\mu_1 t}, \quad t \ge 0.$$

By the similar method of Lemma 3.6, we can prove the following convergence in $[x_0,+\infty)$,

$$\|v(t)\|_{L^\infty[x_0,+\infty)} \le Ce^{-\mu_1 t/3}, \quad t \ge 0.$$

This completes the proof. □

Combining Lemmas 3.6 and 3.7, we can immediately get the following result.

Lemma 3.8. *For any $c > c^*$, it holds that*

$$\sup_{x\in\mathbb{R}}\left|U^+(x,t) - \phi(x+ct)\right| = \|v(t)\|_{L^\infty(\mathbb{R})} \le Ce^{-\mu_1 t/3}, \quad t \ge 0.$$

Next, we are going to prove the fact that the critical traveling wave with $c = c^*$ is time-asymptotically stable.

Lemma 3.9. *For any $c \ge c^*$, it holds that*

$$\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|v(s)\|^2_{L^2}(\mathbb{R})\right|\mathrm{d}s + \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|v_\xi(s)\|^2_{L^2(\mathbb{R})}\right|\mathrm{d}s \le C. \tag{3.22}$$

Proof. Multiplying (3.4) by $v(\xi,t)$ and integrating it over $\mathbb{R}$ with respect to ξ, then we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2(\mathbb{R})} &= 2d\int_{\mathbb{R}}(J*v - v)\,\mathrm{d}\xi + 2\int_{\mathbb{R}}\partial_1 f(0,0)v^2(\xi,t)\,\mathrm{d}\xi \\ &\quad + 2\int_{\mathbb{R}}\partial_2 f(0,0)k*v_\tau v(\xi,t)\,\mathrm{d}\xi + 2\int_{\mathbb{R}}Q(\xi,t)v(\xi,t)\,\mathrm{d}\xi.\end{aligned}$$

Noticing the fact that $Q(\xi,t) \le 0$ and $v(\xi,t) \ge 0$, we obtain

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{L^2(\mathbb{R})}\right| &\le \left|2d\int_{\mathbb{R}}(J*v - v)\,\mathrm{d}\xi\right| + \left|2\int_{\mathbb{R}}\partial_1 f(0,0)v^2(\xi,t)\,\mathrm{d}\xi\right| \\ &\quad + \left|2\int_{\mathbb{R}}\partial_2 f(0,0)k*v_\tau v(\xi,t)\,\mathrm{d}\xi\right| \\ &\le C\|v(t)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Integrating the above inequality over $[0,t]$, we obtain

$$\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|v(s)\|^2_{L^2(\mathbb{R})}\right| \mathrm{d}s \le \int_0^t C\,\|v(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s.$$

Noting the result of Lemma 3.4, we have

$$\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|v(s)\|^2_{L^2(\mathbb{R})}\right| \mathrm{d}s \le C. \tag{3.23}$$

Let us differentiate (3.4) with respect to ξ and multiply the resulting equation by $v_\xi(\xi,t)$, and then by the similar method above, we can obtain

$$\int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|v_\xi(s)\|^2_{L^2(\mathbb{R})}\right| \mathrm{d}s \le C. \tag{3.24}$$

Combining (3.23) and (3.24), we can immediately get (3.22) and we complete the proof. □

Lemma 3.10. *For $c=c^*$, it holds that*

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left|U^+(x,t)-\phi(x+c^*t)\right| = 0.$$

Proof. From Lemmas 3.4, 3.5 and 3.9, we have

$$\|v(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|v(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|v(s)\|^2_{L^2(\mathbb{R})}\right| \mathrm{d}s \le C \tag{3.25}$$

and

$$\|v_\xi(t)\|^2_{L^2(\mathbb{R})} + \int_0^t \|v_\xi(s)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}s + \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s}\|v_\xi(s)\|^2_{L^2(\mathbb{R})}\right| \mathrm{d}s \le C,$$

for $t\in[0,+\infty)$ and $c=c^*$. Let $g(t)=\|v(t)\|^2_{L^2(\mathbb{R})}$. From (3.25), we know that

$$0\le g(t)\le C, \quad \int_0^{+\infty} g(t)\,\mathrm{d}t \le C, \quad \int_0^{+\infty}\left|g'(t)\right|\mathrm{d}t\le C.$$

This implies

$$\lim_{t\to+\infty}\|v(t)\|^2_{L^2(\mathbb{R})} = \lim_{t\to+\infty} g(t) = 0. \tag{3.26}$$

