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On Constraint Qualification for an Infinite System of Quasiconvex Inequalities in Normed Linear Space

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Abstract. The constraint qualification Q-CCCQ plays an important role in quasiconvex programming and has been developed by many authors to investigate the set containment problem, duality and optimality conditions for quasiconvex programming. In this paper, we consider an infinite quasiconvex inequality system defined by a family of proper lower semicontinuous quasiconvex functions $\{h_i : i \in I\}$ and establish some sufficient conditions for ensuring the Q-CCCQ in terms of the interior-point condition together with approximate continuity assumption of the function $i \mapsto h_i(x)$.

1. Introduction

Let X be a real normed linear space, $D \subseteq X$ be a closed convex set, I be an arbitrary index set, $\{h_i : i \in I\}$ be a family of proper lower semicontinuous quasiconvex functions on X and h be a proper convex function on X. Consider the following inequality system:

$$x \in D; \quad h_i(x) \le 0 \quad \text{for each } i \in I, \tag{1.1}$$

and the minimization problem:

$$\text{Minimize } h(x) \text{ subject to } h_i(x) \le 0,\ i \in I,\ x \in D.$$

These two types of problems are important in quasiconvex programming. Since the constraint qualifications play an important role in the study of these two problems, the constraint qualifications for quasiconvex programming and their applications were widely studied and extensively developed, see, e.g., [12–14] and the references therein. Among these constraint qulifications, the closed cone constraint qualification (Q-CCCQ) for quasiconvex programming is a kind of constraint qualification that of much importance.

The Q-CCCQ was introduced in [12] for a finite quasiconvex system in a locally convex Hausdorff topological space to study the Lagrange-type duality and the authors showed

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that the Q-CCCQ is the weakest constraint qualification for the duality. Then, this condition was extended in [14] to the general case when the involved functions are infinitely many.

In view of the importance of this constraint qualification, it is not only natural but also useful to investigate the sufficient conditions for ensuring it. However, to our best knowledge, very few papers are devoted to discussing when the Q-CCCQ holds except the work in [14], where Suzuki and Kuroiwa established a necessary and sufficient condition for one lower semicontinuous quasiconvex function to satisfy the Q-CCCQ; further, by applying that equivalent condition, they provided some sufficient conditions for ensuring that function to satisfy the Q-CCCQ in the special case when $X = \mathbb{R}^n$.

Note that in convex programming, to establish and develop the sufficient conditions for ensuring the constraint qualifications have attracted a lot of attention from many mathematicians in recent years, and one of the most well-known conditions is the Slater type condition. The Slater condition was introduced in [8] for the semi-infinite convex inequality system in the Euclidean space. Then, Li and Ng in [4] introduced the concepts of Slater condition and weak Slater condition for an infinite system of convex continuous functions on a Banach space, and the authors applied these conditions to provide some sufficient conditions to ensure the basic constraint qualification. Particularly, to meet much broader class of problems, in our recent work [9], we introduced the concept of quasi-Slater condition for a family of continuous convex functions on $\mathbb{R}^n$, which is much weaker than the weak Slater condition; moreover, we showed in [9] that the quasi-Slater condition implies the Farkas-Minkowski (FM) qualification under some appropriate continuity assumption. In the special case when each of the involved function in the convex inequality system is the indicator function of a closed convex set, Li, Ng and Pong [6] provided some sufficient conditions for the closed convex sets system to satisfy the sum of epigraphs constraint qualification (SECQ) in terms of the interior-point conditions together with appropriate continuity of the associated set-valued function on the (topologized) index set. However, as far as we know, in quasiconvex programming, not many results are known to provide sufficient conditions for Q-CCCQ in terms of interior-point condition.

The main objective of this paper is to establish some sufficient conditions of the Q-CCCQ in the normed linear space in terms of the interior-point condition. Note that the Q-CCCQ for quasiconvex programming is closely related to the FM qualification for convex programming. By applying these relations, we will study the property of the Q-CCCQ in Section 3 and will give some alternative form of the Q-CCCQ, which extends the known results.

