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Random Attractors for Non-autonomous Stochastic Lattice FitzHugh-Nagumo Systems with Random Coupled Coefficients

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Abstract. In this paper, we study the non-autonomous stochastic lattice FitzHugh-Nagumo system with random coupled coefficients and multiplicative white noise. We consider the existence of random attractors in a weighted space $l^2_\rho \times l^2_\rho$ for this system, and establish the upper semicontinuity of random attractors as the intensity of noise approaches zero.

1. Introduction

The dynamics of infinite lattice dynamical systems (LDSs) have drawn much attention of mathematicians and physicists, see [5,30–32,41,42,44–49] and the references therein. Since most of the realistic systems involve random effects which may play an important role as intrinsic phenomena rather than just compensation of defects in deterministic models, stochastic LDSs (SLDSs) then arise naturally while these uncertainties or noise are taken into account. Recently, many works have been done regarding the existence of random attractors for SLDSs (see e.g., [6, 8, 11, 21–23, 26, 27, 39, 40, 43]). Of those, Bates, Lu and Wang [8] considered the existence of random attractors for first-order non-autonomous stochastic lattice system driven by multiplicative white noise. Han, Shen and Zhou [22] considered the existence of random attractors for first-order SLDSs with random coupled coefficients and multiplicative/additive white noise.

Motivated by [8,22], we will focus our study on the asymptotic behavior of solutions of the following non-autonomous stochastic lattice FitzHugh-Nagumo system with random coupled coefficients and multiplicative white noise: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\tag{1.1} \begin{cases} du_i = \left(-f_i(u_i) + g_i(t) + \sum_{j=-q}^{q} \eta_{i,j}(\theta_t\omega)u_{i+j} - v_i\right) dt + \epsilon u_i \circ dw(t), & i \in \mathbb{Z}, \\ dv_i = (\sigma u_i - \delta v_i + h_i(t))\, dt + \epsilon v_i \circ dw(t), & i \in \mathbb{Z}, \end{cases}$$

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with the initial data

$$u_i(\tau) = u_{i,\tau}, \quad v_i(\tau) = v_{i,\tau}, \quad i \in \mathbb{Z},$$

where $\epsilon, \sigma > 0$ and $\delta > 0$ are constants; $u_i, v_i \in \mathbb{R}$; $f_i(u_i), g_i(t), h_i(t) \in \mathbb{R}$; $\eta_{i,j}(\omega)$ ($j \in \{-q, \ldots, 0, \ldots, q\}$, $q \in \mathbb{N}$) are random variables; $(\theta_t)_{t\in\mathbb{R}}$ is a metric dynamical system defined on proper probability space $(\Omega, \mathcal{F}, \mathcal{P})$; $w(t)$ is an independent two-sided real-valued Wiener processes; and $\circ$ denotes the Stratonovich sense of the stochastic term. If g_i and h_i do not depend on t for all $i \in Z$, then we say (1.1) is an autonomous stochastic system. System (1.1) can be regarded as a discrete analogue of the following non-autonomous stochastic FitzHugh-Nagumo system on $\mathbb{R}$

$$\begin{cases} du = (\mathbb{B}(\theta_t\omega)u + f(u) - v + g(t))\,dt + \epsilon u \circ dw(t), \\ dv = (\sigma u - \delta v + h(t))\,dt + \epsilon v \circ dw(t), \\ u(\tau) = u_\tau, \quad v(\tau) = v_\tau. \end{cases}$$

The FitzHugh-Nagumo system is used to describe the signal transmission across axons in neurobiology (see [24]). The long time behavior of FitzHugh-Nagumo system is investigated in both deterministic [35] and stochastic cases [2, 3, 25, 36]. The lattice of FitzHugh-Nagumo system is used to stimulate the propagation of action potentials in myelinated nerve axons (see [16]). The attractor of lattice FitzHugh-Nagumo system has been investigated in [30, 33, 35] in the deterministic case, in [18, 19, 23] in the autonomous stochastic case. Of those, Huang [23] obtained the random attractor for stochastic lattice FitzHugh-Nagumo system with additive noise in $l^2 \times l^2$; the same result was obtained by Gu [19] for stochastic lattice FitzHugh-Nagumo system driven by α-stable Lévy noises in $l^2 \times l^2$.

In practice, the coupled mode between two nodes (say, adjacent nodes) is usually random. It is then of great importance to investigate stochastic lattice system with random coupled coefficients. To the best of our knowledge, there are no results on stochastic lattice FitzHugh-Nagumo system with random coupled coefficients.

In this paper, we consider the existence of a compact global random attractor in a weighted space $l^2_\rho \times l^2_\rho$ for non-autonomous stochastic lattice FitzHugh-Nagumo system (1.1) with random coupled coefficients, which attracts the random tempered bounded sets in pullback sense, and establish the upper semicontinuity of the random attractor as the intensity ϵ of noise approaches zero.

The rest of this paper is organized as follows. In the next section, we recall some basic concepts related to the random attractor for non-autonomous RDSs. In section 3, we mainly consider the existence and upper semicontinuity of a compact global random attractor in a weighted space $l^2_\rho \times l^2_\rho$ for system (1.1). In Section 4, we give some remarks.

2. Preliminaries

In this section, we recall some known results from [37] regarding random attractors for non-autonomous RDSs. All results given in this section are not original and they are presented here just for the reader's convenience. The theory of random attractors for autonomous RDSs can be found in [4, 9, 12, 13, 17].

Let $(X, \|\cdot\|_X)$ be a separable Banach space, $(\Omega, \mathcal{F}, \mathcal{P})$ be a standard probability space and $\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}$ be a family of measure preserving transformations such that $(t, \omega) \mapsto \theta_t\omega$ is measurable, $\theta_0 = \mathrm{Id}_\Omega$, $\theta_{t+s} = \theta_t\theta_s$ for all $s, t \in \mathbb{R}$. We often say that $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ is a metric dynamical system.

Definition 2.1. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ is called a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, the following conditions (1)–(4) are satisfied:

(1) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;

(2) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X;

(3) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s\omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$;

(4) $\Phi(t, \tau, \omega, \cdot) : X \to X$ is continuous.

Hereafter, we assume Φ is a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$, and $\mathcal{D}(X)$ is the collection of all tempered families of nonempty bounded subsets of X. Remember that a family $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ of nonempty bounded subsets of X is said to be tempered if for every $\sigma > 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, the following holds:

$$\lim_{t\to-\infty} e^{\sigma t} \sup_{x\in D(\tau+t,\theta_t\omega)} \|x\|_X = 0.$$

Definition 2.2. A family $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$ is called a random absorbing set for Φ if for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and for every $D \in \mathcal{D}(X)$, there exists $T = T(D, \tau, \omega) > 0$ such that

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega) \quad \text{for all } t \geq T.$$

If, in addition, $K(\tau, \omega)$ is closed in X and is measurable in ω with respect to $\mathcal{F}$, then K is called a closed measurable random absorbing set for Φ.

Definition 2.3. A family $\mathcal{A} = \{\mathcal{A}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$ is called a random attractor for Φ if the following conditions (1)–(3) are fulfilled: for all $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(1) $\mathcal{A}(\tau,\omega)$ is compact in X and is measurable in ω with respect to $\mathcal{F}$;

(2) $\mathcal{A}$ is invariant, that is,

$$\Phi(t,\tau,\omega,\mathcal{A}(\tau,\omega)) = \mathcal{A}(\tau+t,\theta_t\omega);$$

(3) For every $D = \{D(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$,

$$\lim_{t\to\infty} d_H(\Phi(t,\tau-t,\theta_{-t}\omega, D(\tau-t,\theta_{-t}\omega)), \mathcal{A}(\tau,\omega)) = 0,$$

where d_H is the Hausdorff semi-distance given by $d_H(F,G) = \sup_{u\in F}\inf_{v\in G}\|u-v\|_X$, for any $F, G \subset X$.

Next, we provide some sufficient conditions for the existence of random attractors for non-autonomous RDSs in weighted spaces of infinite sequences.

We first introduce a weighted space l^p_ρ of infinite sequences. Let $p \geq 1$ be a real number, and ρ be a positive-valued function from $\mathbb{Z}$ into $(0, M_0] \subset \mathbb{R}^+$, where M_0 is a positive constant. Define $\rho_i = \rho(i)$, $\forall\, i \in \mathbb{Z}$ and $l^p_\rho = \left\{u = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} \rho_i |u_i|^p < \infty, u_i \in \mathbb{R}\right\}$ with the norm $\|u\|_{\rho,p} = \left(\sum_{i\in\mathbb{Z}} \rho_i |u_i|^p\right)^{\frac{1}{p}}$ for $u = (u_i)_{i\in\mathbb{Z}} \in l^p_\rho$. In particular, l^2_ρ is a separable Hilbert space with the inner product $(u,v)_{\rho,2} = \sum_{i\in\mathbb{Z}} \rho_i u_i v_i$ and norm $\|u\|_{\rho,2} = \left(\sum_{i\in\mathbb{Z}} \rho_i |u_i|^2\right)^{\frac{1}{2}}$ for $u = (u_i)_{i\in\mathbb{Z}}, v = (v_i)_{i\in\mathbb{Z}} \in l^2_\rho$. We write $\|\cdot\|_{\rho,2}$ as $\|\cdot\|_\rho$, $(\cdot,\cdot)_{\rho,2}$ as $(\cdot,\cdot)_\rho$, and $\|\cdot\|_\rho$ as $\|\cdot\|$ if $\rho(i) \equiv 1$.

