TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 5, pp. 1117–1137, October 2016
DOI: 10.11650/tjm.20.2016.6758
This paper is available online at http://journal.tms.org.tw

On Focusing Entropy at a Point

Ewa Korczak-Kubiak, Anna Loranty* and Ryszard J. Pawlak

Abstract. In the paper we consider points focusing entropy and such that this fact is influenced exclusively by the behaviour of the function around these points (i.e., it is independent from the form of the function at any distance from these points). Thus the notion of an $\mathcal{F}$-focal entropy point has been introduced. We prove that each edge periodic tree function and each continuous function mapping the unit interval into itself have such points. Moreover, we discuss the possibility of improving functions defined on some topological manifolds so that any fixed point of the function becomes its focal entropy point.

1. Introduction

Entropy for discrete dynamical systems may be considered in terms of topological or measure approach. In 1971 T. Goodman [8] proved the variational principle determining the relationship between these two approaches. In this paper we will consider exclusively topological entropy.

It is commonly accepted that if the entropy is positive, the function is chaotic. The analysis of different examples of functions lead us to the interesting observation that chaos, and thereby entropy of a function, may be focused around one point. The basic problem in this case is connected with the possible best description of this situation. There are some papers connected with the problem of focusing entropy at a point (e.g., [13, 14, 16, 19]). However, in this paper we do not want to consider in detail the earlier propositions related to this topic. We now present a completely new approach to this issue. There are two basic reasons for our research.

The first one will not be reflected in this paper, but we can read these intentions comparing our solution with results contained in [15]. We would like to be able to use our definitions to a more general case (for example, generalized topological space and moreover, for multifunctions). This may be connected with the theory of information flow.

Received September 4, 2015; Accepted May 2, 2016.
Communicated by Yingfei Yi.
2010 *Mathematics Subject Classification*. 37B40, 54C70.
Key words and phrases. Entropy, $\mathcal{F}$-focal entropy point, Nonwandering point, Tree, Tree function, Edge periodic tree function, m-dimensional manifold with boundary.
*Corresponding author.

The second reason is more complex and it is connected with the expectations related to such points. For example, if we consider a function $f\colon [0,1] \to [0,1]$ such that $f(x) = -x+1$, for $x \in [0, \frac{1}{4}]$ and $f([\frac{3}{4}, 1]) \subset [\frac{3}{4}, 1]$, then the value of the entropy "around" 0 depends exclusively on the behaviour of function f on $[\frac{3}{4}, 1]$. Obviously, if we change the function f around 1 (quite far from 0), it also changes the value of entropy of f around 0. Discussing points focusing entropy we want to avoid the inconvenience described in this example. It seems appropriate to assume that the essence of such points should be connected with the behaviour of functions exclusively around this point.

That is why we will consider such a definition that being a "focal entropy point" or not will depend exclusively on the behaviour of the function around the given point. Obviously, the question of existence of such a point is essential. For this reason, directly after introducing definition and presenting basic properties of focal entropy points we will put the section in which we will prove, among others, that each continuous function mapping the unit interval into itself and each edge periodic tree function have such a point. The last part of the paper will be devoted to possibility of improving continuous functions defined on some topological manifolds so that a given fixed point of the function becomes its focal entropy point.

2. Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the sets of positive integers and real numbers, respectively. The symbol $\mathbb{I}$ stands for the unit interval. In this paper it will be always considered with the natural topology. Moreover, all functions considered in the paper are always assumed to be continuous. So writing that f is a function we have in mind that f is a continuous function.

If $x_0 = (x_0^1, x_0^2, \dots, x_0^n) \in \mathbb{R}^n$ and $r > 0$, then $C[x_0, r]$ is the Cartesian product $\prod_{i=1}^n [x_0^i - r, x_0^i + r]$ and $R[x_0, r] = C[x_0, r] \setminus \prod_{i=1}^n (x_0^i - r, x_0^i + r)$. Moreover, we will write the cardinality of a set A as $\#(A)$.

Let (X, ρ) be a metric space and $A \subset X$. The interior (closure, diameter) of A will be denoted by $\operatorname{int}(A)$ $(\operatorname{cl}(A), \operatorname{diam}(A))$. The symbol $\operatorname{dist}(A, B)$ stands for the distance between the sets $A, B \subset X$. Moreover, if $x_0 \in X$ then the family of all open neighbourhoods of x_0 will be denoted by $O(x_0)$ and the open ball of radius $r > 0$ centered at x_0 will be denoted by $B_\rho(x_0, r)$. The symbol $\exp(X)$ will stand for the family of all subsets of X.

Let $f\colon X \to X$ and $A, B \subset X$. We say that A f-covers B ($A \underset{f}{\to} B$ for short) if $B \subset f(A)$. If $A \underset{f}{\to} A$ then we say that A is a set f-covering itself. Moreover, we say that A is an f-invariant set if $f(A) \subset A$. The symbol $f \restriction A$ will stand for the restriction of f to A. The metric of uniform convergence in the space of all continuous functions mapping compact space X into itself will be denoted by ρ_u. What is more, if $f, g\colon X \to Y$ then $\nabla(f,g) = \{x \in X : f(x) \neq g(x)\}$.

Additionally, we will use the classical definitions of nonwandering, recurrent and fixed points [2, 4, 6, 19]. The sets of all nonwandering, recurrent and fixed points of a function f will be denoted by $\Omega(f)$, $R(f)$ and $\mathrm{Fix}(f)$, respectively.

Now, we shortly recall the concepts associated with an entropy of a function. In the paper we will use two equivalent concepts of this notion. The first one was introduced by R. L. Adler, A. G. Konheim and M. H. McAndrew in the paper [1] and the second one is the Bowen-Dinaburg version of the definition of entropy of a function defined on a metric space [5, 7] (see also [6, 10]).

In some parts of the proof of Theorem 4.4 we will use directly the results contained in [2, 4, 12, 15], so in this connection, we will present basic definitions and some statements (without proofs) from these papers adjusted to the notations used in this paper.

Let α be an open cover of a compact space X. Then $f^{-k}(\alpha) = \big\{f^{-k}(A) : A \in \alpha\big\}$ and $\Lambda^k = \alpha \vee f^{-1}(\alpha) \vee \cdots \vee f^{-k+1}(\alpha) = \{A_1 \cap f^{-1}(A_2) \cap \cdots \cap f^{-k+1}(A_k) : A_i \in \alpha \text{ for } i \in \{1, \ldots, k\}\}$ are open covers of X for any $k \in \mathbb{N}$. Moreover, if β is an open cover of X and α is an open cover of X such that for any set $B \in \beta$ there exists a set $A \in \alpha$ such that $B \subset A$ then we will write $\alpha < \beta$.

An entropy of a cover α is the number $H(\alpha) = \log N(\alpha)$, where $N(\alpha)$ denotes the minimal number of sets in any finite subcover chosen from α.

An entropy of a function $f\colon X \to X$ with respect to an open cover α is the number $h(f,\alpha) = \lim_{n\to\infty} \frac{1}{n} H(\Lambda^n)$. Obviously, if $\alpha < \beta$, then $h(f,\alpha) \leq h(f,\beta)$.

Let us put $h(f) = \sup h(f,\alpha)$, where supremum is taken over all open covers α of X. The number $h(f)$ is called the (topological) entropy of f.

Now, let λ be a finite family of pairwise disjoint intervals contained in $\mathbb{I}$ and $f\colon \mathbb{I} \to \mathbb{I}$ be a function.

Put $\lambda^k = \{(A_1, \ldots, A_k) : A_i \in \lambda \text{ for } i \in \{1, \ldots, k\} \text{ and } A_1 \cap f^{-1}(A_2) \cap \cdots \cap f^{-k+1}(A_k) \neq \emptyset\}$, for $k \in \mathbb{N}$. Let $c_k(\lambda)$ denote the cardinality of the family λ^k. If $A \in \lambda$, then $\lambda^k|A = \big\{(A_1, \ldots, A_k) \in \lambda^k : A_1 = A\big\}$ and $c_k(\lambda|A)$ denotes the cardinality of the family $\lambda^k|A$. Put $h^*(f,\lambda) = \lim_{k\to\infty} \frac{\log c_k(\lambda)}{k}$. Clearly, since the sequence $\{\log c_k(\lambda)\}_{k\in\mathbb{N}}$ is subadditive, i.e., $\log c_{k+\ell}(\lambda) \leq \log c_k(\lambda) + \log c_\ell(\lambda)$ for all $k, \ell \in \mathbb{N}$, so Lemma 4.1.1 in [2] gives that the above limit exists. Let $\lambda^\star$ be the family of all the intervals $A \in \lambda$ for which the following condition is fulfilled $\limsup_{k\to\infty} \frac{\log c_k(\lambda|A)}{k} = h^*(f,\lambda)$. Notice that $\lambda^\star \neq \emptyset$. Using our notations we have (by Proposition 25 in [4, Chapter 8])

$$\limsup_{k\to\infty} \frac{\log c_k(\lambda^\star|A)}{k} = h^*(f,\lambda).$$

Now, let us define inductively a sequence $\{\lambda_k^\star\}_{k\in\mathbb{N}}$ consisting of finite families of pairwise disjoint intervals. Put $\lambda_1^\star = \lambda^\star$. If $\lambda_k^\star$ has been defined, then for any interval $J \in \lambda_k^\star$ and any $A \in \lambda^\star$ such that $f^k(J) \cap A \neq \emptyset$ there exists interval $I(J,A)$ such that $I(J,A) \subset J$, $f^k(I(J,A)) = f^k(J) \cap A$. Put $\lambda_{k+1}^\star = \{I(J,A) : J \in \lambda_k^\star, A \in \lambda^\star, f^k(J) \cap A \neq \emptyset\}$.