Furthermore, note that in the special case when each function in the convex inequality system is the indicator function of a closed convex set, the FM qualification reduces to the

SECQ. Thus, by applying the sufficient conditions that originally proposed in [6] to ensure the SECQ, we will provide some sufficient conditions for ensuring the Q-CCCQ in Section 4 in terms of the interior-point condition together with suitable continuity assumption of the function $i \mapsto h_i(x)$ and some property of some finite subsystems of (1.1).

2. Notations and preliminary results

The notations used in the present paper are standard (cf. [2, 15]). Throughout the whole paper, we assume that X is a real normed linear space, and we let X^* denote the dual space of X, and $\langle x^*, x\rangle$ denote the value of a functional x^* in X^* at $x \in X$, i.e., $\langle x^*, x\rangle = x^*(x)$. The dual X^* is endowed with the weak*-topology. Thus, if $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the weak* closure of W. We use $\boldsymbol{B}(x, \varepsilon)$ to denote the closed ball with center x and radius ε. Let C be a nonempty subset of X. The interior (resp. relative interior, convex cone hull, linear hull, affine hull) of C is denoted by $\operatorname{int} C$ (resp. $\operatorname{ri} C, \operatorname{cone} C, \operatorname{span} C, \operatorname{aff} C$). We shall adopt the convention that $\operatorname{cone} C = \{0\}$ when C is an empty set. Consider a closed convex nonempty subset Z of X. The interior of C relative to Z is denoted by $\operatorname{rint}_Z C$ and defined to be the interior of the set $\operatorname{aff} Z \cap C$ in the metric space $\operatorname{aff} Z$. Thus, a point $z \in \operatorname{rint}_Z C$ if and only if there exists $\varepsilon > 0$ such that

$$z \in \operatorname{aff} Z \cap \boldsymbol{B}(z, \varepsilon) \subseteq C,$$

and $\operatorname{ri} C = \operatorname{rint}_C C$. The indicator function δ_C and the support function σ_C of C are, respectively, defined by

$$\delta_C(x) := \begin{cases} 0, & x \in C, \\ +\infty, & \text{otherwise}, \end{cases}$$

and

$$\sigma_C(x^*) := \sup_{x \in C} \langle x^*, x\rangle \quad \text{for each } x^* \in X^*.$$

The distance function of C, denoted by $d(\cdot, C) \colon X \to \mathbb{R}$, is defined by

$$d(x, C) := \inf \{\|x - c\| \mid c \in C\} \quad \text{for each } x \in X.$$

Let $\{A_i : i \in I\}$ be a system of subsets of X. The set $\sum_{i \in I} A_i$ is defined by

$$\sum_{i \in I} A_i := \begin{cases} \left\{\sum_{i \in I_0} a_i : a_i \in A_i, I_0 \subseteq I \text{ being finite}\right\}, & I \neq \emptyset, \\ \{0\}, & I = \emptyset. \end{cases}$$

In particular, we adopt the convention that $\sum_{i \in I} a_i = 0$ if $I = \emptyset$. For a proper convex function $f \colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, the effective domain of f is denoted by $\operatorname{dom} f :=$

$\{x \in X : f(x) < +\infty\}$. The epigraph and conjugate of a function f on X, denoted by $\operatorname{epi} f$ and f^*, are defined respectively by

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$$

and

$$f^*(x^*) := \sup \{\langle x^*, x\rangle - f(x) : x \in X\} \quad \text{for each } x^* \in X^*.$$

If f and g are proper lower semicontinuous (l.s.c. in short) extended real-valued convex functions on X, then we have

$$f \leq g \iff f^* \geq g^* \iff \operatorname{epi} f^* \subseteq \operatorname{epi} g^*.$$

In particular, for closed convex sets A and B, the following assertions are well-known and easy to verify:

$$\sigma_A = \delta_A^*,$$

and

$$\operatorname{epi} \delta_A^* \subseteq \operatorname{epi} \delta_B^* \iff A \supseteq B. \tag{2.1}$$

Recall that a function $f\colon X \to \overline{\mathbb{R}}$ is said to be quasiconvex if for all $x_1, x_2 \in X$, and $\alpha \in [0, 1]$,

$$f((1-\alpha)x_1 + \alpha x_2) \leq \max\{f(x_1), f(x_2)\}.$$

Then, one has that any convex function is quasiconvex, but the opposite is not true. A function is said to be quasiaffine if it is quasiconvex and quasiconcave. In the present paper, we denote

$$Q = \left\{k\colon \mathbb{R} \to \overline{\mathbb{R}} \mid k \text{ is l.s.c. and non-decreasing}\right\}.$$