Theorem 2.4. *Let Φ be a continuous cocycle on l^p_ρ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$, and $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Suppose that*

(a) *there exists a bounded closed random absorbing set $B_0 = \{B_0(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l^p_\rho)$ such that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l^p_\rho)$, there exists $T_1 = T_1(\tau,\omega,B) > 0$ yielding $\Phi(t,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)) \subset B_0(\tau,\omega)$, $\forall\, t \geq T_1$;*

(b) *for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for any $\varepsilon > 0$, there exist $T_2 = T_2(\tau,\varepsilon,\omega,B_0) > 0$ and $I_0 = I_0(\tau,\varepsilon,\omega,B_0) \in \mathbb{N}$ such that*

$$\sum_{|i|>I_0} \rho_i |\Phi_i(t,\tau-t,\theta_{-t}\omega,u_{\tau-t})|^p \leq \varepsilon, \quad \forall\, t \geq T_2,\ u_{\tau-t} \in B_0(\tau-t,\theta_{-t}\omega).$$

Then Φ possesses a unique random attractor $\mathcal{A}$ in $\mathcal{D}(l^p_\rho)$ given by, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\mathcal{A}(\tau,\omega) = \bigcap_{\tau \geq T_1} \overline{\bigcup_{t\geq\tau} \Phi(t,\tau-t,\theta_{-t}\omega, B_0(\tau-t,\theta_{-t}\omega))}.$$

Proof. The proof is based on Theorem 3.1 in [22] and Theorem 2.23 in [37] under slight modifications, thus we omit it here. □

3. Random attractors for stochastic lattice FitzHugh-Nagumo system

In this section, we study the existence of a random attractor for non-autonomous stochastic lattice FitzHugh-Nagumo system (1.1) in $l^2_\rho \times l^2_\rho$.

Throughout this section, a positive weight function $\rho\colon \mathbb{Z} \to \mathbb{R}^+$ is chosen to satisfy

$$0 < \rho(i) \le M_0, \quad \rho(i) \le c_0\rho(i \pm 1), \quad \forall\, i \in \mathbb{Z}, \tag{3.1}$$

where M_0 and c_0 are positive constants. For example, for $i \in \mathbb{Z}$, $\rho(i) = \frac{1}{(1+\sigma^2 i^2)^q}$ $(q > \frac{1}{2})$ [31] and $\rho(i) = e^{-\sigma|i|}$ satisfy condition (3.1), where $\sigma > 0$.

3.1. Mathematical setting

Let Ω be defined by

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact open topology of Ω, and $\mathcal{P}$ is the Wiener measure on $(\Omega, \mathcal{F})$ (see [4]). Consider the following non-autonomous stochastic lattice FitzHugh-Nagumo system with random coupled coefficients and multiplicative white noise: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{cases} du_i = \left(\sum_{j=-q}^{q} \eta_{i,j}(\theta_t\omega)u_{i+j} + f_i(u_i) - v_i + g_i(t)\right) dt + \epsilon u_i \circ dw(t), & i \in \mathbb{Z}, \\ dv_i = (\sigma u_i - \delta v_i + h_i(t))\, dt + \epsilon v_i \circ dw(t), & i \in \mathbb{Z}, \\ u_i(\tau) = u_{i,\tau}, \quad v_i(\tau) = v_{i,\tau}, & i \in \mathbb{Z}, \end{cases} \tag{3.2}$$

where $\epsilon, \sigma > 0$ and $\delta > 0$ are constants, $u_i, v_i \in \mathbb{R}$; $f_i(u_i), g_i(t), h_i(t) \in \mathbb{R}$; $\eta_{i,j}(\omega)$ $(j \in \{-q, \dots, 0, \dots, q\}$, $q \in \mathbb{N})$ are random variables on probability space $(\Omega, \mathcal{F}, \mathcal{P})$;

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega,\ t \in \mathbb{R};$$

$w(t)$ is an independent two-sided real-valued Wiener process on $(\Omega, \mathcal{F}, \mathcal{P})$.

Note that system (3.2) can be rewritten as abstract ODEs: for every $\tau \in \mathbb{R}$ and $t > \tau$,

$$\begin{cases} du = (\mathbb{B}(\theta_t\omega)u + f(u) - v + g(t))\, dt + \epsilon u \circ dw(t), \\ dv = (\sigma u - \delta v + h(t))\, dt + \epsilon v \circ dw(t), \\ u(\tau) = u_\tau, \quad v(\tau) = v_\tau, \end{cases} \tag{3.3}$$

where $u = (u_i)_{i\in\mathbb{Z}}$, $v = (v_i)_{i\in\mathbb{Z}}$, $f(u) = (f_i(u_i))_{i\in\mathbb{Z}}$ is a nonlinearity satisfying certain conditions, $g(t) = (g_i(t))_{i\in\mathbb{Z}}$ and $h(t) = (h_i(t))_{i\in\mathbb{Z}}$ are given time dependent sequences, $\mathbb{B}(\omega)$ is a linear operator defined by

$$(\mathbb{B}(\omega)u)_i = \sum_{j=-q}^{q} \eta_{i,j}(\omega)u_{i+j}. \tag{3.4}$$

To convert the problem (3.3) into random differential equations, let

$$z(\theta_t\omega) := -\int_{-\infty}^{0} e^{s}(\theta_t\omega)(s)\,ds, \quad t\in\mathbb{R},\ \omega\in\Omega,$$

which is an Ornstein-Uhlenbeck process on $(\Omega,\mathcal{F},\mathcal{P})$ and solves the following Ornstein-Uhlenbeck equation:

$$dz + zdt = dw(t),$$

see [4, 11] for detail, where $w(t)(\omega) = w(t,\omega) = \omega(t)$. From [4, 7, 15], it is known that the random variable $|z(\omega)|$ is tempered,

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{t} = \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_s\omega)\,ds = 0, \tag{3.5}$$

and there is a θ_t-invariant set $\widetilde{\Omega}\subset\Omega$ of full $\mathcal{P}$ measure such that $t\mapsto z(\theta_t\omega)$ is continuous in t for every $\omega\in\widetilde{\Omega}$.

Let $\mathcal{H}(\theta_t\omega) := e^{\epsilon z(\theta_t\omega)}\,\mathrm{Id}_{l^2_\rho}$, which is a homeomorphism in l^2_ρ and the inverse operator of $\mathcal{H}$ is defined as $\mathcal{H}^{-1}(\theta_t\omega) := e^{-\epsilon z(\theta_t\omega)}\,\mathrm{Id}_{l^2_\rho}$. By (3.5), both $\|\mathcal{H}(\theta_t\omega)\|$ and $\left\|\mathcal{H}^{-1}(\theta_t\omega)\right\|$ have sub-exponential growth as $t\to\pm\infty$ for $\omega\in\widetilde{\Omega}$, then, $\|\mathcal{H}(\omega)\|$ and $\left\|\mathcal{H}^{-1}(\omega)\right\|$ are tempered.

Let $\widetilde{u}(t,\omega) = \mathcal{H}^{-1}(\theta_t\omega)u(t,\omega) = e^{-\epsilon z(\theta_t\omega)}u(t,\omega)$, $\widetilde{v}(t,\omega) = \mathcal{H}^{-1}(\theta_t\omega)v(t,\omega) = e^{-\epsilon z(\theta_t\omega)}$ $\cdot v(t,\omega)$, $\omega\in\Omega$, $t\in\mathbb{R}$, then (3.3) can be written as the following equivalent random system with random coefficients: for every $\tau\in\mathbb{R}$ and $t>\tau$,

$$\begin{cases} \dfrac{d\widetilde{u}}{dt} = \mathbb{B}(\theta_t\omega)\widetilde{u} + e^{-\epsilon z(\theta_t\omega)}f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}) - \widetilde{v} + e^{-\epsilon z(\theta_t\omega)}g(t) + \epsilon\widetilde{u}z(\theta_t\omega), \\ \dfrac{d\widetilde{v}}{dt} = \sigma\widetilde{u} - \delta\widetilde{v} + e^{-\epsilon z(\theta_t\omega)}h(t) - \epsilon\widetilde{v}z(\theta_t\omega), \\ \widetilde{u}(\tau) = \widetilde{u}_\tau, \quad \widetilde{v}(\tau) = \widetilde{v}_\tau. \end{cases} \tag{3.6}$$

We will consider (3.6) for $\omega\in\widetilde{\Omega}$ and write $\widetilde{\Omega}$ as Ω from now on. In order to obtain the existence and uniqueness of solutions to problem (3.6), we make the following assumptions on g_i, h_i, f_i and the coefficients $\eta_{i,j}(\omega)$, $j\in -q,\dots,0,\dots,q$, for $i\in\mathbb{Z}$:

(A1) Letting $\widehat{\beta}(\omega) = \sup\{|\eta_{i,j}(\omega)| : j\in\{-q,\dots,0,\dots,q\}, q\in\mathbb{N} \text{ and } i\in\mathbb{Z}\}\geq 0$, $\widehat{\beta}(\theta_t\omega)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ with respect to $t\in\mathbb{R}$ for each $\omega\in\Omega$,

$$\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t \widehat{\beta}(\theta_s\omega)\,ds = 0; \tag{3.7}$$

and $\widehat{\beta}(\omega)$ is tempered.