If $A, B \in \lambda^\star$, then put $\gamma(A,B,k) = \#\{J \in \lambda_k^\star : J \subset A, B \subset f^k(J)\}$. Using our assumptions and notations we have (by Proposition 28 in [4, Chapter 8])

(2.1) If $h^\star(f,\lambda) > 2$, then there is $A \in \lambda^\star$ such that $\limsup_{k\to\infty} \dfrac{\log \gamma(A,A,k)}{k} = h^*(f,\lambda)$.

Now let us concentrate on the Bowen-Dinaburg version of the definition of an entropy of a function [5, 7] (see also [10] for entropy on a subset and [6] for an arbitrary function).

Let (X,ρ) be a compact metric space and $f\colon X \to X$. If $\varepsilon > 0$ and $n \in \mathbb{N}$, then we say that $M \subset X$ is an (n,ε)-*separated set* for f if for every different points $x, y \in M$ we have that there is $i \in \{0, 1, \ldots, n-1\}$ such that $\rho(f^i(x), f^i(y)) > \varepsilon$. The symbol $s_n(\varepsilon, f)$ denotes the cardinality of an (n,ε)-separated set for f with maximal possible number of points.

Putting $\overline{s}(\varepsilon, f) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, f)$, we obtain $\overline{s}(\varepsilon, f) \le \overline{s}(\varepsilon_1, f)$ if $\varepsilon_1 < \varepsilon$ (see Remarks in [18, p. 169]). The topological entropy of f is the number

$$h(f) = \lim_{\varepsilon\to 0^+} \overline{s}(\varepsilon, f) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, f).$$

The topological entropy of $f \restriction A$ for any nonempty set $A \subset X$ is given by the formula

$$h(f \restriction A) = \lim_{\varepsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log s_n^A(\varepsilon, f),$$

where $s_n^A(\varepsilon, f)$ denotes the cardinality of an (n,ε)-separated set for f contained in A with maximal possible number of points. Taking into account remark after Lemma 4.4 in [10] it is easy to see that $h(f^{n_0} \restriction A) = n_0 \cdot h(f \restriction A)$.

The problem connected with the existence of $\mathcal{F}$-focal entropy points will be examined in connection with the spaces named trees. Following [2, 3], we assume the definitions below. A continuum X is uniquely arcwise connected if for any $x, y \in X$ and $x \neq y$, there is a unique arc in X with endpoints x and y (from now on, by an arc we mean any space homeomorphic to [0,1]). A tree is a compact uniquely arcwise connected space which is a union of a finite number of arcs. If $\mathcal{T}$ is a tree and $x \in \mathcal{T}$, then the number of connected components of $\mathcal{T} \setminus \{x\}$ is called the valance of x. We say that $x \in \mathcal{T}$ is a vertex of $\mathcal{T}$ if the valance of x is different from 2. The closure of each connected component of $\mathcal{T} \setminus V(\mathcal{T})$, where $V(\mathcal{T})$ denotes the set of all vertices of $\mathcal{T}$, is called an edge of $\mathcal{T}$. Any continuous map f from a tree into itself will be called a tree map. Moreover, if for any edge E of a tree there exists $m \in \mathbb{N}$ such that E is an f^m-invariant set, then f will be called an edge periodic tree function.

In Section 5 we will consider an m-dimensional manifold with boundary X ($\dim(X) = m$). Our terminology and notations related to m-dimensional manifolds will coincide with these of [11]. We say that a topological space X is an m-dimensional topological manifold (manifold with boundary) if X is a second countable Hausdorff space and for every point $x_0 \in X$ there exist a set $W_{x_0} \in O(x_0)$, a set U open in $\mathbb{R}^m$ (open in the m-dimensional upper half space $\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m \geq 0\}$) and a homeomorphism $h_{x_0} \colon W_{x_0} \to U$. From now on, writing W_{x_0} and h_{x_0} for a point $x_0 \in X$ we will always have in mind the set and the homeomorphism defined above. Moreover, we will write $W_{h_{x_0}}$ instead of $h_{x_0}(W_{x_0})$. We will call any homeomorphism from an open subset V of X to an open subset of $\mathbb{H}^m$ a *chart* on U. A point which belongs to the inverse image of $\delta\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m = 0\}$ ($\mathrm{int}(\mathbb{H}^m) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m > 0\}$) under some chart is called a *boundary* (*interior*) *point* of X. The boundary of X (i.e., the set of all its boundary points) will be denoted by δX. Moreover, we will use the symbol $\mathrm{Int}(X)$ to denote the set of all interior points of a manifold X.

3. $\mathcal{F}$-focal entropy point: Definitions and basic properties

In this section, we will introduce a notion of an $\mathcal{F}$-focal entropy point and we will give basic properties of this kind of points. These properties will situate $\mathcal{F}$-focal entropy points among the families of other important points considered in the theory of discrete dynamical systems. The motivation for this definition (more precisely sequence of definitions) was discussed in detail in the introduction to this paper. For this reason, now we turn to the basic agreements and notations.

Our main discussion will focus on established families of sets (arcs, continua). But even easy observations lead us to the conclusion, that the considered families of sets can be replaced with another (larger) family of sets (for example Borel sets). However, limiting statements to fixed sets permits to avoid too complicated notations in proofs. The above observations indicate a necessity of formulation of definitions in general case.

So let (X, ρ) be a compact metric space (from now on writing X we will always mean such a space) and let $\mathcal{F} \subset \exp(X) \setminus \{\emptyset\}$ be a family such that each open set contains some element of $\mathcal{F}$. Further, writing about family $\mathcal{F}$ we will always assume that it fulfils the above condition. By $\vartheta_{\mathcal{F}}^{Y}$ we will denote the family of all finite sequences of sets from $\mathcal{F}$ contained in $Y \subset X$ such that their closures are pairwise disjoint i.e., $F = (A_1, \ldots, A_m) \in \vartheta_{\mathcal{F}}^{Y}$ if and only if $A_i \in \mathcal{F}$, $A_i \subset Y$ for any $i \in \{1, \ldots, m\}$ and $\mathrm{cl}(A_i) \cap \mathrm{cl}(A_j) = \emptyset$ for any $i, j \in \{1, \ldots, m\}$ and $i \neq j$. For simplicity of notation, let $\vartheta_{\mathcal{F}}$ stand for $\vartheta_{\mathcal{F}}^{X}$. Moreover, $\mathcal{F}|Y = \{K \cap Y : K \in \mathcal{F}\}$.

Let $n, m \in \mathbb{N}$, $f \colon X \to X$ and $F = (F_1, F_2, \ldots, F_{m-1}, F_1)$, where $F_i \in \mathcal{F}$ for $i \in \{1, \ldots, m-1\}$. If $F_i \underset{f^n}{\to} F_{i+1}$ for $i \in \{1, \ldots, m-2\}$ and $F_{m-1} \underset{f^n}{\to} F_1$ then we will denote this sequence F by $[F_1, F_2, \dots, F_{m-1}, F_1]_{f^n}$.

We shall say that $f \in J(\mathcal{F})$ if for any $n, m \in \mathbb{N}$ and any sequence $[F_1, F_2, \dots, F_{m-1}, F_1]_{f^n}$ there exists $x_0 \in F_1$ such that $f^{n(m-1)}(x_0) = x_0$ and $f^{n \cdot i}(x_0) \in F_{i+1}$ for $i \in \{1, 2, \dots, m-1\}$, where $F_m = F_1$. This property is closely connected with so called Itinerary Lemma [4].

If $F = (A_1, \dots, A_m) \in \vartheta_{\mathcal{F}}$ and $f \colon X \to X$ is a function then we define so called structural matrix $\mathcal{M}_{F,f} = [a_{ij}]_{i,j=1}^m$ in the following way: $a_{ij} = 1$ if $A_i \underset{f}{\to} A_j$ and $a_{ij} = 0$ otherwise.

A generalized entropy of f with respect to the sequence $F \in \vartheta_{\mathcal{F}}$ is the number

$$H_f(F) = \begin{cases} \log \sigma(\mathcal{M}_{F,f}) & \text{if } \sigma(\mathcal{M}_{F,f}) > 0, \\ 0 & \text{if } \sigma(\mathcal{M}_{F,f}) = 0, \end{cases}$$

where $\sigma(\mathcal{M}_{F,f}) = \limsup_{n \to \infty} \sqrt[n]{\operatorname{tr}(\mathcal{M}_{F,f}^n)}$ (cf. [15]).

Let $Y \subset X$ be a nonempty open set. An entropy of f on Y with respect to the family $\mathcal{F}$ is the number

$$H_{\mathcal{F},f}(Y) = \sup \left\{ \frac{1}{n} H_{f^n}(F) : F \in \vartheta_{\mathcal{F}}^Y, n \in \mathbb{N} \right\}.$$

The following lemma will be very useful in the proofs of theorems in further parts of this paper.

Lemma 3.1. *If $f \colon X \to X$ is any function, then*

$$H_{f^n}(F) \le h(f^n) = n \cdot h(f), \tag{3.1}$$

for any $n \in \mathbb{N}$ and $F \in \vartheta_{\mathcal{F}}$.[1]

Proof. It is sufficient to prove (e.g., [2, Lemma 4.1.2]) that

$$H_{f^n}(F) \le h(f^n), \tag{3.2}$$

for any $n \in \mathbb{N}$ and $F \in \vartheta_{\mathcal{F}}$. So, let us fix $n \in \mathbb{N}$ and $F \in \vartheta_{\mathcal{F}}$. The cases when $\#(F) = 1$ or $H_{f^n}(F) = 0$ are obvious. So, let us assume that $H_{f^n}(F) > 0$ and $F = (F_1, \dots, F_s) \in \vartheta_{\mathcal{F}}$, where $s > 1$. This means that $\sigma(\mathcal{M}_{F,f^n}) > 1$. Fix $\beta \in (0, \sigma(\mathcal{M}_{F,f^n}))$.