Then, by the result in [10], one has that a function f is l.s.c. quasiaffine if and only if there exist $k \in Q$ and $\omega \in X^*$ such that $f = k \circ \omega$. The following characterization theorem of the quasiconvex function was also proved by Penot and Volle in [10], which indicates that a l.s.c. quasiconvex function f consists of a supremum of some family of l.s.c. quasiaffine functions.

Theorem 2.1. [10] *Let f be a function from X to $\overline{\mathbb{R}}$. Then, the following statements are equivalent:*

(i) *f is l.s.c. quasiconvex;*

(ii) *there exist a set I, $\{k_i\} \subseteq Q$ and $\{\omega_i\} \subseteq X^*$ such that $f = \sup_{i \in I} k_i \circ \omega_i$.*

Based on this result, Suzuki and Kuroiwa [12] defined the notion of a generator for quasiconvex functions, that is, $G = \{(k_i, \omega_i) \mid i \in I\} \subseteq Q \times X^*$ is said to be a generator of f if $f = \sup_{i\in I} k_i \circ \omega_i$. Then, by Penot and Volle's result, one has that any l.s.c. quasiconvex function has at least one generator. Moreover, Penot and Volle [10] studied generalized concepts of the inverse of non-decreasing functions. For a function $k \in Q$, the hypo-epi-inverse of k is defined by

$$k^{-1}(r) = \inf \{t \in \mathbb{R} \mid r < k(t)\} = \sup \{s \in \mathbb{R} \mid k(s) \leq r\}.$$

In the present paper, we denote the hypo-epi-inverse of k by k^{-1}, since if k has an inverse function, the inverse function and the hypo-epi-inverse of k are the same.

3. The Q-CCCQ

Throughout this paper, we consider the following quasiconvex inequality system:

$$x \in D; \quad h_i(x) \leq 0 \quad \text{for each } i \in I, \tag{3.1}$$

where $D \subseteq X$ is a closed convex set in the normed linear space X, I is an arbitrary index set and $\{h_i : i \in I\}$ are a family of proper l.s.c. quasiconvex functions on X. For each $i \in I$, let $\left\{(k_{(i,j)}, \omega_{(i,j)}) \mid j \in J_i\right\} \subseteq Q \times X^*$ be a generator of h_i and let $T = \{t = (i,j) \mid i \in I, j \in J_i\}$. We use A to denote the feasible solution set of the system (3.1), that is,

$$A := \{x \in D : h_i(x) \leq 0, \forall\, i \in I\},$$

and we always assume that $A \neq \emptyset$.

The following constraint qualification for quasiconvex programming extends the one introduced in [14], where the authors only considered the case when D is the whole space. For convenience, we set

$$\widetilde{T} := \left\{t \in T : k_t^{-1}(0) \in \mathbb{R}\right\}$$

and we always assume that $\widetilde{T} \neq \emptyset$.

Definition 3.1. The system $\{h_i : i \in I\}$ is said to satisfy the closed cone constraint qualification for quasiconvex programming (the Q-CCCQ) w.r.t. $\{(k_t, \omega_t) \mid t \in T\}$ relative to D if

$$\operatorname{epi} \delta_D^* + \sum_{t\in T} \operatorname{cone} \left\{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \leq \delta\right\}$$

is closed.

For each $t \in T$, we define $f_t \colon X \to \overline{\mathbb{R}}$ by

$$f_t(x) = \langle \omega_t, x\rangle - k_t^{-1}(0) \quad \text{for each } x \in X, \tag{3.2}$$

then, the system (3.1) can be rewritten as the following convex system

$$x \in D; \quad f_t(x) \le 0 \quad \text{for each } t \in T \tag{3.3}$$

and thus the feasible solution set of (3.1) reduces to

$$A = \{x \in D : f_t(x) \le 0 \text{ for all } t \in T\}.$$

Note that the assumption $A \neq \emptyset$ ensures that $k_t^{-1}(0) > -\infty$ for each $t \in T$. Thus, $T_\infty := \left\{t \in T : k_t^{-1}(0) = +\infty\right\}$ is the complementary set of $\widetilde{T}$ in T, that is, $T = \widetilde{T} \cup T_\infty$. Moreover, the convex system (3.3) is equivalent to

$$x \in D; \quad f_t(x) \le 0 \quad \text{for each } t \in \widetilde{T} \tag{3.4}$$

and the set A can be rewritten as $A = \left\{x \in D : f_t(x) \le 0 \text{ for all } t \in \widetilde{T}\right\}$.