(A2) For some positive constants α, β and κ,

$$f_i(0) = 0, \quad f_i(u)u \leq -\alpha u^2 + \beta, \quad f_i'(u) \leq \kappa, \quad \forall\, i\in\mathbb{Z},\ u\in\mathbb{R}.$$

(A3) $g = (g_i)_{i\in\mathbb{Z}} \in L^2_{\mathrm{loc}}(\mathbb{R}, l^2_\rho)$, $h = (h_i)_{i\in\mathbb{Z}} \in L^2_{\mathrm{loc}}(\mathbb{R}, l^2_\rho)$.

(A4) Let $\lambda = \min\left\{\frac{\alpha}{2}, \delta\right\}$. There exists a positive constant $a \in (0, \lambda)$ such that

$$\int_{-\infty}^{0} e^{as}\left(\|g(s+\tau)\|_\rho^2 + \|h(s+\tau)\|_\rho^2\right) ds < \infty, \quad \forall\, \tau \in \mathbb{R}.$$

3.2. Existence and uniqueness of solutions

In this subsection, we first consider the existence and uniqueness of solutions of (3.6), then define a continuous cocycle for the non-autonomous stochastic LDSs in a weighted space.

We call $v\colon [\tau, \tau+T) \to l^2_\rho$ a mild solution of the following random lattice differential equations

$$\frac{dv}{dt} = G(v, t, \theta_t\omega), \quad v = (v_i)_{i\in\mathbb{Z}}, \quad G = (G_i)_{i\in\mathbb{Z}}, \quad t \in [\tau, \tau+T),\ \tau \in \mathbb{R},$$

where $\omega \in \Omega$, if $v \in C([\tau, \tau+T), l^2_\rho)$ and

$$v_i(t) = v_i(\tau) + \int_\tau^t G_i(v(s), s, \theta_s\omega)\, ds, \quad i \in \mathbb{Z},\ t \in [\tau, \tau+T),\ \tau \in \mathbb{R}.$$

Theorem 3.1. *Let $T > 0$ and* (A1)–(A3) *hold. Then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$ and any initial data $(\widetilde{u}_\tau, \widetilde{v}_\tau) \in l^2_\rho \times l^2_\rho$, problem* (3.6) *admits a unique mild solution $(\widetilde{u}(\cdot, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(\cdot, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau)) \in C([\tau, \tau+T), l^2_\rho \times l^2_\rho)$ with $(\widetilde{u}(\tau, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(\tau, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau)) = (\widetilde{u}_\tau, \widetilde{v}_\tau)$, $(\widetilde{u}(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau))$ being continuous in $(\widetilde{u}_\tau, \widetilde{v}_\tau) \in l^2_\rho \times l^2_\rho$; $(\widetilde{u}(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau)) \in l^2 \times l^2$ if $(\widetilde{u}_\tau, \widetilde{v}_\tau) \in l^2 \times l^2$. Then* (3.6) *generates a continuous cocycle Ψ_ϵ over $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ with state space $l^2_\rho \times l^2_\rho$: for $(\widetilde{u}_\tau, \widetilde{v}_\tau) \in l^2_\rho \times l^2_\rho$, $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$,*

$$\Psi_\epsilon(t, \tau, \omega, \widetilde{u}_\tau, \widetilde{v}_\tau) := (\widetilde{u}(t+\tau, \tau, \theta_{-\tau}\omega, \widetilde{u}_\tau, \widetilde{v}_\tau), \widetilde{v}(t+\tau, \tau, \theta_{-\tau}\omega, \widetilde{u}_\tau, \widetilde{v}_\tau)).$$

Moreover, for $(u_\tau, v_\tau) \in l^2_\rho \times l^2_\rho$, $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega$,

$$\Phi_\epsilon(t, \tau, \omega, u_\tau, v_\tau) = \mathcal{H}(\theta_t\omega)\Psi_\epsilon(t, \tau, \omega, \mathcal{H}^{-1}(\omega)u_\tau, \mathcal{H}^{-1}(\omega)v_\tau),$$

defines a continuous cocycle Φ_ϵ on $l^2_\rho \times l^2_\rho$ over $(\Omega, \mathcal{F}, \mathcal{P}, (\theta_t)_{t\in\mathbb{R}})$ associated with (3.3).

We can prove Theorem 3.1 directly by Theorem 6.1.7 in [28] and Definition 2.1. We omit it here.

Note that the above two cocycles Ψ_ϵ and Φ_ϵ are equivalent. Therefore cocycle Φ_ϵ has a random attractor provided cocycle Ψ_ϵ possesses a random attractor. Then, we only need to consider the cocycle Ψ_ϵ.

3.3. Existence of tempered random bounded absorbing set

In this subsection, we study the existence of tempered random bounded absorbing set for the cocycle Ψ_ϵ in $l^2_\rho \times l^2_\rho$.

Theorem 3.2. *If* (A1)–(A4) *hold, then for every* $\epsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and for any* $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l^2_\rho \times l^2_\rho)$, *there exists* $T = T(\tau,\omega,B,\epsilon) > 0$ *such that for all* $t \geq T$ *and* $(\widetilde{u}_{\tau-t}, \widetilde{v}_{\tau-t}) \in B(\tau - t, \theta_{-t}\omega)$, *the solution* $(\widetilde{u},\widetilde{v})$ *of* (3.6) *satisfies*

$$
\begin{aligned}
&\|\widetilde{u}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\|^2_\rho + \|\widetilde{v}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\|^2_\rho \\
&+ \int_{-t}^0 e^{\int_s^0 \left(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)\right)dr} \|\widetilde{u}(s+\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\|^2_\rho \, ds \\
&\leq I(\epsilon,\tau,\omega),
\end{aligned}
$$

where $I(\epsilon,\tau,\omega) > 0$ *is given by*

$$
\begin{aligned}
I(\epsilon,\tau,\omega) = c + c\int_{-t}^0 & e^{\int_s^0 \left(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)\right)dr - 2\epsilon z(\theta_s\omega)} \\
& \times \left(\|g(s+\tau)\|^2_\rho + \|h(s+\tau)\|^2_\rho\right) ds \\
+ c\sum_{i\in\mathbb{Z}} \rho_i \int_{-\infty}^0 & e^{\int_s^0 \left(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)\right)dr - 2\epsilon z(\theta_s\omega)} \, ds,
\end{aligned}
\tag{3.8}
$$

where c *is a positive constant independent of* τ, ω, B *and* ϵ.

Proof. For each $\omega \in \Omega$, there exists a sequence $\eta^{(m)}_{i,j}(t,\omega)$ of continuous functions in $t \in \mathbb{R}$ (see [1]) such that

$$
\lim_{m\to\infty} \int_\tau^t \left|\eta^{(m)}_{i,j}(s,\omega) - \eta_{i,j}(\theta_s\omega)\right| ds = 0,
$$

$\forall\, t > 0$, $\tau \in \mathbb{R}$, $j \in \{-q,\dots,0,\dots,q\}$, $q \in \mathbb{N}$ and $i \in \mathbb{Z}$, and $\left|\eta^{(m)}_{i,j}(t,\omega)\right| \leq |\eta_{i,j}(\theta_t\omega)| \leq \widehat{\beta}(\theta_t\omega)$ for $t \in \mathbb{R}$.

Consider the following random differential equations:

$$
\begin{cases}
\dfrac{d\widetilde{u}^{(m)}}{dt} = \mathbb{B}_m(t,\omega)\widetilde{u}^{(m)} + e^{-\epsilon z(\theta_t\omega)} f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(m)}) - \widetilde{v}^{(m)} \\
\qquad\qquad + e^{-\epsilon z(\theta_t\omega)} g(t) + \epsilon \widetilde{u}^{(m)} z(\theta_t\omega), \\
\dfrac{d\widetilde{v}^{(m)}}{dt} = \sigma\widetilde{u}^{(m)} - \delta\widetilde{v}^{(m)} + e^{-\epsilon z(\theta_t\omega)} h(t) + \epsilon\widetilde{v}^{(m)} z(\theta_t\omega), \\
\widetilde{u}^{(m)}(\tau) = \widetilde{u}_\tau, \quad \widetilde{v}^{(m)}(\tau) = \widetilde{v}_\tau,
\end{cases}
\tag{3.9}
$$

where $(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)})_i = \sum_{j=-q}^q \eta^{(m)}_{i,j}(t,\omega)\widetilde{u}^{(m)}_{i+j}$. It is easy to see that (3.9) has a unique mild solution $(\widetilde{u}^{(m)}(\cdot,\tau,\omega,\widetilde{u}_\tau,\widetilde{v}_\tau), \widetilde{v}^{(m)}(\cdot,\tau,\omega,\widetilde{u}_\tau,\widetilde{v}_\tau)) \in C([\tau,+\infty), l^2 \times l^2) \cap C^1((\tau,+\infty),$