There exists a sequence n_k of positive integers such that

$$\sqrt[n_k]{\operatorname{tr}(\mathcal{M}_{F,f^n}^{n_k})} > \beta \quad \text{for any } k \in \mathbb{N}. \tag{3.3}$$

Let us denote the diagonal entries of a matrix $\mathcal{M}_{F,f^n}^{n_k}$ by $a_1^{n_k}, \dots, a_s^{n_k}$. Moreover, put

$$\varepsilon_0 = \frac{1}{2} \min \left\{ \operatorname{dist}(\operatorname{cl}(F_i), \operatorname{cl}(F_j)) : i, j \in \{1, \dots, s\}, i \ne j \right\}.$$

[1] Let us remind that according to the agreement adopted at the beginning of this section, X denotes a compact metric space (X, ρ) and $\mathcal{F} \subset \exp(X) \setminus \{\emptyset\}$ is a family such that each open set contains some element of $\mathcal{F}$.

Obviously, $\varepsilon_0 > 0$. The condition (3.3) implies that

$$T_{n_k} = \{i \in \{1,\dots,s\} : a_i^{n_k} \neq 0\} \neq \emptyset,$$

for any $k \in \mathbb{N}$. If $i \in T_{n_k}$ then there exist $a_i^{n_k}$ different sequences $F_{i,j} = [F_{i,j}^1, F_{i,j}^2, \dots, F_{i,j}^{n_k}, F_{i,j}^{n_k+1}]_{f^n}$ ($j \in \{1,\dots,a_i^{n_k}\}$) such that $F_{i,j}^1 = F_{i,j}^{n_k+1} = F_i$ and $F_{i,j}^\ell \in \{F_1,\dots,F_s\}$ for $\ell \in \{2,\dots,n_k\}$. Then for any $i \in T_{n_k}$ and any $j \in \{1,\dots,a_i^{n_k}\}$ one can find a point $x_{i,j}^{n_k} \in F_i$ such that $(f^n)^\ell(x_{i,j}^{n_k}) \in F_{i,j}^{\ell+1}$ for $\ell \in \{1,\dots,n_k\}$.

Note that $x_{i,j_1}^{n_k} \neq x_{i,j_2}^{n_k}$ for any $j_1, j_2 \in \{1,\dots,a_i^{n_k}\}$ and $j_1 \neq j_2$. Indeed, suppose contrary to our claim that there are $j_1, j_2 \in \{1,\dots,a_i^{n_k}\}$ and $j_1 \neq j_2$ such that $x_{i,j_1}^{n_k} = x_{i,j_2}^{n_k}$. Since $F_{i,j_1} \neq F_{i,j_2}$, one can find $m \in \{2,\dots,n_k\}$ such that $F_{i,j_1}^m \cap F_{i,j_2}^m = \emptyset$. Obviously, $(f^n)^{m-1}(x_{i,j_1}^{n_k}) \in F_{i,j_1}^m$ and $(f^n)^{m-1}(x_{i,j_1}^{n_k}) = (f^n)^{m-1}(x_{i,j_2}^{n_k}) \in F_{i,j_2}^m$, which is impossible.

Therefore, putting $A_{n_k} = \left\{x_{i,j}^{n_k} : i \in T_{n_k}, j \in \{1,\dots,a_i^{n_k}\}\right\}$ we obtain $\#(A_{n_k}) = a_1^{n_k} + \dots + a_s^{n_k}$. Let $x_{i_0,j_0}^{n_k}$, $x_{i_1,j_1}^{n_k}$ be different elements of the set A_{n_k}. Clearly, if $i_0 \neq i_1$, then $\rho(x_{i_0,j_0}^{n_k}, x_{i_1,j_1}^{n_k}) \geq \operatorname{dist}(\operatorname{cl}(F_{i_0}), \operatorname{cl}(F_{i_1})) > \varepsilon_0$. If $i_0 = i_1$ then there exists $\ell \in \{1,\dots,n_k - 1\}$ such that $F_{i_0,j_0}^{\ell+1} \cap F_{i_0,j_1}^{\ell+1} = \emptyset$. Since $(f^n)^\ell(x_{i_0,j_0}^{n_k}) \in F_{i_0,j_0}^{\ell+1}$ and $(f^n)^\ell(x_{i_0,j_1}^{n_k}) \in F_{i_0,j_1}^{\ell+1}$, we have $\rho((f^n)^\ell(x_{i_0,j_o}^{n_k}), (f^n)^\ell(x_{i_0,j_1}^{n_k})) > \varepsilon_0$. From the above consideration we get $s_{n_k}(\varepsilon_0, f^n) \geq \#(A_{n_k}) = a_1^{n_k} + \dots + a_s^{n_k}$. Thus for any $\varepsilon \in (0, \varepsilon_0)$ we have

$$\overline{s}(\varepsilon, f^n) \geq \limsup_{k\to\infty} \log \sqrt[n_k]{\mathcal{M}_{F,f^n}^{n_k}} \geq \log\beta.$$

Therefore $h(f^n) = \lim_{\varepsilon\to 0^+} \overline{s}(\varepsilon, f^n) \geq \log\beta$. From arbitrariness of β we obtain $h(f^n) \geq \log\sigma(\mathcal{M}_{F,f^n}) = H_{f^n}(F)$. □

Summarizing, for any nonempty open set $Y \subset X$ we have

$$H_{\mathcal{F},f}(Y) \leq h(f). \tag{3.4}$$

Taking into account the possible values of $H_{\mathcal{F},f}(Y)$ and $h(f)$ let us introduce the following notation

$$d(\mathcal{F}, f, Y) = \begin{cases} \frac{H_{\mathcal{F},f}(Y)}{h(f)} & \text{if } h(f) \in (0,\infty), \\ 1 & \text{if } H_{\mathcal{F},f}(Y) = \infty \text{ or } h(f) = 0, \\ 0 & \text{if } H_{\mathcal{F},f}(Y) \in [0,\infty) \text{ and } h(f) = \infty. \end{cases}$$

A density of entropy of f with respect to $\mathcal{F}$ at the point x_0 is the number

$$E_{\mathcal{F},f}(x_0) = \inf\{d(\mathcal{F}, f, V) : V \in O(x_0)\}.$$

Obviously (by (3.4) and by the definition of d), we have

$$0 \leq E_{\mathcal{F},f}(x_0) \leq 1.$$

We say that $x_0 \in X$ is an *$\mathcal{F}$-focal entropy point of f* if $E_{\mathcal{F},f}(x_0) = 1$. The set of all $\mathcal{F}$-focal entropy points of f will be denoted by $E_{\mathcal{F}}(f)$.

In order to avoid lengthening the paper we will omit elementary properties connected with $\mathcal{F}$-focal entropy points (e.g., if $\mathcal{F} \subset \mathcal{F}_1$ then $E_{\mathcal{F}}(f) \subset E_{\mathcal{F}_1}(f)$). In this section we will present only these properties, which permit to notice relations with some kinds of points considered in dynamical systems theory or these which refer to further considerations.

Proposition 3.2. *Let $f \colon X \to X$ be a function. Then the set $E_{\mathcal{F}}(f)$ is closed.*

Proposition 3.3. *Let $f \colon X \to X$ be a function such that $h(f) > 0$. Then $E_{\mathcal{F}}(f) \subset \Omega(f)$.*

Proof. Suppose, contrary to our claim, that there is $x_0 \in E_{\mathcal{F}}(f) \setminus \Omega(f)$. Thus there exists $U \in O(x_0)$ such that $U \cap f^i(U) = \emptyset$ for any $i \in \mathbb{N}$. Let $F = (F_1, \ldots, F_m) \in \vartheta^U_{\mathcal{F}}$. Clearly, for any $n \in \mathbb{N}$ all entries of the matrix $\mathcal{M}_{F,f^n}$ are equal to 0. This means that $H_{f^n}(F) = 0$ for any $n \in \mathbb{N}$, and in consequence, we obtain $H_{\mathcal{F},f}(U) = 0$, so $d(\mathcal{F}, f, U) = 0$. Finally we have that $E_{\mathcal{F},f}(x_0) = 0$, which is impossible. $\square$

Note that the assumption $h(f) > 0$ can not be omitted. Indeed, if we consider the function $f(x) = 1$ for all $x \in \mathbb{I}$, then it is easy to see that $0 \in E_{\mathcal{F}}(f) \setminus \Omega(f)$.

Moreover, we have the following fact.

Theorem 3.4. *If x_0 is an $\mathcal{F}$-focal entropy point of $f \colon X \to X$, then*

$$h(f \restriction U) = h(f) \quad \text{for any } U \in O(x_0). \tag{3.5}$$

Proof. Certainly, it is sufficient to consider the case $h(f) > 0$. To prove (3.5) we will show that for any $U \in O(x_0)$ we have

$$h(f \restriction U) \geq \beta \quad \text{for any } \beta \in (0, h(f)).$$

So, fix $U \in O(x_0)$ and let $\beta \in (0, h(f))$. It is easy to see that $H_{\mathcal{F},f}(U) > \beta$. Consequently, one can find $n_0 \in \mathbb{N}$ and $K = (A_1, \ldots, A_k) \in \vartheta^U_{\mathcal{F}}$, where $k > 1$, such that

$$\frac{1}{n_0} H_{f^{n_0}}(K) > \beta.$$

Put $\delta_0 = \min\left\{\rho(\mathrm{cl}(A_i), \mathrm{cl}(A_j)) : i, j \in \{1, \ldots, k\}, i \neq j\right\} > 0$ and $g = f^{n_0}$. Obviously, $H_g(K) > n_0 \cdot \beta$. Thus there exists a strictly increasing sequence $\{d_n\}_{n \in \mathbb{N}}$ of positive integers such that

$$\log \sqrt[d_n]{\mathrm{tr}(\mathcal{M}^{d_n}_{K,g})} > n_0 \cdot \beta. \tag{3.6}$$

Let us introduce the following notation $\mathcal{M}^{d_n}_{K,g} = [a^{d^n}_{i,j}]_{1 \leq i,j \leq k}$. Fix $n_* \in \mathbb{N}$. We write d_* instead of d_{n_*} for short. Clearly, by (3.6), we have

$$\log \mathrm{tr}(\mathcal{M}^{d_*}_{K,g}) > d_* \cdot n_0 \cdot \beta. \tag{3.7}$$

Put $\pi = \{m \in \{1, \ldots, k\} : a^{d_*}_{m,m} > 0\}$. The condition (3.7) implies that $\pi \neq \emptyset$. If $m \in \pi$, then there are $a^{d_*}_{m,m}$ different sequences $[A_m, A_{s_1}, \ldots, A_{s_{d_*-1}}, A_m]_g$, where $s_1, \ldots, s_{d_*-1} \in \{1, \ldots, k\}$. Let $\mathbb{S}^{d_*}_m = \left\{S^m_1, \ldots, S^m_{a^{d_*}_{m,m}}\right\}$ be the set of these sequences. Denote $S^m_i = [A_m, A^i_{s_1}, \ldots, A^i_{s_{d_*-1}}, A_m]_g$ for $i \in \{1, \ldots, a^{d_*}_{m,m}\}$. Then for any $i \in \{1, \ldots, a^{d_*}_{m,m}\}$ one can find points y^i_p for $p \in \{0, 1, \ldots, d_*\}$ such that $y^i_0, y^i_{d_*} \in A_m$, $y^i_j \in A^i_{s_{d_*-j}}$ for $j \in \{1, \ldots, d_* - 1\}$ and $g(y^i_j) = y^i_{j-1}$ for $j \in \{1, \ldots, d_*\}$.