Following [3], we define the characteristic cone K of (3.4) by

$$K := \operatorname{cone}\left\{(\operatorname{epi}\delta_D^*) \cup \bigcup_{t\in\widetilde{T}} \operatorname{epi} f_t^*\right\}.$$

Since each f_t with $t \in \widetilde{T}$ is an affine function, it can be verified by definition that

$$\operatorname{epi} f_t^* = \{\omega_t\} \times [k_t^{-1}(0), +\infty).$$

Granting this and taking into account that $\operatorname{epi}\delta_D^*$ is a convex cone, one has that

$$K = \operatorname{epi}\delta_D^* + \sum_{t\in\widetilde{T}} \operatorname{cone}(\operatorname{epi} f_t^*) = \operatorname{epi}\delta_D^* + \sum_{t\in\widetilde{T}} \operatorname{cone}\left\{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \le \delta\right\}.$$

Furthermore, it follows from [7, Proposition 4.1] (applied to the system $\left\{\delta_D, f_t : t \in \widetilde{T}\right\}$) that

$$\operatorname{epi}\delta_A^* = \operatorname{cl} K = \operatorname{cl}\left(\operatorname{epi}\delta_D^* + \sum_{t\in\widetilde{T}} \operatorname{cone}\left\{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \le \delta\right\}\right); \tag{3.5}$$

see also [3, (3.1)]. Recall that the family $\left\{D; f_t : t \in \widetilde{T}\right\}$ is said to be Farkas-Minkowski (FM) if K is closed (see [3] for example). By (3.5) and note that $K \subseteq \operatorname{epi}\delta_A^*$, one has the following equivalences for the system $\left\{D; f_t : t \in \widetilde{T}\right\}$:

$$\left\{D; f_t : t \in \widetilde{T}\right\} \text{ is FM} \iff K \text{ is closed} \iff \operatorname{epi}\delta_A^* \subseteq K. \tag{3.6}$$

Note that for each $t \in T_\infty$, $\operatorname{epi} f_t^* = \emptyset$. Thus, we further have

$$K = \operatorname{epi}\delta_D^* + \sum_{t\in T} \operatorname{cone}\left\{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \le \delta\right\}.$$

Consequently, combining this and (3.6), we can obtain the following theorem, which gives the alternative form of Q-CCCQ. The first equivalence in the following theorem was given in [14] for the case when D is the whole space.

Theorem 3.2. *For each $t \in T$, let $f_t\colon X \to \overline{\mathbb{R}}$ be defined by* (3.2). *Then the following equivalences are true:*

$$\begin{aligned} &\{h_i : i \in I\} \text{ satisfies the Q-CCCQ w.r.t. } \{(k_t, \omega_t) \mid t \in T\} \text{ relative to } D \\ \Longleftrightarrow\ &\operatorname{epi} \delta_A^* \subseteq \operatorname{epi} \delta_D^* + \sum_{t \in T} \operatorname{cone}\left\{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \le \delta\right\} \\ \Longleftrightarrow\ &\left\{D; f_t : t \in \widetilde{T}\right\} \text{ is FM.} \end{aligned}$$

4. Sufficient conditions for Q-CCCQ

Recall that D is a closed convex subset of the normed linear space X, I is an index set, $\{h_i : i \in I\}$ are a family of proper l.s.c. quasiconvex functions on X, for each $i \in I$, $\left\{(k_{(i,j)}, \omega_{(i,j)}) \mid j \in J_i\right\} \subseteq Q \times X^*$ is a generator of h_i, the index set $T = \{t = (i,j) \mid i \in I, j \in J_i\}$ and the feasible solution set $A = \{x \in D : h_i(x) \le 0 \text{ for each } i \in I\} \neq \emptyset$. Furthermore, throughout the remainder of this section, we always assume that I is a compact metric space. The main objective of this section is to provide sufficient conditions for ensuring the Q-CCCQ. Before going further, we give some notation and definition which will be used in the rest of this paper.