$l^2 \times l^2$) satisfying (3.9). Taking the inner product of (3.9) with $\widetilde{u}^{(m)}(t)$ and $\widetilde{v}^{(m)}(t)$ respectively in l^2_ρ, we have that

$$\begin{aligned} \frac{d}{dt}\left\|\widetilde{u}^{(m)}\right\|_\rho^2 = {} & 2\left(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)},\widetilde{u}^{(m)}\right)_\rho + 2\left(e^{-\epsilon z(\theta_t\omega)} f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(m)}),\widetilde{u}^{(m)}\right)_\rho \\ & - 2\left(\widetilde{v}^{(m)},\widetilde{u}^{(m)}\right)_\rho + 2\left(e^{-\epsilon z(\theta_t\omega)} g(t),\widetilde{u}^{(m)}\right)_\rho \\ & + 2\epsilon\left(z(\theta_t\omega)\widetilde{u}^{(m)},\widetilde{u}^{(m)}\right)_\rho, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \frac{1}{\sigma}\frac{d}{dt}\left\|\widetilde{v}^{(m)}\right\|_\rho^2 = {} & 2\left(\widetilde{u}^{(m)},\widetilde{v}^{(m)}\right)_\rho - \frac{2\delta}{\sigma}\left\|\widetilde{v}^{(m)}\right\|^2 + \frac{2}{\sigma}\left(e^{-\epsilon z(\theta_t\omega)} h(t),\widetilde{v}^{(m)}\right)_\rho \\ & + \frac{2\epsilon}{\sigma}\left(z(\theta_t\omega)\widetilde{v}^{(m)},\widetilde{v}^{(m)}\right)_\rho . \end{aligned} \tag{3.11}$$

Summing the two equations of (3.10) and (3.11), we find that

$$\begin{aligned} & \frac{d}{dt}\left(\left\|\widetilde{u}^{(m)}\right\|_\rho^2 + \frac{1}{\sigma}\left\|\widetilde{v}^{(m)}\right\|_\rho^2\right) \\ & = 2\left(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)},\widetilde{u}^{(m)}\right)_\rho + 2\left(e^{-\epsilon z(\theta_t\omega)} f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(m)}),\widetilde{u}^{(m)}\right)_\rho \\ & \quad + 2\left(e^{-\epsilon z(\theta_t\omega)} g(t),\widetilde{u}^{(m)}\right)_\rho + \frac{2}{\sigma}\left(e^{-\epsilon z(\theta_t\omega)} h(t),\widetilde{v}^{(m)}\right)_\rho \\ & \quad + 2\epsilon z(\theta_t\omega)\left\|\widetilde{u}^{(m)}\right\|_\rho^2 + \frac{2}{\sigma}(\epsilon z(\theta_t\omega) - \delta)\left\|\widetilde{v}^{(m)}\right\|_\rho^2 . \end{aligned} \tag{3.12}$$

Note that

$$2\left(e^{-\epsilon z(\theta_t\omega)} g(t),\widetilde{u}^{(m)}\right)_\rho \le \frac{e^{-2\epsilon z(\theta_t\omega)}}{\alpha}\|g(t)\|_\rho^2 + \alpha\left\|\widetilde{u}^{(m)}\right\|_\rho^2, \tag{3.13}$$

$$\frac{2}{\sigma}\left(e^{-\epsilon z(\theta_t\omega)} h(t),\widetilde{v}^{(m)}\right)_\rho \le \frac{e^{-2\epsilon z(\theta_t\omega)}}{\delta\sigma}\|h(t)\|_\rho^2 + \frac{\delta}{\sigma}\left\|\widetilde{v}^{(m)}\right\|_\rho^2 . \tag{3.14}$$

By (3.1), we get

$$\begin{aligned} 2\left(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)},\widetilde{u}^{(m)}\right)_\rho & = 2\sum_{i\in\mathbb{Z}}\left(\rho_i\widetilde{u}_i^{(m)}\cdot\sum_{j=-q}^{q}\eta_{i,j}^{(m)}(t,\omega)\widetilde{u}_{i+j}^{(m)}\right) \\ & \le 2\widehat{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\left(\rho_i\left|\widetilde{u}_i^{(m)}\right|\cdot\sum_{j=-q}^{q}\left|\widetilde{u}_{i+j}^{(m)}\right|\right) \\ & \le 2(1+q+\widetilde{q})\widehat{\beta}(\theta_t\omega)\left\|\widetilde{u}^{(m)}\right\|_\rho^2, \end{aligned} \tag{3.15}$$

where $\widetilde{q}=\sum_{k=1}^{q}c_0^k$. By (A2), we have

$$
\begin{aligned}
2\left(e^{-\epsilon z(\theta_t\omega)}f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(m)}),\widetilde{u}^{(m)}\right)_\rho &= 2\sum_{i\in\mathbb{Z}}\rho_i e^{-\epsilon z(\theta_t\omega)}f_i(e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(m)})\cdot\widetilde{u}_i^{(m)}\\
&= 2e^{-2\epsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}}\rho_i f_i(e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(m)})\cdot e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(m)}\\
&\le 2e^{-2\epsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}}\rho_i\left(-\alpha(e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(m)})^2+\beta\right)\\
&\le -2\alpha\left\|\widetilde{u}^{(m)}\right\|_\rho^2+2\beta e^{-2\epsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}}\rho_i.
\end{aligned}
\tag{3.16}
$$

From (3.12)–(3.16), we obtain that for $t>0$,

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\widetilde{u}^{(m)}\right\|_\rho^2+\frac{1}{\sigma}\left\|\widetilde{v}^{(m)}\right\|_\rho^2\right)+\frac{\alpha}{2}\left\|\widetilde{u}^{(m)}\right\|_\rho^2\\
&\le\left(2\epsilon z(\theta_t\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_t\omega)-\frac{\alpha}{2}\right)\left\|\widetilde{u}^{(m)}\right\|_\rho^2+\frac{1}{\sigma}\left(2\epsilon z(\theta_t\omega)-\delta\right)\left\|\widetilde{v}^{(m)}\right\|_\rho^2\\
&\quad+\frac{e^{-2\epsilon z(\theta_t\omega)}}{\alpha}\|g(t)\|_\rho^2+\frac{e^{-2\epsilon z(\theta_t\omega)}}{\delta\sigma}\|h(t)\|_\rho^2+2\beta e^{-2\epsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}}\rho_i.
\end{aligned}
$$

Recalling that $\lambda=\min\left\{\frac{\alpha}{2},\delta\right\}$, then we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\widetilde{u}^{(m)}\right\|_\rho^2+\frac{1}{\sigma}\left\|\widetilde{v}^{(m)}\right\|_\rho^2\right)+\frac{\alpha}{2}\left\|\widetilde{u}^{(m)}\right\|_\rho^2\\
&\le\left(-\lambda+2\epsilon z(\theta_t\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_t\omega)\right)\left(\left\|\widetilde{u}^{(m)}\right\|_\rho^2+\frac{1}{\sigma}\left\|\widetilde{v}^{(m)}\right\|_\rho^2\right)\\
&\quad+\frac{e^{-2\epsilon z(\theta_t\omega)}}{\alpha}\|g(t)\|_\rho^2+\frac{e^{-2\epsilon z(\theta_t\omega)}}{\delta\sigma}\|h(t)\|_\rho^2+2\beta e^{-2\epsilon z(\theta_t\omega)}\sum_{i\in\mathbb{Z}}\rho_i.
\end{aligned}
$$

Then we obtain that for $t>0$,

$$
\begin{aligned}
&\left\|\widetilde{u}^{(m)}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2+\frac{1}{\sigma}\left\|\widetilde{v}^{(m)}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2\\
&\quad+\frac{\alpha}{2}\int_{\tau-t}^{\tau}e^{\int_s^\tau(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr}\left\|\widetilde{u}^{(m)}(s,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2 ds\\
&\le e^{\int_{\tau-t}^\tau(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|_\rho^2+\frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|_\rho^2\right)\\
&\quad+\int_{\tau-t}^{\tau}e^{\int_s^\tau(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr-2\epsilon z(\theta_s\omega)}\left(\frac{1}{\alpha}\|g(s)\|_\rho^2+\frac{1}{\delta\sigma}\|h(s)\|_\rho^2\right)ds\\
&\quad+2\beta\sum_{i\in\mathbb{Z}}\rho_i\int_{\tau-t}^{\tau}e^{\int_s^\tau(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr-2\epsilon z(\theta_s\omega)}\,ds.
\end{aligned}
\tag{3.17}
$$