Putting $\xi_m(S^m_i) = y^i_{d_*}$ for $m \in \pi$ and $i \in \{1, \ldots, a^{d_*}_{m,m}\}$ we obtain a function $\xi_m \colon \mathbb{S}^{d_*}_m \to A_m$ for any $m \in \pi$. Moreover, for any $m \in \pi$ we have

$$\begin{aligned} &\xi_m(S^m_i) \in A_m, \quad g^j(\xi_m(S^m_i)) \in A^i_{s_j} \text{ for any } j \in \{1, \ldots, d_* - 1\}, \text{ and} \\ &\qquad\qquad g^{d_*}(\xi_m(S^m_i)) \in A_m, \end{aligned} \tag{3.8}$$

for $i \in \{1, \ldots, a^{d_*}_{m,m}\}$. Furthermore, for each $m \in \pi$ a function ξ_m is injective, so

$$\#(\xi_m(\mathbb{S}^{d_*}_m)) = a^{d_*}_{m,m} \quad \text{for } m \in \pi. \tag{3.9}$$

Put $\Delta_{d_*} = \bigcup_{m\in\pi} \xi_m(\mathbb{S}^{d_*}_m)$. Obviously $\Delta_{d_*} \subset U$. Since $\xi_{m_1}(\mathbb{S}^{d_*}_{m_1}) \cap \xi_{m_2}(\mathbb{S}^{d_*}_{m_2}) = \emptyset$ for any $m_1, m_2 \in \pi$ and $m_1 \neq m_2$, we obtain, by (3.9), $\#(\Delta_{d_*}) = a^{d_*}_{1,1} + \cdots + a^{d_*}_{k,k} = \mathrm{tr}(\mathcal{M}^{d_*}_{K,g})$.

Moreover, we have that Δ_{d_*} is a (d_*, δ)-separated set for g for any $\delta \in (0, \delta_0)$. Indeed, let $x, y \in \Delta_{d_*}$ and $x \neq y$. There exist $m_x, m_y \in \pi$ such that $x \in \xi_{m_x}(\mathbb{S}^{d_*}_{m_x}) \subset A_{m_x}$ and $y \in \xi_{m_y}(\mathbb{S}^{d_*}_{m_y}) \subset A_{m_y}$. Clearly, if $m_x \neq m_y$, then $\rho(x, y) \geq \delta_0 > \delta$. If $m_x = m_y = t$, then one can find $i_x, i_y \in \left\{1, \ldots, a^{d_*}_{t,t}\right\}$ such that $x = \xi_t(S^t_{i_x})$ and $y = \xi_t(S^t_{i_y})$. Obviously, $S^t_{i_x} \neq S^t_{i_y}$. Thus there is $p \in \{1, \ldots, d_* - 1\}$ such that $A^{i_x}_{s_p} \cap A^{i_y}_{s_p} = \emptyset$. By (3.8) we have $g^p(x) \in A^{i_x}_{s_p}$ and $g^p(y) \in A^{i_y}_{s_p}$, so $\rho(g^p(x), g^p(y)) \geq \delta_0 > \delta$. Finally, we obtain $\rho(g^\ell(x), g^\ell(y)) > \delta$ for some $\ell \in \{0, \ldots, d_* - 1\}$. This means that Δ_{d_*} is a (d_*, δ)-separated set for g, for any $\delta \in (0, \delta_0)$ and, in consequence, we have

$$\log s^U_{d_*}(\delta, g) \geq \log(\#(\Delta_{d_*})) = \log(\mathrm{tr}(\mathcal{M}^{d_*}_{K,g})) > d_* \cdot n_0 \cdot \beta$$

for any $\delta \in (0, \delta_0)$. Repeating the previous reasoning for any d_n ($n \in \mathbb{N}$), we obtain $\log s^U_{d_n}(\delta, g) > d_n \cdot n_0 \cdot \beta$ for any $\delta \in (0, \delta_0)$. Thus

$$\overline{s}^U(\delta, g) = \limsup_{n\to\infty} \frac{1}{\ell} \log s^U_\ell(\delta, g) \geq \limsup_{n\to\infty} \frac{1}{d_n} \log s^U_{d_n}(\delta, g) \geq n_0 \cdot \beta$$

for any $\delta \in (0, \delta_0)$. Therefore,

$$h(g \restriction U) = \lim_{\delta\to 0^+} \overline{s}^U(\delta, g) \geq n_0 \cdot \beta,$$

so $n_0 \cdot h(f \restriction U) \geq n_0 \cdot \beta$ which finishes the proof. $\square$

It is easy to notice that the above theorem shows some kind of relation with full entropy points considered in [19].

The following result may be proved in similar way as Theorem 3.4. Choosing the points y_p^i for $i \in \{1, \ldots, a_{m,m}^{d_*}\}$ and $p \in \{0, \ldots, d_*\}$ (see the above proof) we need to make sure that $g^{d_*+1}(y_{d_*}^i) = y_{d_*}^i$.

Theorem 3.5. *Let $f\colon X \to X$ be a function such that $h(f) > 0$ and $f^n \in J(\mathcal{F})$ for any $n \in \mathbb{N}$. If x_0 is an $\mathcal{F}$-focal entropy point of f, then $h(f \restriction (U \cap R(f))) = h(f)$ for any $U \in O(x_0)$.*

4. The existence of focal entropy points

In this part of the paper we will discuss the problem of existence of $\mathcal{F}$-focal entropy points for continuous functions. Chapter 5 of [2] and the monograph [3] have directed our considerations to spaces called trees. Obviously, one can easily generalize Theorem 4.4 for example for the case of the graph-like spaces [2, 3]. The idea of the proof would stay almost the same, so, in order to make notations more readable, we decided to leave the theorem under the assumption of considering trees. In all the above cases, a special role is played by arcs. That is why in this section we consider the family of all arcs as a family $\mathcal{F}$ being a basis of our discussion. It is easy to notice that Theorem 4.4 will still be true if we use other families of sets (e.g., continua, Borel sets etc.). Therefore, we formulate lemmas as general as possible, i.e., $\mathcal{F} \subset \exp(X) \setminus \{\emptyset\}$ is a family such that each open set contains some element of $\mathcal{F}$.

Lemma 4.1. *Let $f\colon X \to X$ be a function, $E \subset X$ be an f^n-invariant set for some $n \in \mathbb{N}$, $h(f) = h(f \restriction E)$, $\mathcal{F}_E = \{A \in \mathcal{F} : A \subset E\}$ and $x_0 \in E$. If for any $V \in O(x_0)$ there is $B \in \mathcal{F}_E$ such that $B \subset V$ and x_0 is an $\mathcal{F}_E$-focal entropy point of $f^n \restriction E$, then x_0 is an $\mathcal{F}$-focal entropy point of f.*

Proof. If $h(f) = 0$ then the proof is obvious.

Let us first assume that $0 < h(f) < +\infty$ and let $V \in O(x_0)$. We will show that

$$\frac{H_{\mathcal{F},f}(V)}{h(f)} = 1. \tag{4.1}$$

Let $\varepsilon > 0$. Since x_0 is an $\mathcal{F}_E$-focal entropy point of $f^n \restriction E$, we obtain $\frac{H_{\mathcal{F}_E,f^n\restriction E}(V\cap E)}{h(f^n\restriction E)} = 1$. Thus $H_{\mathcal{F}_E,f^n\restriction E}(V \cap E) > n \cdot h(f) - n \cdot \varepsilon$. Therefore, there exist $\mathcal{K} = (K_1, \ldots, K_\ell) \in \vartheta_{\mathcal{F}_E}^{V\cap E} \subset \vartheta_{\mathcal{F}}^{V}$ and $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} H_{(f^n\restriction E)^{m_0}}(\mathcal{K}) > n \cdot h(f) - n \cdot \varepsilon$. Moreover, $(f^n \restriction E)^{m_0}(K_i) = f^{n\cdot m_0}(K_i)$ for $i \in \{1, \ldots, \ell\}$. Thus $\mathcal{M}_{\mathcal{K},f^{n\cdot m_0}} = \mathcal{M}_{\mathcal{K},(f^n\restriction E)^{m_0}}$, so $\frac{1}{m_0} H_{(f^n\restriction E)^{m_0}}(\mathcal{K}) = \frac{1}{m_0} H_{f^{n\cdot m_0}}(\mathcal{K})$. This means that $H_{\mathcal{F},f}(V) > h(f) - \varepsilon$. From arbitrariness of ε we obtain (4.1) and, in consequence, we have that x_0 is an $\mathcal{F}$-focal entropy point of f.