Consider the metric space I. Recall that a function $f\colon I \mapsto \overline{\mathbb{R}}$ is upper semicontinuous at $i_0 \in I$ if for any $\epsilon > 0$, there exists a neighborhood $U(i_0)$ of i_0 such that

$$f(i) < f(i_0) + \epsilon \quad \text{for all } i \in U(i_0),$$

and that f is upper semicontinuous on I if it is upper semicontinuous at each $i \in I$.

Let $|J|$ denote the cardinality of the set J. The following interior-point condition was introduced in [6].

Definition 4.1. Let $\{D, C_i : i \in I\}$ be a system of closed convex sets in X and let m be a positive integer. Then the system $\{D, C_i : i \in I\}$ is said to satisfy the m-D-interior-point condition if, for any subset J of I with $|J| \le m$,

$$D \cap \bigcap_{j \in J} \operatorname{rint}_D C_j \neq \emptyset.$$

For the following notion of semicontinuity of set-valued maps, readers may refer to standard texts such as [1, 11].

Definition 4.2. Let $F\colon I \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping and $i_0 \in I$. The mapping F is said to be

(i) lower semicontinuous at i_0 if, for any $x_0 \in F(i_0)$ and any $\varepsilon > 0$, there exists a neighborhood $U(i_0)$ of i_0 such that $\boldsymbol{B}(x_0, \varepsilon) \cap F(i) \neq \emptyset$ for each $i \in U(i_0)$;

(ii) lower semicontinuous on I if it is lower semicontinuous at each $i \in I$.

The following proposition, which was given in [5, Proposition 3.1], provides some useful reformulations regarding the lower semicontinuity.

Proposition 4.3. *Let $F\colon I \to 2^X \setminus \{\emptyset\}$ be a set-valued mapping and let $i_0 \in I$. Then the following statements are equivalent:*

(i) *F is lower semicontinuous at i_0.*

(ii) *For any $x_0 \in F(i_0)$, there exists $x_i \in F(i)$ for each $i \in I$ such that $\lim_{i \to i_0} x_i = x_0$.*

(iii) *For any $x_0 \in F(i_0)$, $\lim_{i \to i_0} d(x_0, F(i)) = 0$.*

Now, we give the following useful lemma. For this, we write

$$C_i := \{x \in X : h_i(x) \le 0\} \quad \text{for each } i \in I.$$

For any proper function $f\colon X \to \overline{\mathbb{R}}$ and $x \in X$, let $[f(x)]_+ := \max\{f(x), 0\}$.

Lemma 4.4. *Let $i_0 \in I$. Suppose that for each $x \in \operatorname{aff} D$, the function $i \mapsto h_i(x)$ is upper semicontinuous at i_0. Suppose further that at least one of the following conditions is satisfied:*

(a) $\emptyset \neq \operatorname{rint}_D C_{i_0} \subseteq \{x \in X : h_{i_0}(x) < 0\}$.

(b) *For each $x \in X$, there exists $\tau_x > 0$ such that*

$$d(x, \operatorname{aff} D \cap C_i) \le \tau_x [h_i(x)]_+ \quad \text{for each } i \in I.$$

Then the set-valued mapping $i \mapsto (\operatorname{aff} D) \cap C_i$ is lower semicontinuous at i_0.