By the similar method, we obtain

$$\lim_{t\to+\infty}\|v_\xi(t)\|^2_{L^2(\mathbb{R})} = 0. \tag{3.27}$$

Due to the inequality

$$\begin{aligned} v^2(\xi,t) &= \int_{-\infty}^{\xi} \partial_\xi(v^2)\,\mathrm{d}\xi = 2\int_{-\infty}^{\xi} v(\xi,t)v_\xi(\xi,t)\,\mathrm{d}\xi \\ &\le 2\int_{\mathbb{R}} |v(\xi,t)v_\xi(\xi,t)|\,\mathrm{d}\xi \le 2\|v(t)\|_{L^2(\mathbb{R})}\cdot\|v_\xi(t)\|_{L^2(\mathbb{R})}, \end{aligned}$$

we have

$$\|v(t)\|^2_{L^\infty(\mathbb{R})} \le 2\|v(t)\|_{L^2(\mathbb{R})}\cdot\|v_\xi(t)\|_{L^2(\mathbb{R})}. \tag{3.28}$$

By (3.26), (3.27) and (3.28), we immediately obtain

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}\left|U^+(x,t)-\phi(x+c^*t)\right| = \lim_{t\to+\infty}\|v(t)\|_{L^\infty(\mathbb{R})} = 0.$$

□

Step 2. The convergence of $U^-(x,t)$ to $\phi(x+ct)$. For any given $c \geq c^*$, let $\xi = x+ct$ and

$$v(\xi,t) = \phi(x+ct) - U^-(x,t), \quad v_0(\xi,s) = \phi(x+cs) - U_0^-(s,x).$$

As in Step 1, we can similarly prove that $U^-(x,t)$ converges to $\phi(x+ct)$ as follows.

Lemma 3.11. *For any $c > c^*$, it holds that*

$$\sup_{x\in\mathbb{R}} \left|U^-(x,t) - \phi(x+ct)\right| \leq Ce^{-\mu t}, \quad t > 0,$$

and for $c = c^*$, it holds that

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left|U^-(x,t) - \phi(x+c^*t)\right| = 0.$$

Step 3. The convergence of $u(x,t)$ to $\phi(x+ct)$. Here, we present the proof of Theorem 3.1 based on lemmas in Steps 1 and 2.

Proof. From Lemmas 3.8, 3.10 and 3.11, we obtain the following results

$$\sup_{x\in\mathbb{R}} \left|U^{\pm}(x,t) - \phi(x+ct)\right| \leq Ce^{-\mu t}, \quad t > 0,\ c > c^*,$$

and

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left|U^{\pm}(x,t) - \phi(x+c^*t)\right| = 0, \quad c = c^*.$$

By (3.1) and the squeezing argument, we immediately obtain

$$\sup_{x\in\mathbb{R}} |u(x,t) - \phi(x+ct)| \leq Ce^{-\mu t}, \quad t > 0,\ c > c^*,$$

and

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} avu(x,t) - \phi(x+c^*t) = 0, \quad c = c^*.$$

The proof is completed. □

4. Applications

In this section, we apply our main result — Theorem 3.1 to some monostable evolution equations to obtain the global stability of traveling waves, including the critical one.

4.1. Nonlocal Nicholson's blowflies equation

If we take the nonlocal diffusion rate $d = 1$ and $f(u, k*u) = -du(x,t) + b(u(x,t-\tau))$, the equation (1.1) will reduce to (1.3). That is, we consider the following initial value problem

$$\text{(4.1)} \quad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \displaystyle\int_{\mathbb{R}} J(y)[u(x-y,t) - u(x,t)]\,\mathrm{d}y - du(x,t) + b(u(x,t-\tau)), & t > 0,\ x \in \mathbb{R}, \\ u(x,s) = u_0(x,s), & s \in [0,\tau],\ x \in \mathbb{R}. \end{cases}$$

Here, we assume that the function $f(u,v)$ and kernel function $J(x)$ satisfy the assumptions (H1)–(H4). It is easy to see that (4.1) has two equilibria $u_- = 0$ and $u_+ > 0$. The following result is a straightforward consequence of Theorem 3.1