Now, assume that $h(f) = +\infty$. Clearly, $h(f^n \restriction E) = +\infty$. Let $V \in O(x_0)$. We will show that

$$H_{\mathcal{F},f}(V) = +\infty. \tag{4.2}$$

Let $r > 0$. Since x_0 is an $\mathcal{F}_E$-focal entropy point of $f^n \restriction E$, we obtain $H_{\mathcal{F}_E, f^n\restriction E}(V \cap E) = +\infty$, so $H_{\mathcal{F}_E, f^n\restriction E}(V \cap E) > n \cdot r$. Thus there exist $\mathcal{K} = (K_1, \dots, K_\ell) \in \vartheta^{V\cap E}_{\mathcal{F}_E} \subset \vartheta^{V}_{\mathcal{F}}$ and $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0} H_{(f^n\restriction E)^{m_0}}(\mathcal{K}) > n \cdot r$. Analysis similar to that in the case $h(f) \in (0, \infty)$ shows $\frac{1}{m_0} H_{(f^n\restriction E)^{m_0}}(\mathcal{K}) = \frac{1}{m_0} H_{f^{n\cdot m_0}}(\mathcal{K})$. Therefore $H_{\mathcal{F},f}(V) > r$. Since r was arbitrary, we obtain (4.2). This means that x_0 is an $\mathcal{F}$-focal entropy point of f. □

Let us introduce the following notations. If L $(\mathcal{T})$ is an arc (a tree), then $\mathcal{A}_L$ $(\mathcal{A}_\mathcal{T})$ is the family of sets containing all arcs contained in L $(\mathcal{T})$.

Lemma 4.2. *Let L be any arc, $\psi\colon \mathbb{I} \to L$ be a homeomorphism, $f\colon L \to L$ be a function and $g = \psi^{-1} \circ f \circ \psi$. A point x_0 is an $\mathcal{A}_\mathbb{I}$-focal entropy point of g if and only if $\psi(x_0)$ is an $\mathcal{A}_L$-focal entropy point of f.*

Proof. Note first that

$$\text{if } \mathbb{A} = (A_1, A_2, \dots, A_m) \in \vartheta_{\mathcal{A}_\mathbb{I}}, \text{ then } \psi(\mathbb{A}) = (\psi(A_1), \psi(A_2), \dots, \psi(A_m)) \in \vartheta_{\mathcal{A}_L}.$$

Indeed. Let $\mathbb{A} = (A_1, A_2, \dots, A_m) \in \vartheta_{\mathcal{A}_\mathbb{I}}$. By assumption $\psi(A_i) \in \mathcal{A}_L$. Suppose, contrary to our claim, that $\psi(\mathbb{A}) = (\psi(A_1), \psi(A_2), \dots, \psi(A_m)) \notin \vartheta_{\mathcal{A}_L}$. Thus there exist $i, j \in \{1, 2, \dots, m\}$ such that $\mathrm{cl}(A_i) \cap \mathrm{cl}(A_j) \neq \emptyset$, which is impossible.

By a similar argument, we obtain the following fact

$$\text{if } \mathbb{A} = (A_1, A_2, \dots, A_m) \in \vartheta_{\mathcal{A}_L}, \text{ then } \psi^{-1}(\mathbb{A}) = (\psi^{-1}(A_1), \psi^{-1}(A_2), \dots, \psi^{-1}(A_m)) \in \vartheta_{\mathcal{A}_\mathbb{I}}.$$

Obviously

$$\mathcal{M}_{\mathbb{A},g^n} = \mathcal{M}_{\psi(\mathbb{A}),f^n} \quad \text{for any } n \in \mathbb{N} \text{ and } \mathbb{A} \in \vartheta_{\mathcal{A}_\mathbb{I}}.$$

Thus

$$H_{g^n}(\mathbb{A}) = H_{f^n}(\psi(\mathbb{A})) \quad \text{for any } n \in \mathbb{N} \text{ and } \mathbb{A} \in \vartheta_{\mathcal{A}_\mathbb{I}}. \tag{4.3}$$

Finally, we get

$$H_{\mathcal{A}_L,f}(V) = H_{\mathcal{A}_\mathbb{I},g}(\psi^{-1}(V)) \quad \text{for any } V \in O(\psi(x_0)). \tag{4.4}$$

From (4.4) it may be concluded that x_0 is an $\mathcal{A}_\mathbb{I}$-focal entropy point of g if and only if $\psi(x_0)$ is an $\mathcal{A}_L$-focal entropy point of f. □

Moreover, Lemma 4.1 in [10] gives

Lemma 4.3. *If* $f\colon X \to X$ *and* $X = A_1 \cup A_2 \cup \dots \cup A_k$, *then* $h(f) = \max\{h(f \restriction A_1), h(f \restriction A_2), \dots, h(f \restriction A_k)\}$.

Now, one can prove

Theorem 4.4. *Let* $g\colon \mathcal{T} \to \mathcal{T}$ *be an edge periodic tree function. Then there exists a point* x_0 *being an* $\mathcal{A}_{\mathcal{T}}$*-focal entropy point of* g.

Proof. If $h(g) = 0$, then the proof is obvious. So, let us assume that $h(g) > 0$.

According to Lemma 4.3 let E be an edge of $\mathcal{T}$ such that $h(g) = h(g \restriction E)$. Let $\Phi\colon \mathbb{I} \to E$ be a homeomorphism and $n_E \in \mathbb{N}$ be such that $g^{n_E}(E) \subset E$.

Put $g_E = g^{n_E} \restriction E\colon E \to E$ and $f = \Phi^{-1} \circ g_E \circ \Phi\colon \mathbb{I} \to \mathbb{I}$. We start with showing that

$$\text{there exists an } \mathcal{A}_{\mathbb{I}}\text{-focal entropy point of } f. \tag{4.5}$$

To prove (4.5) it suffices to consider the case $h(f) > 0$. First we will show that:

$$\begin{gathered}\text{for any } z \in \mathbb{N} \text{ there is an interval } U \subset \mathbb{I} \text{ such that } \operatorname{diam}(U) < \tfrac{1}{z} \text{ and} \\ H_{\mathcal{A}_{\mathbb{I}},f}(U) \in (h(f) - \tfrac{1}{z}, h(f)] \quad \text{if } h(f) < +\infty\end{gathered} \tag{4.6}$$

and

$$\begin{gathered}\text{for any } z \in \mathbb{N} \text{ there is an interval } U \subset \mathbb{I} \text{ such that } \operatorname{diam}(U) < \tfrac{1}{z} \text{ and} \\ H_{\mathcal{A}_{\mathbb{I}},f}(U) > 2 + z + \frac{1}{6z} \quad \text{if } h(f) = +\infty.\end{gathered} \tag{4.7}$$

Let $z_0 \in \mathbb{N}$. Without loss of generality we can assume that $\frac{1}{z_0} < h(f)$. Moreover, put $\beta = h(f) - \frac{1}{z_0}$ if $h(f) < +\infty$ and $\beta = 2 + z_0 + \frac{1}{6z_0}$ if $h(f) = +\infty$. Clearly, if $h(f) < +\infty$ one can find $m_0 \in \mathbb{N} \setminus \{1\}$ such that

$$m_0 h(f) > 2 + \frac{1}{3z_0}. \tag{4.8}$$

To simplify further notation put: $\varphi = f^{m_0}$ if $h(f) < +\infty$ and $\varphi = f$ if $h(f) = +\infty$.

Let for any $n \in \mathbb{N}$, α_n be a cover of $\mathbb{I}$ consisting of open balls in $\mathbb{I}$ having diameter less than $\frac{1}{n}$. Obviously, $h(\varphi) = \lim_{n\to\infty} h(\varphi, \alpha_n)$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{1}{3z_0}$ and

$$h(\varphi, \alpha_{n_0}) \geq h(\varphi) - \frac{1}{3z_0} \quad \text{if } h(f) < +\infty \tag{4.9}$$

and

$$h(\varphi, \alpha_{n_0}) \geq 2 + z_0 + \frac{2}{3z_0} \quad \text{if } h(f) = +\infty. \tag{4.10}$$

Now, we will show that there exists a finite cover α_1 consisting of disjoint intervals having diameter less than $\frac{1}{3z_0}$ and such that

$$h^*(\varphi, \alpha_1) \geq h(\varphi, \alpha_{n_0}). \tag{4.11}$$

Let α_0 be a finite subcover of the cover α_{n_0}. Obviously, all sets from α_0 are intervals. Create a finite sequence $\{b_i\}_{i=1}^s$ of all endpoints of these intervals such that $0 = b_1 < b_2 < \cdots < b_{s-1} < b_s = 1$. We see at once that $b_i - b_{i-1} < \frac{1}{3z_0}$ for $i \in \{2, 3, \ldots, s\}$. Put

$$\alpha_1 = \left\{ \left[b_1, \frac{b_2}{2}\right), \left[\frac{b_2}{2}, b_2\right), \ldots, \left[b_{s-1}, \frac{b_{s-1}+b_s}{2}\right), \left[\frac{b_{s-1}+b_s}{2}, b_s\right] \right\}.$$

It is sufficient to show that α_1 fulfills the condition (4.11).

Fix $A \in \alpha_1$. Then there is an interval $B_A \in \alpha_0$ such that $A \subset B_A$.