Proof. Let $\overline{x} \in \operatorname{aff} D \cap C_{i_0}$. By Proposition 4.3, it suffices to show that

$$\lim_{i \to i_0} d(\overline{x}, \operatorname{aff} D \cap C_i) = 0.$$

Given $\epsilon > 0$. We need to show that there exits a neighborhood $U(i_0)$ of i_0 such that

$$d(\overline{x}, \operatorname{aff} D \cap C_i) \le \epsilon \quad \text{for each } i \in U(i_0). \tag{4.1}$$

We now assume that (a) holds. By [15, Theorem 1.1.2(iv)], we can take a point $\overline{y} \in \operatorname{rint}_D C_{i_0}$ such that $\|\overline{x} - \overline{y}\| \le \epsilon$. Moreover, by (a), one has $h_{i_0}(\overline{y}) < 0$. Let $0 < \overline{\epsilon} < -h_{i_0}(\overline{y})$. By the assumption, the function $i \mapsto h_i(\overline{y})$ is upper semicontinuous at i_0. Then, there exists a neighborhood $U(i_0)$ of i_0 such that for each $i \in U(i_0)$,

$$h_i(\overline{y}) < h_{i_0}(\overline{y}) + \overline{\epsilon} < 0;$$

and hence $\overline{y} \in \operatorname{aff} D \cap C_i$. Thus,

$$d(\overline{x}, \operatorname{aff} D \cap C_i) \leq \|\overline{x} - \overline{y}\| \leq \epsilon \quad \text{for each } i \in U(i_0).$$

Therefore, (4.1) holds and we establish the result under assumption (a).

Below we assume that (b) holds. Then there exists $\tau_{\overline{x}} > 0$ such that

$$d(\overline{x}, \operatorname{aff} D \cap C_i) \leq \tau_{\overline{x}}[h_i(\overline{x})]_+ \quad \text{for each } i \in I. \tag{4.2}$$

On the other hand, by the assumption, the function $i \mapsto h_i(\overline{x})$ is upper semicontinuous at i_0. Then, there exists a neighborhood $U(i_0)$ of i_0 such that

$$h_i(\overline{x}) < h_{i_0}(\overline{x}) + \frac{\epsilon}{\tau_{\overline{x}}} \leq \frac{\epsilon}{\tau_{\overline{x}}} \quad \text{for each } i \in U(i_0).$$

This, together with (4.2), implies that

$$d(\overline{x}, \operatorname{aff} D \cap C_i) \leq \tau_{\overline{x}}[h_i(\overline{x})]_+ \leq \epsilon \quad \text{for each } i \in U(i_0).$$

Thus (4.1) holds and the proof is complete. □

Now we are going to state and prove the main result of this section.

Theorem 4.5. *Let m be a positive integer. Suppose that $0 \in D$ and consider the following conditions:*

(a) *D is of finite dimension m.*

(b) *For each $x \in \operatorname{aff} D$, the function $i \mapsto h_i(x)$ is upper semicontinuous on I.*

(c) *Either for each $i \in I$, $\emptyset \neq \operatorname{rint}_D C_i \subseteq \{x \in X : h_i(x) < 0\}$ or for each $x \in X$, there exists $\tau_x > 0$ such that*

$$d(x, \operatorname{aff} D \cap C_i) \leq \tau_x[h_i(x)]_+ \quad \textit{for each } i \in I.$$

(d) *The system $\{D, C_i : i \in I\}$ satisfies the $(m+1)$-D-interior-point condition.*

(e) *For each $i \in I$, h_i satisfies the Q-CCCQ w.r.t. $\{(k_{i,j}, \omega_{i,j}) \mid j \in J_i\}$ relative to D, that is,*

$$\operatorname{epi} \delta^*_{D\cap C_i} \subseteq \operatorname{epi} \delta^*_D + \operatorname{cone}\left(\bigcup_{j\in J_i} \left\{(\omega_{i,j}, \delta) \in X^* \times \mathbb{R} \mid k^{-1}_{i,j}(0) \leq \delta\right\}\right).$$

(d*) *The system $\{D, C_i : i \in I\}$ satisfies the m-D-interior-point condition.*

(e*) *For each finite subset J of I with $|J| = \min\{m+1, |I|\}$, the subsystem $\{h_i : i \in J\}$ satisfies the Q-CCCQ w.r.t. $\{(k_t, \omega_t) \mid t \in T_J\}$ relative to D, where $T_J = \{t = (i,j) \mid i \in J, j \in J_i\}$, that is,*

$$\operatorname{epi} \delta^*_{D \cap (\cap_{i\in J} C_i)} \subseteq \operatorname{epi} \delta^*_D + \operatorname{cone}\left(\bigcup_{t \in T_J} \{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \le \delta\} \right).$$