From (3.17) and by replacing ω by $\theta_{-\tau}\omega$, we have

$$\begin{aligned}
&\left\|\widetilde{u}^{(m)}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2+\frac{1}{\sigma}\left\|\widetilde{v}^{(m)}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2\\
&+\frac{\alpha}{2}\int_{\tau-t}^{\tau}e^{\int_s^\tau(-\lambda+2\epsilon z(\theta_{r-\tau}\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_{r-\tau}\omega))\,dr}\\
&\qquad\times\left\|\widetilde{u}^{(m)}(s,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\right\|_\rho^2\,ds\\
&\le e^{\int_{-t}^0(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|_\rho^2+\frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|_\rho^2\right)\\
&+\int_{-t}^0 e^{\int_s^0(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr-2\epsilon z(\theta_s\omega)}\\
&\qquad\times\left(\frac{1}{\alpha}\|g(s+\tau)\|_\rho^2+\frac{1}{\delta\sigma}\|h(s+\tau)\|_\rho^2\right)ds\\
&+2\beta\sum_{i\in\mathbb{Z}}\rho_i\int_{-t}^0 e^{\int_s^0(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr-2\epsilon z(\theta_s\omega)}\,ds.
\end{aligned}\tag{3.18}$$

Note that (3.18) holds with $\widetilde{u}^{(m)}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})$ and $\widetilde{v}^{(m)}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})$ being replaced by $\widetilde{u}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})$ and $\widetilde{v}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})$, then we have

$$\begin{aligned}
&\|\widetilde{u}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\|_\rho^2+\frac{1}{\sigma}\|\widetilde{v}(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\|_\rho^2\\
&+\frac{\alpha}{2}\int_{\tau-t}^{\tau}e^{\int_s^\tau(-\lambda+2\epsilon z(\theta_{r-\tau}\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_{r-\tau}\omega))\,dr}\\
&\qquad\times\|\widetilde{u}(s,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})\|_\rho^2\,ds\\
&\le e^{\int_{-t}^0(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|_\rho^2+\frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|_\rho^2\right)\\
&+\int_{-t}^0 e^{\int_s^0(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr-2\epsilon z(\theta_s\omega)}\\
&\qquad\times\left(\frac{1}{\alpha}\|g(s+\tau)\|_\rho^2+\frac{1}{\delta\sigma}\|h(s+\tau)\|_\rho^2\right)ds\\
&+2\beta\sum_{i\in\mathbb{Z}}\rho_i\int_{-t}^0 e^{\int_s^0(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr-2\epsilon z(\theta_s\omega)}\,ds.
\end{aligned}\tag{3.19}$$

By (3.5) and (3.7), we find that there exists $T_1=T_1(\omega)>0$ such that for $t>T_1$,

$$\int_{-t}^0\widehat{\beta}(\theta_s\omega)\,ds\le\frac{\lambda}{4(1+q+\widetilde{q})}t,$$

and

$$\int_{-t}^0 z(\theta_s\omega)\,ds\le\frac{\lambda}{8\varepsilon}t.$$

By (A4) and $(\widetilde{u}_{\tau-t}, \widetilde{v}_{\tau-t}) \in B(\tau - t, \theta_{-t}\omega) \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$, we have

$$\begin{aligned}&\lim_{t\to+\infty} e^{\int_{-t}^{0}(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|_\rho^2 + \frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|_\rho^2\right)\\ &\le \limsup_{t\to+\infty} e^{\frac{-\lambda t}{4}}\|B(\tau-t,\theta_{-t}\omega)\|_\rho^2\\ &\le 0.\end{aligned}$$

Therefore, there exists $T_2 = T_2(\tau, \omega, B, \epsilon) > 0$ such that for all $t \ge T_2$,

$$e^{\int_{-t}^{0}(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr}\left(\|\widetilde{u}_{\tau-t}\|_\rho^2 + \frac{1}{\sigma}\|\widetilde{v}_{\tau-t}\|_\rho^2\right) \le 1. \tag{3.20}$$

Note that $z(\theta_t\omega)$ and $\widehat{\beta}(\theta_t\omega)$ are tempered. Then by (A4), we can verify the following integrals are convergent

$$\begin{aligned}&2\int_{-t}^{0} e^{\int_s^0(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr-2\epsilon z(\theta_s\omega)}\\ &\quad\times\left(\frac{1}{\alpha}\|g(s+\tau)\|_\rho^2 + \frac{1}{\delta\sigma}\|h(s+\tau)\|_\rho^2\right)ds\\ &+2\beta\sum_{i\in\mathbb{Z}}\rho_i\int_{-t}^{0} e^{\int_s^0(-\lambda+2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega))\,dr-2\epsilon z(\theta_s\omega)}\,ds\\ &<\infty.\end{aligned} \tag{3.21}$$

Thus the theorem follows from (3.19), (3.20) and (3.21). □

3.4. Existence of random attractor

Theorem 3.3. *Assume that* (A1)–(A4) *hold. Then the continuous cocycle* Ψ_ϵ *associated with* (3.6) *has a unique random attractor* $\mathscr{A}_\epsilon = \{\mathscr{A}_\epsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$.

Proof. By Theorem 2.4, it suffices to prove that for every $\varepsilon > 0$, $\epsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and for any $B = \{B(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(l_\rho^2 \times l_\rho^2)$, there exist $T = T(\tau, \omega, B, \epsilon, \varepsilon) > 0$ and $R = R(\tau, \omega, \epsilon, \varepsilon) > 1$ such that for all $t \ge T$ and $(\widetilde{u}_{\tau-t}, \widetilde{v}_{\tau-t}) \in B(\tau - t, \theta_{-t}\omega)$, the solution $(\widetilde{u}, \widetilde{v})$ of (3.6) satisfies

$$\sum_{|i|>R}\rho_i\left(|\widetilde{u}_i(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})|^2 + |\widetilde{v}_i(\tau,\tau-t,\theta_{-\tau}\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t})|^2\right) \le \varepsilon. \tag{3.22}$$

Choose a smooth increasing function $\chi\colon \mathbb{R}^+ \to [0,1]$ such that

$$\chi(s) = \begin{cases} 0, & 0 \le s \le 1,\\ 1, & s \ge 2,\end{cases}$$

and there exists a positive constant c_χ such that $|\chi'(s)| \le c_\chi$ for $s \in \mathbb{R}^+$.

Let $(\widetilde{u}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t}),\widetilde{v}(\tau,\tau-t,\omega,\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t}))$ be a mild solution of (3.6) with $(\widetilde{u}_{\tau-t},\widetilde{v}_{\tau-t}) \in l^2_\rho \times l^2_\rho$. For any given $N > 0$ define $\mathcal{Q}_N\colon l^2_\rho \times l^2_\rho \to l^2 \times l^2$, $(\widetilde{u},\widetilde{v}) = (\widetilde{u}_i,\widetilde{v}_i)_{i\in\mathbb{Z}} \mapsto \mathcal{Q}_N(\widetilde{u},\widetilde{v}) = ((\mathcal{Q}_N\widetilde{u})_i,(\mathcal{Q}_N\widetilde{v})_i)_{i\in\mathbb{Z}}$ by $((\mathcal{Q}_N\widetilde{u})_j,(\mathcal{Q}_N\widetilde{v})_j) = (\widetilde{u}_j,\widetilde{v}_j)$ if $|j| \le N$ and $((\mathcal{Q}_N\widetilde{u})_j,(\mathcal{Q}_N\widetilde{v})_j) = (0,0)$ otherwise.

For any $n \ge 1$, let $(\widetilde{u}^{(m)},\widetilde{v}^{(m)}) = (\widetilde{u}^{(m)}(\tau,\tau-t,\omega,\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t}),\widetilde{v}^{(m)}(\tau,\tau-t,\omega,$ $\mathcal{Q}_n\widetilde{u}_{\tau-t},\mathcal{Q}_n\widetilde{v}_{\tau-t})) = (\widetilde{u}^{(m)}_i,\widetilde{v}^{(m)}_i)_{i\in\mathbb{Z}}$ be the solution of (3.9). Then taking the inner product $\left(\chi\left(\frac{|i|}{r}\right)\widetilde{u}^{(m)},\chi\left(\frac{|i|}{r}\right)\widetilde{v}^{(m)}\right)$ of (3.9) in $l^2_\rho \times l^2_\rho$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\left(\left|\widetilde{u}^{(m)}_i\right|^2+\frac{1}{\sigma}\left|\widetilde{v}^{(m)}_i\right|^2\right)\\
&= 2\left(\mathcal{B}_m(t,\omega)\widetilde{u}^{(m)},\chi\left(\frac{|i|}{r}\right)\widetilde{u}^{(m)}\right)_\rho + 2\left(e^{-\epsilon z(\theta_t\omega)}f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(m)}),\chi\left(\frac{|i|}{r}\right)\widetilde{u}^{(m)}\right)_\rho\\
&\quad + 2\left(e^{-\epsilon z(\theta_t\omega)}g(t),\chi\left(\frac{|i|}{r}\right)\widetilde{u}^{(m)}\right)_\rho + \frac{2}{\sigma}\left(e^{-\epsilon z(\theta_t\omega)}h(t),\chi\left(\frac{|i|}{r}\right)\widetilde{v}^{(m)}\right)_\rho\\
&\quad + 2\epsilon z(\theta_t\omega)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)(\widetilde{u}^{(m)}_i)^2 + \frac{2}{\sigma}(\epsilon z(\theta_t\omega)-\delta)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)(\widetilde{v}^{(m)}_i)^2.
\end{aligned}
\tag{3.23}
$$