Now, fix $v \in \mathbb{N}$. By the above, we conclude that for any sequence $(A_1, \ldots, A_v)$ of sets from α_1 such that $A_1 \cap \varphi^{-1}(A_2) \cap \cdots \cap \varphi^{-v+1}(A_v) \neq \emptyset$ there are sets $B_k \in \alpha_0$ (for $k \in \{1, \ldots, v\}$) such that

$$A_1 \cap \varphi^{-1}(A_2) \cap \cdots \cap \varphi^{-v+1}(A_v) \subset B_1 \cap \varphi^{-1}(B_2) \cap \cdots \cap \varphi^{-v+1}(B_v). \tag{4.12}$$

We see at once that $\varphi^{-i}(\alpha_1)$ is a cover of $\mathbb{I}$ for $i \in \{1, \ldots, v-1\}$ and consequently $\{A_1 \cap \varphi^{-1}(A_2) \cap \cdots \cap \varphi^{-v+1}(A_v) : A_i \in \alpha_1 \text{ for } i = 1, 2, \ldots, v\}$ is a cover of $\mathbb{I}$. Consider the number $c_v(\alpha_1)$. Clearly, $c_v(\alpha_1)$ is not less than the number of nonempty sets from the cover $\Lambda_1^v = \alpha_1 \vee \varphi^{-1}(\alpha_1) \vee \cdots \vee \varphi^{-v+1}(\alpha_1)$. The condition (4.12) implies that for any nonempty set E from the cover Λ_1^v there exists at least one set F from the cover $\Lambda_0^v = \alpha_0 \vee \varphi^{-1}(\alpha_0) \vee \cdots \vee \varphi^{-v+1}(\alpha_0)$ such that $E \subset F$. For each $E \in \Lambda_1^v$ we assign a set $F_E \in \Lambda_0^v$ such that $E \subset F_E$. Putting $\mathfrak{R}^v = \{F_E \in \Lambda_0^v : E \in \Lambda_1^v \setminus \{\emptyset\}\}$ we notice that $\mathfrak{R}^v$ is a finite subcover of the cover Λ_0^v. Moreover, the number of elements of $\mathfrak{R}^v$ is not greater than the number of nonempty elements of Λ_1^v. Thus $0 < N(\Lambda_0^v) \leq \#(\mathfrak{R}^v) \leq c_v(\alpha_1)$, so $H(\Lambda_0^v) \leq \log c_v(\alpha_1)$. From arbitrariness of v we obtain

$$h(\varphi, \alpha_0) \leq h^*(\varphi, \alpha_1).$$

Moreover, since $\alpha_{n_0} < \alpha_0$, we have $h(\varphi, \alpha_{n_0}) \leq h(\varphi, \alpha_0)$, and in consequence (4.11).

Now, from (4.11), (4.9), [2, Lemma 4.1.2] and (4.8) we conclude that if $h(f) < +\infty$ then

$$h^*(\varphi, \alpha_1) \geq h(\varphi, \alpha_{n_0}) \geq m_0 h(f) - \frac{1}{3z_0} > 2.$$

Moreover, if $h(f) = +\infty$, then (4.11) and (4.10) give that $h^*(\varphi, \alpha_1) \geq h(\varphi, \alpha_{n_0}) > 2$. Thus by (2.1), we get that there exists $A_1 \in \alpha_1$ such that

$$\limsup_{\ell \to \infty} \frac{\log \gamma(A_1, A_1, \ell)}{\ell} = h^*(\varphi, \alpha_1).$$

Let us recall that $\gamma(A_1, A_1, \ell)$ denotes the number of intervals $J \in (\alpha_1)^\star_\ell$ such that $J \subset A_1$ and $J \underset{\varphi^\ell}{\longrightarrow} A_1$.

Fix $\tau \in \mathbb{N}$. For any $\tau' \in \mathbb{N}$ such that $\tau' > \tau$ there exists $m_{\tau'} \in \mathbb{N}$ such that $m_{\tau'} > \tau' > \tau$ and

$$\log \gamma(A_1, A_1, m_{\tau'}) \geq m_{\tau'} \cdot \left(h^*(\varphi, \alpha_1) - \frac{1}{3z_0}\right) \geq m_{\tau'} \cdot \left(m_0 h(f) - \frac{2}{3z_0}\right) \quad \text{if } h(f) < +\infty$$

and

$$\log \gamma(A_1, A_1, m_{\tau'}) \geq m_{\tau'} \cdot \left(h^*(\varphi, \alpha_1) - \frac{1}{3z_0}\right) \geq m_{\tau'} \cdot \left(2 + z_0 + \frac{1}{3z_0}\right) \quad \text{if } h(f) = +\infty.$$

Put $n = m_{\tau'} \cdot m_0$ if $h(f) < +\infty$ and $n = m_{\tau'}$ if $h(f) = +\infty$. Let $\mathfrak{J}^n$ be a family of disjoint intervals J such that $J \subset A_1$ and $J \underset{\varphi^{m_{\tau'}}}{\longrightarrow} A_1$. Set $p_n = \#(\mathfrak{J}^n)$. Obviously, p_n is a number greater than 1. According to the continuity of φ, we have that p_n is a finite number. Since the family $\mathfrak{J}^n$ may contain intervals that do not belong to $(\alpha_1)^\star_{m_{\tau'}}$, we have $p_n \geq \gamma(A_1, A_1, m_{\tau'})$. Therefore,

$$\frac{\log p_n}{n} \geq \frac{\log \gamma(A_1, A_1, m_{\tau'})}{m_0 \cdot m_{\tau'}} > \beta + \frac{2}{3z_0} - \frac{2}{3z_0 \cdot m_0} > \beta \quad \text{if } h(f) < +\infty \tag{4.13}$$

and

$$\frac{\log p_n}{n} \geq \frac{\log \gamma(A_1, A_1, m_{\tau'})}{m_{\tau'}} > 2 + z_0 + \frac{1}{3z_0} > \beta \quad \text{if } h(f) = +\infty. \tag{4.14}$$

In both cases we have $\log p_n > n \cdot \beta$. Thus if n tends to $+\infty$ then p_n tends to $+\infty$, too. Without loss of generality we can assume that n is such that

$$p_n > 2 \text{ and } n > \frac{-3m_0 z_0 \log(1 - \frac{2}{p_n})}{2(m_0 - 1)} \quad \text{if } h(f) < +\infty \tag{4.15}$$

and

$$p_n > 2 \text{ and } \left|\log\left(1 - \frac{2}{p_n}\right)\right| < \frac{1}{6z_0} \quad \text{if } h(f) = +\infty. \tag{4.16}$$

Let $\{J_i\}_{i=1}^{p_n}$ be a sequence of all elements of $\mathfrak{J}^n$ such that for any $i \in \{1, \ldots, p_n - 1\}$ if $x \in J_i$ and $y \in J_{i+1}$ then $x < y$. Moreover, for any $i \in \{1, \ldots, p_n - 1\}$ denote by a_i, b_i the left and right end of the interval J_i, respectively.

Since $A_1 \in \alpha_1$, so $\mathrm{int}(A_1) \neq \emptyset$ and, in consequence, $\mathrm{int}(J_i) \neq \emptyset$ for $i \in \{1, \ldots, p_n\}$. Taking into account (4.15) if $h(f) < +\infty$ and (4.16) if $h(f) = +\infty$ we infer that there are at least 3 disjoint intervals: J_1, J_2 and J_{p_n}. Note that $b_1, a_{p_n} \in A_1$ and $J_2 \underset{\varphi^{m_{\tau'}}}{\longrightarrow} A_1$. Put $U = (a_1, b_{p_n}) \subset A_1$. Clearly, $\mathrm{diam}(U) < \frac{1}{z_0}$.

Since $\varphi^{m_{\tau'}}$ is a Darboux function, it follows that there are $d_{b_1}, d_{a_{p_n}} \in \mathrm{int}(J_2)$ such that $\varphi^{m_{\tau'}}(d_{b_1}) = b_1$ and $\varphi^{m_{\tau'}}(d_{a_{p_n}}) = a_{p_n}$. Let I_2 be a closed interval with ends d_{b_1}, $d_{a_{p_n}}$. Then $I_2 \underset{\varphi^{m_{\tau'}}}{\longrightarrow} [b_1, a_{p_n}]$. Hence $\varphi^{m_{\tau'}}(I_2) \supset J_2 \cup \cdots \cup J_{p_n-1}$. Moreover, we have that $I_2 \subset \mathrm{int}(J_2) \subset \mathrm{int}(\varphi^{m_{\tau'}}(I_2))$. Repeating the above reasoning for the sets $J_3, \ldots, J_{p_n-1}$ we obtain the sequence $K = (I_2, \ldots, I_{p_n-1})$ of closed, disjoint intervals such that $I_i \subset U$ and

$$\varphi^{m_{\tau'}}(I_i) \supset \mathrm{int}(J_2 \cup \cdots \cup J_{p_n-1}) \supset I_2 \cup \cdots \cup I_{p_n-1} \quad \text{for } i \in \{2, \ldots, p_n - 1\}. \tag{4.17}$$

Thus for any $i, j \in \{2, \ldots, p_n - 1\}$ we have $I_i \underset{f^n}{\longrightarrow} I_j$, so $\sigma(\mathcal{M}_{K,f^n}) = p_n - 2 > 0$. Hence

$$H_{f^n}(K) = \log(p_n - 2). \tag{4.18}$$

Now, we will show that

$$\frac{\log(p_n - 2)}{n} > \beta. \tag{4.19}$$

First, consider the case $h(f) < +\infty$. The condition (4.15) implies

$$\frac{\log(1 - \frac{2}{p_n})}{n} + \frac{2}{3z_0} - \frac{2}{3z_0 \cdot m_0} > 0.$$

From this and (4.13) it follows that

$$\frac{\log(p_n - 2)}{n} = \frac{\log p_n}{n} + \frac{\log(1 - \frac{2}{p_n})}{n} > \beta.$$

If $h(f) = +\infty$, then (4.14) and (4.16) give

$$\frac{\log(p_n - 2)}{n} = \frac{\log p_n}{n} + \frac{\log(1 - \frac{2}{p_n})}{n} > \beta.$$

By (4.18) and (4.19) we may conclude that $H_{f^n}(K) > n\beta$. This and Lemma 3.1 give $\beta < \frac{1}{n} H_{f^n}(K) \leq h(f)$ and, in consequence, $\beta < H_{\mathcal{A}_{\mathbb{I}},f}(U) \leq h(f)$, which finishes the proofs of (4.6) and (4.7).