Then the following assertions hold:

(i) *If* (a), (b), (c), (d), (e) *are satisfied, then the system $\{h_i : i \in I\}$ satisfies the Q-CCCQ w.r.t. $\{(k_t, \omega_t) \mid t \in T\}$ relative to D.*

(ii) *If D is bounded and* (a), (b), (c), (d*), (e*) *are satisfied, then the system $\{h_i : i \in I\}$ satisfies the Q-CCCQ w.r.t. $\{(k_t, \omega_t) \mid t \in T\}$ relative to D.*

Proof. (i) Since $0 \in D$, we have $\operatorname{aff} D = \operatorname{span} D$. Then, by Lemma 4.4, the conditions (b) and (c) ensure that the set-valued mapping $i \mapsto (\operatorname{span} D) \cap C_i$ is lower semicontinuous on I. Moreover, the condition (d) implies that for each finite subset J of I with $|J| \le m+1$,

$$D \cap \bigcap_{j \in J} \operatorname{rint}_D C_j \neq \emptyset.$$

Thus, by applying [6, Theorem 5.3(i)], we know that the conditions (a)–(d) imply that the system $\{D, (\operatorname{span} D) \cap C_i : i \in I\}$ satisfies the SECQ, that is,

$$\operatorname{epi} \delta^*_{D \cap (\cap_{i \in I} (\operatorname{span} D \cap C_i))} = \operatorname{epi} \delta^*_A = \operatorname{epi} \delta^*_D + \sum_{i \in I} \operatorname{epi} \delta^*_{(\operatorname{span} D) \cap C_i}.$$

Granting this and the condition (e), we obtain that

$$\operatorname{epi} \delta^*_A \subseteq \operatorname{epi} \delta^*_D + \sum_{i \in I} \operatorname{epi} \delta^*_{D \cap C_i} \subseteq \operatorname{epi} \delta^*_D + \sum_{t \in T} \operatorname{cone}\left\{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \le \delta\right\},$$

where the first inclusion holds thanks to (2.1) and the fact that $D \subseteq \operatorname{span} D$. By Theorem 3.2, this shows that the system $\{h_i : i \in I\}$ satisfies the Q-CCCQ w.r.t. $\{(k_t, \omega_t) \mid t \in T\}$ relative to D.

(ii) Now suppose that (a), (b), (c), (d*), (e*) are satisfied. Without loss of generality, we may assume that $|I| > m+1$, otherwise, the conclusion follows from assumption (e*). Note that the conditions (b) and (c) ensure that the set-valued mapping $i \mapsto (\operatorname{span} D) \cap C_i$ is lower semicontinuous on I and the condition (d*) implies that for each finite subset J of I with $|J| \le m$,

$$D \cap \bigcap_{j \in J} \operatorname{rint}_D C_j \neq \emptyset.$$

Moreover, one can verify by definition that for each $i \in I$,

$$\operatorname{cone}\left(\bigcup_{j\in J_i}\left\{(\omega_{i,j},\delta)\in X^*\times\mathbb{R}\mid k_{i,j}^{-1}(0)\leq\delta\right\}\right)\subseteq\operatorname{epi}\delta_{C_i}^*.$$

Thus, the condition (e*) implies that for each finite subset J of I with $|J| = \min\{m+1, |I|\}$, the system $\{D, C_i : i \in J\}$ satisfies the following condition

$$\operatorname{epi}\delta^*_{D\cap(\cap_{i\in J}C_i)}\subseteq\operatorname{epi}\delta^*_D+\sum_{i\in J}\operatorname{epi}\delta^*_{C_i}.$$

Consequently, by applying [6, Theorem 5.3(iii)], we know that the conditions (a), (b), (c), (d*), (e*) assert that

$$\operatorname{epi}\delta^*_A\subseteq\operatorname{epi}\delta^*_D+\sum_{i\in I}\operatorname{epi}\delta^*_{C_i}. \tag{4.3}$$