For each term of (3.23), it has been checked that

$$
\begin{aligned}
&2\left(\mathbb{B}_m(t,\omega)\widetilde{u}^{(m)},\chi\left(\frac{|i|}{r}\right)\widetilde{u}^{(m)}\right)_\rho\\
&= 2\sum_{i\in\mathbb{Z}}\left(\rho_i\chi\left(\frac{|i|}{r}\right)\widetilde{u}^{(m)}_i\sum_{j=-q}^{q}\eta^{(m)}_{i,j}(t,\omega)\widetilde{u}^{(m)}_{i+j}\right)\\
&\le \widehat{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\left((1+2q)\left|\widetilde{u}^{(m)}_i\right|^2+\sum_{j=-q}^{q}\left|\widetilde{u}^{(m)}_{i+j}\right|^2\right)\\
&\le \widehat{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\left[\rho_i\sum_{j=-q}^{q}\left(\left(\chi\left(\frac{|i|}{r}\right)-\chi\left(\frac{|i+j|}{r}\right)\right)\left|\widetilde{u}^{(m)}_{i+j}\right|^2+\chi\left(\frac{|i+j|}{r}\right)\left|\widetilde{u}^{(m)}_{i+j}\right|^2\right)\right]\\
&\quad + (1+2q)\widehat{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}^{(m)}_i\right|^2\\
&\le \frac{c_\chi q(1+2\widetilde{q})\widehat{\beta}(\theta_t\omega)}{r}\left\|\widetilde{u}^{(m)}\right\|^2_\rho + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_t\omega)\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}^{(m)}_i\right|^2,
\end{aligned}
$$

$$
\begin{aligned}
&2\left(e^{-\epsilon z(\theta_t\omega)}f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(m)}),\chi\left(\frac{|i|}{r}\right)\widetilde{u}^{(m)}\right)_\rho\\
&= 2\sum_{i\in\mathbb{Z}}\rho_i\chi\left(\frac{|i|}{r}\right)e^{-\epsilon z(\theta_t\omega)}f_i(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(m)}_i)\cdot\widetilde{u}^{(m)}_i
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) e^{-2\epsilon z(\theta_t\omega)} f_i(e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(m)}) \cdot e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(m)} \\
&\le 2e^{-2\epsilon z(\theta_t\omega)} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left(-\alpha\left(e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(m)}\right)^2 + \beta\right) \\
&\le -2\alpha \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left(\widetilde{u}_i^{(m)}\right)^2 + 2\beta e^{-2\epsilon z(\theta_t\omega)} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right),
\end{aligned}
$$

$$
\begin{aligned}
2\left(e^{-\epsilon z(\theta_t\omega)} g(t), \chi\left(\frac{|i|}{r}\right)\widetilde{u}^{(m)}\right)_\rho &= 2\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) e^{-\epsilon z(\theta_t\omega)} g_i(t) \cdot \widetilde{u}_i^{(m)} \\
&\le \frac{e^{-2\epsilon z(\theta_t\omega)}}{\alpha} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) |g_i(t)|^2 \\
&\quad + \alpha \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) \left|\widetilde{u}_i^{(m)}\right|^2,
\end{aligned}
$$

$$
\begin{aligned}
\frac{2}{\sigma}\left(e^{-\epsilon z(\theta_t\omega)} h(t), \chi\left(\frac{|i|}{r}\right)\widetilde{v}^{(m)}\right)_\rho &= \frac{2}{\sigma}\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) e^{-\epsilon z(\theta_t\omega)} h_i(t) \cdot \widetilde{v}_i^{(m)} \\
&\le \frac{e^{-2\epsilon z(\theta_t\omega)}}{\delta\sigma} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) |h_i(t)|^2 \\
&\quad + \frac{\delta}{\sigma} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) \left|\widetilde{v}_i^{(m)}\right|^2.
\end{aligned}
$$

Putting above inequalities into (3.23), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left(\left|\widetilde{u}_i^{(m)}\right|^2 + \frac{1}{\sigma}\left|\widetilde{v}_i^{(m)}\right|^2\right) \\
&\le \left(2\epsilon z(\theta_t\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_t\omega) - \alpha\right) \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{u}_i^{(m)}\right|^2 \\
&\quad + \frac{1}{\sigma}(2\epsilon z(\theta_t\omega) - \delta) \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\left|\widetilde{v}_i^{(m)}\right|^2 + \frac{e^{-2\epsilon z(\theta_t\omega)}}{\alpha} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) |g_i(t)|^2 \\
&\quad + \frac{e^{-2\epsilon z(\theta_t\omega)}}{\delta\sigma} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) |h_i(t)|^2 + \frac{c_\chi q(1+2\widetilde{q})\widehat{\beta}(\theta_t\omega)}{r}\left\|\widetilde{u}^{(m)}\right\|_\rho^2 \\
&\quad + 2\beta e^{-2\epsilon z(\theta_t\omega)} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right).
\end{aligned} \tag{3.24}
$$

Recalling $\lambda = \min\left\{\frac{\alpha}{2}, \delta\right\}$, multiplying (3.24) by $e^{\int_0^t (2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) - \lambda)\,dr}$ and then integrating over $[\tau - t, \tau]$ with $t > 0$, we obtain

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) \\
&\quad \times \left(\left|\widetilde{u}_i^{(m)}(\tau, \tau - t, \omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\right|^2 + \frac{1}{\sigma}\left|\widetilde{v}_i^{(m)}(\tau, \tau - t, \omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\right|^2\right) \\
&\le e^{\int_{\tau-t}^{\tau}(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)-\lambda)\,dr}\left(\|\mathcal{Q}_n\widetilde{u}_{\tau-t}\|_\rho^2 + \frac{1}{\sigma}\|\mathcal{Q}_n\widetilde{v}_{\tau-t}\|_\rho^2\right) \\
&\quad + \frac{c_\chi q(1+2\widetilde{q})}{r}\int_{\tau-t}^{\tau} e^{\int_s^{\tau}(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)-\lambda)\,dr} \\
&\qquad\qquad \times \widehat{\beta}(\theta_s\omega)\left\|\widetilde{u}^{(m)}(s, \tau - t, \omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\right\|_\rho^2 ds \\
&\quad + \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\int_{\tau-t}^{\tau} e^{\int_s^{\tau}(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)-\lambda)\,dr-2\epsilon z(\theta_s\omega)} \\
&\qquad\qquad \times \left(\frac{1}{\alpha}|g_i(s)|^2 + \frac{1}{\delta\sigma}|h_i(s)|^2\right) ds \\
&\quad + 2\beta\int_{\tau-t}^{\tau} e^{\int_s^{\tau}(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)-\lambda)\,dr-2\epsilon z(\theta_s\omega)}\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) ds.
\end{aligned}
$$

Replacing ω in the above by $\theta_{-\tau}\omega$, we obtain that

(3.25)

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) \\
&\quad \times \left(\left|\widetilde{u}_i^{(m)}(\tau, \tau - t, \theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\right|^2 + \frac{1}{\sigma}\left|\widetilde{v}_i^{(m)}(\tau, \tau - t, \theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\right|^2\right) \\
&\le e^{\int_{-t}^{0}(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)-\lambda)\,dr}\left(\|\mathcal{Q}_n\widetilde{u}_{\tau-t}\|_\rho^2 + \frac{1}{\sigma}\|\mathcal{Q}_n\widetilde{v}_{\tau-t}\|_\rho^2\right) \\
&\quad + \frac{c_\chi q(1+2\widetilde{q})}{r}\int_{-t}^{0} e^{\int_s^{0}(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)-\lambda)\,dr} \\
&\qquad\qquad \times \widehat{\beta}(\theta_s\omega)\left\|\widetilde{u}^{(m)}(s+\tau, \tau - t, \theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\right\|_\rho^2 ds \\
&\quad + \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right)\int_{-t}^{0} e^{\int_s^{0}(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)-\lambda)\,dr-2\epsilon z(\theta_s\omega)} \\
&\qquad\qquad \times \left(\frac{1}{\alpha}|g_i(s+\tau)|^2 + \frac{1}{\delta\sigma}|h_i(s+\tau)|^2\right) ds \\
&\quad + 2\beta\int_{-t}^{0} e^{\int_s^{0}(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)-\lambda)\,dr-2\epsilon z(\theta_s\omega)}\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) ds.
\end{aligned}
$$

By substituting m by m_k and letting $k \to \infty$ in (3.25), then we have that

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) \\
&\qquad \times \left(|\widetilde{u}_i(\tau,\tau-t,\theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})|^2 + \frac{1}{\sigma}\, |\widetilde{v}_i(\tau,\tau-t,\theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})|^2 \right) \\
&\le e^{\int_{-t}^0 (2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) - \lambda)\, dr} \left(\|\mathcal{Q}_n\widetilde{u}_{\tau-t}\|_\rho^2 + \frac{1}{\sigma}\, \|\mathcal{Q}_n\widetilde{v}_{\tau-t}\|_\rho^2 \right) \\
&\quad + \frac{c_\chi q(1+2\widetilde{q})}{r} \int_{-t}^0 e^{\int_s^0 (2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) - \lambda)\, dr} \\
&\qquad\qquad \times \widehat{\beta}(\theta_s\omega)\, \|\widetilde{u}(s+\tau,\tau-t,\theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\|_\rho^2\, ds \\
&\quad + \sum_{i\in\mathbb{Z}} \rho_i\chi\left(\frac{|i|}{r}\right) \int_{-t}^0 e^{\int_s^0 (2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) - \lambda)\, dr - 2\epsilon z(\theta_s\omega)} \\
&\qquad\qquad \times \left(\frac{1}{\alpha}|g_i(s+\tau)|^2 + \frac{1}{\delta\sigma}|h_i(s+\tau)|^2 \right) ds \\
&\quad + 2\beta \int_{-t}^0 e^{\int_s^0 (2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) - \lambda)\, dr - 2\epsilon z(\theta_s\omega)} \sum_{i\in\mathbb{Z}} \rho_i\chi\left(\frac{|i|}{r}\right) ds.
\end{aligned} \tag{3.26}
$$