Consider a sequence $\{U_z\}_{z\in\mathbb{N}}$ of intervals contained in $\mathbb{I}$ such that for any $z \in \mathbb{N}$ we have $\mathrm{diam}(U_z) < \frac{1}{z}$ and $H_{\mathcal{A}_{\mathbb{I}},f}(U_z) \in (h(f) - \frac{1}{z}, h(f)]$ if $h(f) < +\infty$ ($H_{\mathcal{A}_{\mathbb{I}},f}(U_z) > 2 + z + \frac{1}{6z}$ if $h(f) = +\infty$). For any $\nu \in \mathbb{N}$ fix $x_\nu \in U_\nu$. Since $\mathbb{I}$ is compact then there exists a point x_0 being an accumulation point of the set $\{x_\nu : \nu \in \mathbb{N}\}$. For simplicity of notation, we will assume that $\lim_{\nu\to\infty} x_\nu = x_0$.

Now, we will show that

$$x_0 \text{ is an } \mathcal{A}_{\mathbb{I}}\text{-focal entropy point of } f. \tag{4.20}$$

Suppose, contrary to our claim, that there exists $V \in O(x_0)$ such that $d(\mathcal{A}_{\mathbb{I}}, f, V) = \theta < 1$. Without restriction of generality we can assume that V is an interval, so $V = (x_0 - \kappa, x_0 + \kappa) \cap \mathbb{I}$ ($\kappa > 0$).

Assume first that $h(f) < +\infty$. Clearly, one can find $\nu_0 \in \mathbb{N}$ such that $x_{\nu_0} \in (x_0 - \kappa, x_0 + \kappa)$ and $\frac{1}{\nu_0} < \min\left\{\frac{\kappa}{2}, (1-\theta)h(f)\right\}$. Thus $U_{\nu_0} \subset (x_0 - \kappa, x_0 + \kappa) \cap \mathbb{I} = V$ and, in consequence,

$$\frac{H_{\mathcal{A}_{\mathbb{I}},f}(U_{\nu_0})}{h(f)} \leq \frac{H_{\mathcal{A}_{\mathbb{I}},f}(V)}{h(f)} = \theta.$$

On the other hand

$$\frac{H_{\mathcal{A}_{\mathbb{I}},f}(U_{\nu_0})}{h(f)} > \frac{h(f) - \frac{1}{\nu_0}}{h(f)} \geq \frac{h(f) - h(f) + \theta h(f)}{h(f)} = \theta,$$

which is impossible.

Assume, now that $h(f) = +\infty$. Then $H_{\mathcal{A}_{\mathbb{I}},f}(V) < +\infty$. There exists $\nu_0 \in \mathbb{N}$ such that $x_{\nu_0} \in (x_0 - \kappa, x_0 + \kappa)$, $\frac{1}{\nu_0} < \frac{\kappa}{2}$ and $H_{\mathcal{A}_{\mathbb{I}},f}(V) < \nu_0$. Obviously,

$$U_{\nu_0} \subset (x_0 - \kappa, x_0 + \kappa) \cap \mathbb{I} = V.$$

Therefore, $H_{\mathcal{A}_{\mathbb{I}},f}(U_{\nu_0}) \leq H_{\mathcal{A}_{\mathbb{I}},f}(V) < \nu_0$, which contradicts the fact that $H_{\mathcal{A}_{\mathbb{I}},f}(U_{\nu_0}) > 2 + \nu_0 + \frac{1}{6\nu_0}$. Finally, (4.20) is proved and, in consequence, we obtain (4.5).

Now, by Lemma 4.2, we conclude that

$$\Phi(x_0) \text{ is an } \mathcal{A}_E\text{-focal entropy point of } g_E.$$

Moreover, Lemma 4.1 implies that

$$\Phi(x_0) \text{ is an } \mathcal{A}_{\mathcal{T}}\text{-focal entropy point of } g. \qquad \square$$

If we want to consider more classical spaces, then the following corollary is an easy consequence of the above theorem.

Corollary 4.5. *Let $g\colon [0,1] \to [0,1]$ be a function. Then there exists a point x_0 being an $\mathcal{A}_{[0,1]}$-focal entropy point of g.*

Certainly, in the above corollary the family $\mathcal{A}_{[0,1]}$ consists of all nodegenerated closed intervals.

Within the context of the above considerations the questions connected with possibility of generalizations of Theorem 4.4 for the case of other spaces or under weaker assumptions put on considered functions (e.g., almost continuous functions [17]) seem to be interesting.

5. Improvement

A discussion regarding points focusing entropy and such that this property is independent from the behaviour of a function on the sets which lie far from this point, in a natural way direct our considerations to fixed points. On the other hand if we consider a continuous function $f\colon \mathbb{I} \to \mathbb{I}$ such that $h(f) > 0$, $f(0) = 1$, $f(1) = 0$, $\mathrm{Fix}(f) = \left\{\frac{1}{2}\right\}$ and $f(x) = \frac{1}{2}$ for $x \in [\frac{1}{4}, \frac{3}{4}]$ then we obtain the example of function mapping $[0,1]$ into itself such that no fixed point of this function is its $\mathcal{A}_{[0,1]}$-focal entropy point (Corollary 4.5 shows that such functions have $\mathcal{A}_{[0,1]}$-focal entropy points). This leads us to the natural problem: Is it possible to "improve" a given function in such a way that a given fixed point becomes a focal entropy point with respect to a fixed family of sets? In this section we will analyze this issue.

First, we will explain a notion of a "function improvable at a point x_0". Intuitively, we mean a possibility of changing a function in an arbitrary neighbourhood of x_0 in such a way that a new function "differs little" from the given one. So, we assume the following definitions.

Let (X,ρ) be a metric space. We say that a function $f\colon X \to X$ is improvable at a point $x_0 \in X$ if for any $\varepsilon > 0$ there exists a function $g\colon X \to X$ such that $x_0 \in E_{\mathcal{F}}(g)$, $\rho_u(f,g) < \varepsilon$ and $\nabla(f,g) \subset B_\rho(x_0,\varepsilon)$.

It is important for us to fix a set with respect to which a given function is improvable. This leads to the next definition.

If $A \subset X$ is a set f-covering itself and $x_0 \in A$, then we say that a function f is improvable at a point x_0 with respect to the set A if for any $\varepsilon > 0$ there exists a function $g\colon A \to A$ such that $x_0 \in E_{\mathcal{F}|A}(g)$, $\rho_u(f \restriction A, g) < \varepsilon$ and $\nabla(f \restriction A, g) \subset B_\rho(x_0,\varepsilon)$.

Pointing out a set with respect to which we can improve functions is especially evident in the case of topological manifolds. For that reason, in this section we will concentrate on a compact metric space (X,ρ) being an m-dimensional manifold with boundary. From now on, the symbol X will stand for such a space. Moreover, the symbol $\mathcal{F}$ will denote the family of all nonsingletons continuums in X. In addition, we assume that all functions considered in this section are continuous.

Theorem 5.1. *If $f\colon X \to X$ and $x_0 \in \mathrm{Fix}(f)$ then the function f is improvable at x_0. Moreover,*

(i) *if $x_0 \in \mathrm{Int}(X)$ and $\mathrm{Int}(X)$ is a set f-covering itself, then f is improvable at the point x_0 with respect to $\mathrm{Int}(X)$,*

(ii) *if $\dim(X) > 1$, $x_0 \in \delta X$ and δX is a set f-covering itself, then f is improvable at the point x_0 with respect to δX.*

Proof. Assume first that $x_0 \in \mathrm{Int}(X)$. Let $\varepsilon > 0$. Without loss of generality we can assume that $W_{x_0} \subset B(x_0, \frac{\varepsilon}{2})$. Additionally, to make notations shorter put $h = h_{x_0}$. Let $x_h = h(x_0) = (x_h^1, x_h^2, \ldots, x_h^m)$. Choose a number $\alpha_h > 0$ such that $C[x_h, \alpha_h] \subset W_{h_{x_0}}$. There exist positive integers $m_0 \geq n_0 \geq 2$ such that $\mathrm{cl}(B(x_0, \frac{\varepsilon}{m_0})) \subset B(x_0, \frac{\varepsilon}{n_0}) \subset h^{-1}(\mathrm{int}(C[x_h, \alpha_h])) \subset h^{-1}(C[x_h, \alpha_h]) \subset W_{x_0}$ and $f(\mathrm{cl}(B(x_0, \frac{\varepsilon}{m_0}))) \subset B(x_0, \frac{\varepsilon}{n_0})$.

Let $\alpha \in (0, \alpha_h)$ be a number such that $C[x_h, \alpha] \subset h(B(x_0, \frac{\varepsilon}{m_0}))$. Choose a positive number $\alpha_0 < \alpha$ and put $W^x = h^{-1}(C[x_h, \alpha_0]) \subset h^{-1}(C[x_h, \alpha])$. It is easy to notice that $h^{-1}(R[x_h, \alpha_0]) = \delta(W^x)$ and

$$W^x \text{ is closed in } X. \tag{5.1}$$

Let us now turn to the appropriate construction of the required function. Fix $\varepsilon_h \in (0, \alpha_0)$ and consider sequences $\{s_n\}_{n\in\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}}$ such that $0 < \cdots < r_{n+1} < s_{n+1} < r_n < s_n < \cdots < r_1 < s_1 < \varepsilon_h$ and $s_n \to 0$. Put $A[x_h, r_n, s_n] = \bigcup_{t\in[r_n,s_n]} R[x_h, t]$.

Now, for each $n \in \mathbb{N}$ choose 2^{n+1} numbers $d_1^n, d_2^n, \ldots, d_{n+1}^n$ such that $r_n = d_1^n < d_2^n < \cdots < d_{2^{n+1}-1}^n < d_{2^{n+1}}^n = s_n$. Define the function $f_n^*\colon [r_n, s_n] \to [r_n, s_n]$ in the following way: $f_n^*(d_{2i-1}^n) = r_n$, $f_n^*(d_{2i}^n) = s_n$ for $i = 1, \ldots, 2^n$ and f_n^* linear on the intervals $[d_i^n, d_{i+1}^n]$ for $i = 1, \ldots, 2^{n+1} - 1$.