Let $i \in I$ and let J be any subset of I such that $i \in J$ and $|J| = m+1$. Then, by assumption (e*), we have that

$$\operatorname{epi}\delta^*_{C_i}\subseteq\operatorname{epi}\delta^*_{D\cap(\cap_{i\in J}C_i)}\subseteq\operatorname{epi}\delta^*_D+\operatorname{cone}\left(\bigcup_{t\in T_J}\left\{(\omega_t,\delta)\in X^*\times\mathbb{R}\mid k_t^{-1}(0)\leq\delta\right\}\right).$$

Combining this with (4.3) yields that

$$\operatorname{epi}\delta^*_A\subseteq\operatorname{epi}\delta^*_D+\sum_{t\in T}\operatorname{cone}\left\{(\omega_t,\delta)\in X^*\times\mathbb{R}\mid k_t^{-1}(0)\leq\delta\right\},$$

which shows that the system $\{h_i : i \in I\}$ satisfies the Q-CCCQ w.r.t. $\{(k_t, \omega_t) \mid t \in T\}$ relative to D thanks to Theorem 3.2. The proof is complete. □

Remark 4.6. The assumption $0 \in D$ that required in Theorem 4.5 can be dropped. Indeed, we can take an arbitrary element $x_0 \in D$. Define

$$\widehat{D}:=D-x_0,\quad \widehat{h}_i(\cdot):=h_i(\cdot+x_0)\quad\text{for each } i\in I.$$

Then,

$$\widehat{C}_i:=\left\{x\in X:\widehat{h}_i(x)\leq 0\right\}=C_i-x_0\quad\text{for each } i\in I$$

and

$$\widehat{A}:=\left\{x\in D:\widehat{h}_i(x)\leq 0\right\}=A-x_0.$$

It can be checked that the conditions (a)–(d) and (d*) are also satisfied by replacing D, $\{C_i : i \in I\}$, respectively, $\{h_i : i \in I\}$ by $\widehat{D}$, $\left\{\widehat{C}_i : i \in I\right\}$, respectively, $\left\{\widehat{h}_i : i \in I\right\}$. Moreover, we define the following conditions:

(ê) For each $i \in I$, the pair $\left\{\widehat{D}; \widehat{h}_i\right\}$ has the property

$$\operatorname{epi} \sigma_{\widehat{D} \cap \widehat{C}_i} \subseteq \operatorname{epi} \sigma_{\widehat{D}} + \operatorname{cone}\left(\bigcup_{j \in J_i} \left\{(\omega_{i,j}, \delta) \in X^* \times \mathbb{R} \mid k_{i,j}^{-1}(0) - \langle \omega_{i,j}, x_0 \rangle \leq \delta\right\}\right).$$

(ê*) For each finite subset J of I with $|J| = \min\{m+1, |I|\}$, the family $\left\{\widehat{D}; \widehat{h}_i : i \in J\right\}$ has the property

$$\operatorname{epi} \sigma_{\widehat{D} \cap (\cap_{i \in J} \widehat{C}_i)} \subseteq \operatorname{epi} \sigma_{\widehat{D}} + \operatorname{cone}\left(\bigcup_{t \in T_J} \left\{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) - \langle \omega_t, x_0 \rangle \leq \delta\right\}\right),$$

where $T_J = \{t = (i,j) \mid i \in J, j \in J_i\}$.

Then, it is easy to check that the condition (e), respectively, (e*) is equivalent to (ê), respectively, (ê*). Consequently, by using the similar argument as that in the proof of Theorem 4.5, one can obtain that the system $\left\{\widehat{D}; \widehat{h}_i : i \in I\right\}$ satisfies the following property

$$\operatorname{epi} \delta^*_{\widehat{A}} \subseteq \operatorname{epi} \delta^*_{\widehat{D}} + \sum_{t \in T} \operatorname{cone}\left\{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) - \langle \omega_t, x_0 \rangle \leq \delta\right\},$$

which is equivalent to that

$$\operatorname{epi} \delta^*_A \subseteq \operatorname{epi} \delta^*_D + \sum_{t \in T} \operatorname{cone}\left\{(\omega_t, \delta) \in X^* \times \mathbb{R} \mid k_t^{-1}(0) \leq \delta\right\}.$$

Thus, the system $\{h_i : i \in I\}$ satisfies the Q-CCCQ w.r.t. $\{(k_t, \omega_t) \mid t \in T\}$ relative to D.

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