We now estimate each term on the right-hand side of (3.26). For the first term on the right-hand side of (3.26), since $(\mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t}) \in B(\tau-t, \theta_{-t}\omega)$, and B is tempered, then there exists $T_1 = T_1(\tau,\varepsilon,\omega,B) > 0$ such that if $t > T_1$, then

$$
e^{\int_{-t}^0 (2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) - \lambda)\, dr} \left(\|\mathcal{Q}_n\widetilde{u}_{\tau-t}\|_\rho^2 + \frac{1}{\sigma}\, \|\mathcal{Q}_n\widetilde{v}_{\tau-t}\|_\rho^2 \right) \le \varepsilon. \tag{3.27}
$$

For the second term on the right-hand side of (3.26), by (A1) and Theorem 3.2, there exist $T_2 = T_2(\tau,\varepsilon,\omega,B) > 0$ and $R_1 = R_1(\varepsilon,\omega) > 0$ such that for all $t > T_2$ and $r > R_1$,

$$
\begin{aligned}
&\frac{c_\chi q(1+2\widetilde{q})}{r} \int_{-t}^0 e^{\int_s^0 (2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) - \lambda)\, dr} \\
&\qquad\qquad \times \widehat{\beta}(\theta_s\omega)\, \|\widetilde{u}(s+\tau,\tau-t,\theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})\|_\rho^2\, ds \\
&\le \varepsilon.
\end{aligned} \tag{3.28}
$$

For the third term on the right-hand side of (3.26), by (A4), there exist $R_2 = R_2(\varepsilon,\omega) > 0$ and $T_3 = T_3(\varepsilon,\omega) > 0$, such that if $r > R_2$ and $t > T_3$, then

$$
\begin{aligned}
&\sum_{i\in\mathbb{Z}} \rho_i\chi\left(\frac{|i|}{r}\right) \int_{-t}^0 e^{\int_s^0 (2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) - \lambda)\, dr - 2\epsilon z(\theta_s\omega)} \\
&\qquad\qquad \times \left(\frac{1}{\alpha}|g_i(s+\tau)|^2 + \frac{1}{\delta\sigma}|h_i(s+\tau)|^2 \right) ds \\
&\le \varepsilon.
\end{aligned} \tag{3.29}
$$

For the last term on the right-hand side of (3.26), there exist $R_3 = R_3(\varepsilon, \omega) > 0$ and $T_4 = T_4(\varepsilon, \omega) > 0$, such that if $r > R_3$ and $t > T_4$, then

$$2\beta \int_{-t}^{0} e^{\int_s^0 (2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) - \lambda)\, dr - 2\epsilon z(\theta_s\omega)} \sum_{i\in\mathbb{Z}} \rho_i \chi\left(\frac{|i|}{r}\right) ds \le \varepsilon. \tag{3.30}$$

Let $T = \max\{T_1, T_2, T_3, T_4\}$ and $R = 2\max\{R_1, R_2, R_3\}$. Then it follows from (3.27)–(3.30) that, for all $t > T$ and $r > R$, we obtain

$$\begin{aligned}&\sum_{|i|>R} \rho_i \\ &\quad \times \left(|\widetilde{u}_i(\tau, \tau - t, \theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})|^2 + \frac{1}{\sigma} |\widetilde{v}_i(\tau, \tau - t, \theta_{-\tau}\omega, \mathcal{Q}_n\widetilde{u}_{\tau-t}, \mathcal{Q}_n\widetilde{v}_{\tau-t})|^2 \right) \\ &\le 4\varepsilon\end{aligned}$$

for any $n \ge 1$. Let $n \to \infty$, we have that (3.22) holds. This completes the proof. □

3.5. Upper semicontinuity of random attractors

In this subsection, we consider an upper semicontinuity of random attractors $\mathcal{A}_\epsilon(\tau, \omega)$ to $\mathcal{A}_0(\tau, \omega)$ as $\epsilon \to 0$. First let us present a criteria concerning upper semicontinuity of non-autonomous random attractors with respect to a parameter in [38]. Similar results can be found in [20, 29] for deterministic equations and in [10, 34] for autonomous stochastic equations.

Theorem 3.4. *Suppose Φ_ϵ is a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Suppose that*

(i) *Φ_ϵ has a closed measurable random absorbing set $K_\epsilon = \{K_\epsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(X)$ and a unique random attractor $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(X)$.*

(ii) *For each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$K_0(\tau, \omega) = \{u \in X : \|u\|_X \le r_0(\tau, \omega)\}$$

and

$$\limsup_{\epsilon\to 0} \|K_\epsilon(\tau, \omega)\|_X = \limsup_{\epsilon\to 0} \sup_{x\in K_\epsilon(\tau,\omega)} \|x\|_X \le r_0(\tau, \omega),$$

where $r_0(\tau, \omega)$ is a positive valued tempered random variable.

(iii) *There exists $\epsilon_0 > 0$ such that, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,*

$$\bigcup_{|\epsilon|\le\epsilon_0} \mathcal{A}_\epsilon(\tau, \omega) \quad \textit{is precompact in } X.$$

(iv) *For* $t > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $\epsilon_n \to 0$ *when* $n \to \infty$, *and* $x_n, x_0 \in X$ *with* $x_n \to x_0$ *when* $n \to \infty$, *it holds:*

$$\lim_{n\to\infty} \Phi_{\epsilon_n}(t, \tau, \omega, x_n) = \Phi_0(t, \tau, \omega, x_0).$$

Then for $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$,

$$\mathrm{dist}_X(\mathcal{A}_\epsilon(\tau,\omega), \mathcal{A}_0(\tau,\omega)) = \sup_{u\in\mathcal{A}_\epsilon(\tau,\omega)} \inf_{v\in\mathcal{A}_0(\tau,\omega)} \|u - v\|_X \to 0 \quad \text{as } \epsilon \to 0.$$

The main result of this subsection is as follows. Let $I(\epsilon, \tau, \omega)$ be as in Theorem 2.4. We have

Theorem 3.5. *Assume that* (A1)–(A4) *hold. Then for every* $\tau \in \mathbb{R}$ *and* $\omega \in \Omega$, *we have*

$$\mathrm{dist}_{l_\rho^2\times l_\rho^2}(\mathcal{A}_\epsilon(\tau,\omega), \mathcal{A}_0(\tau,\omega)) = \sup_{(\widetilde{u},\widetilde{v})\in\mathcal{A}_\epsilon(\tau,\omega)} \inf_{(u,v)\in\mathcal{A}_0(\tau,\omega)} \left(\|\widetilde{u} - u\|_\rho^2 + \|\widetilde{v} - v\|_\rho^2\right)^{\frac{1}{2}}$$
$$\to 0 \quad \text{as } \epsilon \to 0.$$

Proof. The proof is based on Theorem 3.4. Let us check that Φ_ϵ satisfies the conditions (i)–(iv) in Theorem 3.4 one by one.

(i) By Theorems 3.2 and 3.3, Φ_ϵ has a closed measurable random absorbing set $B_\epsilon = \{B_\epsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(l_\rho^2 \times l_\rho^2)$,

$$B_\epsilon(\tau,\omega) = \left\{(\widetilde{u}, \widetilde{v}) \in l_\rho^2 \times l_\rho^2 : \|\widetilde{u}\|_\rho^2 + \|\widetilde{v}\|_\rho^2 \le I(\epsilon, \tau, \omega)\right\}$$

and a unique random attractor $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $\mathcal{D}(l_\rho^2 \times l_\rho^2)$, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\mathcal{A}_\epsilon(\tau,\omega) \subseteq B_\epsilon(\tau,\omega).$$

(ii) Given $|\epsilon| < 1$. By (3.8),

$$I(\epsilon, \tau, \omega) \le I(1, \tau, \omega) < \infty,$$

and

$$\limsup_{\epsilon\to 0} I(\epsilon, \tau, \omega) \le I(1, \tau, \omega).$$

So, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\limsup_{\epsilon\to 0} \|B_\epsilon(\tau,\omega)\|_\rho = \limsup_{\epsilon\to 0} \sup_{x\in B_\epsilon(\tau,\omega)} \|x\|_{l_\rho^2\times l_\rho^2} \le I^{\frac{1}{2}}(1, \tau, \omega).$$