Let $x \in R[x_h, t]$ (where $t \in [r_n, s_n]$ for fixed $n \in \mathbb{N}$) and l_x denote the half-line with initial point at x_h and such that x belongs to it. Then, let $f_h(x)$ be the intersection point of l_x and $R[x_h, f_n^*(t)]$. Continuing in this fashion we define the function f_h on the set $\bigcup_{n=1}^{\infty} A[x_h, r_n, s_n]$. Now, if $x \in R[x_h, \alpha_0] \cup (C[x_h, \varepsilon_h] \setminus \bigcup_{n=1}^{\infty} A[x_h, r_n, s_n])$, then put $f_h(x) = x$.

It is easy to see that $f_h\colon C[x_h, \varepsilon_h] \cup R[x_h, \alpha_0] \to C[x_h, \alpha_0]$ is continuous. Denote $Q_x = h^{-1}(C[x_h, \varepsilon_h])$, $Q_{\alpha_0} = h^{-1}(R[x_h, \alpha_0])$ and put $Q = Q_x \cup Q_{\alpha_0}$. Clearly, Q_x and Q_{α_0} are disjoint sets contained in W^x and $\operatorname{int}(Q_x)$ is a neighbourhood of x_0. Moreover, Q is closed in W^x, so by (5.1), it is closed in X.

Consider $f_Q = f_h \circ h\colon Q \to C[x_h, \alpha_0]$. There exists a function $f^*\colon W^x \to C[x_h, \alpha_0]$ being a continuous extension of f_Q.

Put

$$g(x) = \begin{cases} h^{-1}(f^*(x)) & \text{if } x \in W^x, \\ f(x) & \text{if } x \in h^{-1}(R[x_h, \alpha]). \end{cases}$$

Clearly $g\colon W^x \cup h^{-1}(R[x_h, \alpha]) \to h^{-1}(C[x_h, \alpha_h])$ and there exists a function $g^*\colon h^{-1}(C[x_h, \alpha]) \to h^{-1}(C[x_h, \alpha_h])$ being a continuous extension of g.

Putting

$$f_X(x) = \begin{cases} g^*(x) & \text{if } x \in h^{-1}(C[x_h, \alpha]), \\ f(x) & \text{if } x \notin h^{-1}(C[x_h, \alpha]) \end{cases}$$

we notice that $f_X\colon X \to X$ is continuous. Obviously $f_X \in B(f, \varepsilon)$ and $\nabla(f, f_X) \subset B_\rho(x_0, \varepsilon)$.

We will show that

$$x_0 \text{ is an } \mathcal{F}\text{-focal entropy point of } f_X.$$

Let $\delta > 0$ be such that $B(x_0, \delta) \subset W_{x_0}$. Obviously, there exists $n_0 \in \mathbb{N}$ such that $A[x_h, r_n, s_n] \subset h(B(x_0, \delta))$ for $n > n_0$. Fix $n > n_0$ and put $F_n = (A_1^n, \dots, A_{2^n}^n)$, where $A_i^n = h^{-1}(A[x_h, d_{2i-1}^n, d_{2i}^n])$ for $i = 1, \dots, 2^n$. Obviously, $F_n \in \vartheta_{\mathcal{F}}^{B(x_0,\delta)}$. It is easy to see that $A_i^n \underset{f_X}{\to} A_j^n$ for any $i, j \in \{1, 2, \dots, 2^n\}$, so all the elements of the matrix $\mathcal{M}_{F_n, f_X}$ are equal to 1. Hence $\sigma(\mathcal{M}_{F_n, f_X}) = 2^n$ which gives $H_{f_X}(F_n) = n \log 2$. Thus $H_{\mathcal{F}, f_X}(B(x_0, \delta)) = \infty$ which implies that x_0 is an $\mathcal{F}$-focal entropy point of f_X.

For $x_0 \in \delta(X)$, the proof runs in an analogous way. Moreover, similar arguments apply to the condition (i). To prove (ii), it suffices to note that $\delta(X)$ is an $(m-1)$-dimensional manifold and $x_0 \in \mathrm{Int}(\delta(X))$. The condition (i) implies that f is improvable at x_0 with respect to $\delta(X)$. □

It is worth noting that the assumption $x_0 \in \mathrm{Fix}(f)$ in the above theorem is essential. Indeed, consider the function $f\colon [0,1] \to [0,1]$ such that $f(1) = 1$ and for any $i \in \{1, 2, 3, 4\}$ we have that $f(x) = -2x + \frac{i+1}{2}$ for $x \in [\frac{2(i-1)}{8}, \frac{2i-1}{8})$ and $f(x) = 2x + \frac{2-i}{2}$ for $x \in [\frac{2i-1}{8}, \frac{2i}{8})$. Corollary 4.3.13 in [2] implies $h(f) > 0$. Moreover, it is easy to see that $f^n(A) \subset [\frac{3}{4}, 1]$ for any $n \in \mathbb{N}$ and any nonempty set $A \subset (\frac{1}{4}, \frac{3}{4})$. Since $(\frac{1}{4}, \frac{3}{4}) \in O(\frac{1}{2})$, so we deduce immediately that f is not improvable at $\frac{1}{2}$.

We will say, that a function $f\colon X \to X$ is c-improvable at a point $x_0 \in X$, if there exists a function g being improvable at x_0 such that f and g are conjugate (i.e., there exists a homeomorphism $h\colon X \to X$ such that $h \circ g = f \circ h$).

Theorem 5.2. *If X is a connected space having a fixed point property in* $\mathrm{Int}(X)$ *and $f\colon X \to X$, then f is c-improvable at x_0, for any $x_0 \in \mathrm{Int}(X)$.*

Proof. Let $x_0 \in \mathrm{int}(X)$ and $y_0 \in \mathrm{Fix}(f) \cap \mathrm{Int}(X)$. There exists a homeomorphism $h\colon X \to X$ such that $h(x_0) = y_0$. Consider function $g = h^{-1} \circ f \circ h$. It is easy to see that $x_0 \in \mathrm{Fix}(g)$. By Theorem 5.1 the function g is improvable at x_0 which completes the proof. □

References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), no. 2, 309–319.
`http://dx.doi.org/10.1090/s0002-9947-1965-0175106-9`

[2] L. Alsedà, J. Llibre and M. Misiurewicz, *Combinatorial Dynamics and Entropy in Dimension One*, Second edition, Advanced Series in Nonlinear Dynamics **5**, World Scientific, River Edge, NJ, 2000. `http://dx.doi.org/10.1142/1980`

[3] D. J. Barrot, *Set of periods, topological entropy and combinatorial dynamics for tree and graph maps*, Univ. Aut. de Barcelona.
`http://www.tdx.cat/bitstream/handle/10803/3078/djb1de1.pdf?seq`

[4] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Mathematics **1513**, Springer-Verlag, Berlin, 1992. http://dx.doi.org/10.1007/bfb0084762

[5] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401–414; Erratum: **181** (1973), 509–510. http://dx.doi.org/10.1090/s0002-9947-1971-0274707-x; http://dx.doi.org/10.1090/s0002-9947-1973-0320271-8

[6] M. Čiklová, *Dynamical systems generated by functions with connected $\mathcal{G}_\delta$ graphs*, Real Anal. Exchange **30**, (2004), no. 2, 617–637.

[7] E. I. Dinaburg, *The relation between topological entropy and metric entropy*, Soviet Math. Doklady **11** (1970), 13–16.

[8] T. N. T. Goodman, *Relating topological entropy and measure entropy*, Bull. London Math. Soc. **3** (1971), no. 2, 176–180. http://dx.doi.org/10.1112/blms/3.2.176

[9] W. A. Kirk and B. Sims, *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001. http://dx.doi.org/10.1007/978-94-017-1748-9

[10] S. Kolyada and Ĺ. Snoha, *Topological entropy of nonautonomous dynamical systems*, Random Comput. Dynam. **4** 1996, no. 2-3, 205–233.

[11] J. M. Lee, *Introduction to Topological Manifolds*, Graduate Texts in Mathematics **202**, Springer-Verlag, New York, 2000. http://dx.doi.org/10.1007/b98853

[12] M. Misiurewicz, *Horseshoes for mappings of the interval*, Bull. Acad. Polon. Sci. Sér. Sci. Math. **27** (1979), no. 2, 167–169.

[13] Z. H. Nitecki, *Topological entropy and the preimage structure of maps*, Real Anal. Exchange **29** (2003/04), no. 1, 9–41.

[14] R. J. Pawlak, *On the entropy of Darboux functions*, Colloq. Math. **116** (2009), no. 2, 227–241. http://dx.doi.org/10.4064/cm116-2-7

[15] R. J. Pawlak and A. Loranty, *The generalized entropy in the generalized topological spaces*, Topology Appl. **159** (2012), no. 7, 1734–1742. http://dx.doi.org/10.1016/j.topol.2011.05.043

[16] R. J. Pawlak, A. Loranty and A. Bakowska, *On the topological entropy of continuous and almost continuous functions*, Topology Appl. **158** (2011), no. 15, 2022–2033. http://dx.doi.org/10.1016/j.topol.2011.06.049

[17] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.

[18] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics **79**, Springer-Verlag, New Yourk-Berlin, 1982.
http://dx.doi.org/10.1007/978-1-4612-5775-2

[19] X. Ye and G. Zhang, *Entropy points and applications*, Trans. Amer. Math. Soc. **359** (2007), no. 12, 6167–6186. http://dx.doi.org/10.1090/s0002-9947-07-04357-7

Ewa Korczak-Kubiak, Anna Loranty and Ryszard J. Pawlak
Łódź University, Faculty of Mathematics and Computer Science Banacha 22, 90-238
Łódź, Poland
E-mail address: ekor@math.uni.lodz.pl, loranta@math.uni.lodz.pl,
rpawlak@math.uni.lodz.pl