Theorem 4.1. *For a given traveling wave $\phi(x+ct)$ of (4.1) with $c \geq c^*$, if the initial data satisfies*

$$0 = u_- \leq u_0(x,s) \leq u_+, \quad \forall\, (x,s) \in \mathbb{R} \times [-\tau, 0],$$

and the initial perturbation $u_0(\cdot,s) - \phi(\cdot + cs) \in C^1([-\tau,0], H^1_w(\mathbb{R}) \cap H^1(\mathbb{R}))$, then there exists a small $\mu > 0$ such that the solution of (4.1) converges to the traveling wave $\phi(x+ct)$ in the sense that

$$\sup_{x\in\mathbb{R}} |u(x,t) - \phi(x+ct)| \leq Ce^{-\mu t}, \quad \forall\, t > 0,\ for\ c > c^*$$

and

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x + c^* t)| = 0, \quad for\ c = c^*.$$

Remark 4.2. In [24], Pan et al. considered the equation (4.1) and by constructing proper upper and lower solutions with the method of squeezing technique, they obtain the asymptotic stability with phase shift for the noncritical traveling wave in the sense that

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left| \frac{u(x,t)}{\phi(x+ct+\xi_0)} - 1 \right| = 0, \quad \text{for } c > c^*,$$

if the initial value satisfies

$$\lim_{x\to-\infty} \max_{s\in[-r,0]} \left| u_0(s,x)e^{-\lambda_1(c)x} - \rho_0 e^{\lambda_1(c)cs} \right| = 0, \quad \liminf_{x\to\infty} u_0(0,x) > 0,$$

where $\xi_0 = \frac{1}{\lambda_1(c)} \ln \rho_0$ and $\rho > 0$. Here, we not only obtain stability of the noncritical traveling wave for $c > c^*$, but also prove that the decay rate is exponential decay. Furthermore, we also obtain the global asymptotic stability of the critical traveling wave for $c = c^*$.

4.2. A local population model with age-structure

Letting diffusion kernel $J(x) = \delta(x) + \delta''(x)$ ($\delta(x)$ is the Dirac function) and $f(u, k*u) = -\delta u^2(x,t) + pe^{-\gamma\tau} \int_{\mathbb{R}} k(x)u(x-y,t-\tau)\,\mathrm{d}y$ with $\delta > 0$, $p > 0$, $\gamma > 0$, we then reduce (1.1) to the following age-structured population model

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\dfrac{\partial^2 u}{\partial x^2} - \delta u^2 + pe^{-\gamma\tau} \displaystyle\int_{\mathbb{R}} k(y)u(x-y,t-\tau)\,\mathrm{d}y, \\ u(x,s) = u_0(x,s), \quad s \in [-\tau, 0],\ x \in \mathbb{R}. \end{cases} \tag{4.2}$$

It can be easily checked that the constant equilibria of (4.2) are $u_0 = 0$ and $u_+ = \frac{p}{\delta}e^{-\gamma\tau}$, and the assumptions (H1)–(H5) are satisfied. From Theorem 3.1, we immediately obtain the following result.

Theorem 4.3. *For a given traveling wave $\phi(x+ct)$ of (4.2) with $c \geq c^*$, if the initial data satisfies*

$$0 = u_- \leq u_0(x,s) \leq u_+, \quad \forall\,(x,s) \in \mathbb{R} \times [-\tau, 0],$$

and the initial perturbation $u_0(\cdot,s) - \phi(\cdot + cs) \in C^1([-\tau,0], H^1_w(\mathbb{R}) \cap H^1(\mathbb{R}))$, then there exists a small $\mu > 0$ such that the solution of (4.1) converges to the traveling wave $\phi(x+ct)$ in the sense that

$$\sup_{x\in\mathbb{R}} |u(x,t) - \phi(x+ct)| \leq Ce^{-\mu t}, \quad \forall\, t > 0, \ for\ c > c^*$$

and

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x+c^*t)| = 0, \quad for\ c = c^*.$$

Remark 4.4. For the critical traveling wave, in [19], Mei et al. considered the more general form, and proved that the critical traveling wave is global algebraic stable in the sense that

$$\sup_{x\in\mathbb{R}} |u(t,x) - \phi(x+c^*t)| \leq Ct^{-1/2}, \quad \text{for } c = c^*.$$

No doubt that this global algebraic stability is better than time-asymptotic stability given in Theorem 4.3. However, when we introduce the nonlocal diffusion instead of the local diffusion to the equation (4.2), we only obtain the global time-asymptotical stability for the critical traveling wavefront because of the effect of the nonlocal diffusion.

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