Moreover,

$$B_0(\tau,\omega) = \left\{(u, v) \in l_\rho^2 \times l_\rho^2 : \|u\|_\rho^2 + \|v\|_\rho^2 \le I(1, \tau, \omega)\right\}$$

is a closed tempered random absorbing set for the continuous cocycle Φ_0 associated with the limiting system

$$
\begin{cases} du = (\mathbb{B}(\theta_t\omega)u + f(u) - v + g(t))\, dt, \\ dv = (\sigma u - \delta v + h(t))\, dt, \\ u(\tau) = u_\tau, \quad v(\tau) = v_\tau, \end{cases} \tag{3.31}
$$

and

$$
\bigcup_{|\epsilon|\leq 1} \mathcal{A}_\epsilon(\tau,\omega) \subseteq \bigcup_{|\epsilon|\leq 1} B_\epsilon(\tau,\omega) \subseteq B_0(\tau,\omega). \tag{3.32}
$$

(iii) Given $|\epsilon| < 1$. Let us prove the precompactness of $\bigcup_{|\epsilon|\leq 1} \mathcal{A}_\epsilon(\tau,\omega)$ for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, which means that for any $\varepsilon > 0$, the set $\bigcup_{|\epsilon|\leq 1} \mathcal{A}_\epsilon(\tau,\omega)$ has a finite covering of balls of radius $\varepsilon > 0$ in $l^2_\rho \times l^2_\rho$. By Theorem 3.3, for every $\varepsilon > 0$, $\epsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exist $T = T(\tau,\omega,B,\epsilon,\varepsilon) > 0$ and $R = R(\tau,\omega,\epsilon,\varepsilon) > 1$ such that for all $t \geq T$ and $(\widetilde{u}^{(\epsilon)}_{\tau-t}, \widetilde{v}^{(\epsilon)}_{\tau-t}) \in B_0(\tau - t, \theta_{-t}\omega)$, the solution $(\widetilde{u}^{(\epsilon)}, \widetilde{v}^{(\epsilon)})$ of (3.6) satisfies

$$
\sum_{|i|>R} \rho_i \left(\left| \widetilde{u}^{(\epsilon)}_i(\tau,\tau-t,\theta_{-\tau}\omega, \widetilde{u}^{(\epsilon)}_{\tau-t}, \widetilde{v}^{(\epsilon)}_{\tau-t}) \right|^2 + \left| \widetilde{v}^{(\epsilon)}_i(\tau,\tau-t,\theta_{-\tau}\omega, \widetilde{u}^{(\epsilon)}_{\tau-t}, \widetilde{v}^{(\epsilon)}_{\tau-t}) \right|^2 \right) \leq \varepsilon,
$$

which along with (3.32) and the invariance of $\mathcal{A}_\epsilon(\tau,\omega)$, we have for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t \geq T$

$$
\sup_{(\widetilde{u}^{(\epsilon)},\widetilde{v}^{(\epsilon)}) \in \bigcup_{|\epsilon|\leq 1} \mathcal{A}_\epsilon(\tau,\omega)} \sum_{|i|>R} \rho_i \left(\left| \widetilde{u}^{(\epsilon)}_i(\tau,\tau-t,\theta_{-\tau}\omega, \widetilde{u}^{(\epsilon)}_{\tau-t}, \widetilde{v}^{(\epsilon)}_{\tau-t}) \right|^2 + \left| \widetilde{v}^{(\epsilon)}_i(\tau,\tau-t,\theta_{-\tau}\omega, \widetilde{u}^{(\epsilon)}_{\tau-t}, \widetilde{v}^{(\epsilon)}_{\tau-t}) \right|^2 \right) \leq \varepsilon.
$$

On the other hand, by (3.32) we find that the set

$$
\left\{ (\widetilde{u}^{(\epsilon)}_i, \widetilde{v}^{(\epsilon)}_i)_{|i|\leq R} : (\widetilde{u}^{(\epsilon)}, \widetilde{v}^{(\epsilon)}) \in \bigcup_{|\epsilon|\leq 1} \mathcal{A}_\epsilon(\tau,\omega) \right\}
$$

is bounded in a finite-dimensional space and hence $\bigcup_{|\epsilon|\leq 1} \mathcal{A}_\epsilon(\tau,\omega)$ is precompact in $l^2_\rho \times l^2_\rho$.

(iv) Let $\widetilde{\varphi}^{(\epsilon)} = (\widetilde{u}^{(\epsilon)}, \widetilde{v}^{(\epsilon)})$ and $\varphi = (u,v)$ be mild solutions of (3.6) and (3.31) with initial data $(\widetilde{u}^{(\epsilon)}_\tau, \widetilde{v}^{(\epsilon)}_\tau)$ and (u_τ, v_τ), respectively, and set $U = \widetilde{u}^{(\epsilon)} - u$, $V = \widetilde{v}^{(\epsilon)} - v$, $W = (U,V) = \widetilde{\varphi}^{(\epsilon)} - \varphi$. Let $(\widetilde{u}^{(\epsilon)}_\tau, \widetilde{v}^{(\epsilon)}_\tau), (u_\tau, v_\tau) \in l^2_\rho \times l^2_\rho$, then it follows from Theorem 3.1 that $\widetilde{\varphi}^{(\epsilon)}(\cdot,\tau,\omega,\widetilde{u}^{(\epsilon)}_\tau,\widetilde{v}^{(\epsilon)}_\tau), \varphi(\cdot,\tau,\omega,u_\tau,v_\tau), W(\cdot,\tau,\omega,u^{(\epsilon)}_\tau,v^{(\epsilon)}_\tau,u_\tau,v_\tau) \in C([\tau,+\infty), l^2_\rho \times l^2_\rho)$.

Let

$$
\widetilde{\varphi}^{(\epsilon,m)}(t,\tau,\omega,\widetilde{u}^{(\epsilon)}_\tau,\widetilde{v}^{(\epsilon)}_\tau) = (\widetilde{u}^{(\epsilon,m)}(t,\tau,\omega,\widetilde{u}^{(\epsilon)}_\tau,\widetilde{v}^{(\epsilon)}_\tau), \widetilde{v}^{(\epsilon,m)}(t,\tau,\omega,\widetilde{u}^{(\epsilon)}_\tau,\widetilde{v}^{(\epsilon)}_\tau))
$$

and

$$\varphi^{(m)}(t,\tau,\omega,u_\tau,v_\tau)=(u^{(m)}(t,\tau,\omega,u_\tau,v_\tau),v^{(m)}(t,\tau,\omega,u_\tau,v_\tau))$$

be the solutions of the following random differential equations with initial data

$$\begin{cases}\dfrac{d\widetilde{u}^{(\epsilon,m)}}{dt}=\mathbb{B}_m(t,\omega)\widetilde{u}^{(\epsilon,m)}+e^{-\epsilon z(\theta_t\omega)}f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(\epsilon,m)})-\widetilde{v}^{(\epsilon,m)}\\ \qquad\qquad +e^{-\epsilon z(\theta_t\omega)}g(t)+\epsilon\widetilde{u}^{(\epsilon,m)}z(\theta_t\omega),\\ \dfrac{d\widetilde{v}^{(\epsilon,m)}}{dt}=\sigma\widetilde{u}^{(\epsilon,m)}-\delta\widetilde{v}^{(\epsilon,m)}+e^{-\epsilon z(\theta_t\omega)}h(t)+\epsilon\widetilde{v}^{(\epsilon,m)}z(\theta_t\omega),\\ \widetilde{u}^{(\epsilon,m)}(\tau)=\widetilde{u}_\tau^{(\epsilon)},\quad \widetilde{v}^{(\epsilon,m)}(\tau)=\widetilde{v}_\tau^{(\epsilon)},\end{cases}\tag{3.33}$$

and

$$\begin{cases}\dfrac{du^{(m)}}{dt}=\mathbb{B}_m(t,\omega)u^{(m)}+f(u^{(m)})-v^{(m)}+g(t),\\ \dfrac{dv^{(m)}}{dt}=\sigma u^{(m)}-\delta v^{(m)}+h(t),\\ u^{(m)}(\tau)=u_\tau,\quad v^{(m)}(\tau)=v_\tau,\end{cases}\tag{3.34}$$

respectively. Then, $\widetilde{\varphi}^{(\epsilon,m)}(\cdot,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)}),\varphi^{(m)}(\cdot,\tau,\omega,u_\tau,v_\tau)\in C([\tau,+\infty),l_\rho^2\times l_\rho^2)$ and satisfy the differential equations (3.33) and (3.34) respectively. Moreover, $\widetilde{\varphi}^{(\epsilon)}(\cdot,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)})$ and $\varphi(\cdot,\tau,\omega,u_\tau,v_\tau)\in C([\tau,+\infty),l_\rho^2\times l_\rho^2)$ are limit functions of subsequences of $\left\{\widetilde{\varphi}^{(\epsilon,m)}(\cdot,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)})\right\}$ and $\{\varphi^{(m)}(\cdot,\tau,\omega,u_\tau,v_\tau)\}\in C([\tau,+\infty),l_\rho^2\times l_\rho^2)$. So $W(t,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)},u_\tau,v_\tau)$ is a limit function of a subsequence of

$$\left\{W^{(m)}(t,\tau,\omega,\widetilde{u}^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)},u_\tau,v_\tau)=\widetilde{\varphi}^{(\epsilon,m)}(t,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)})-\varphi^{(m)}(t,\tau,\omega,u_\tau,v_\tau)\right\}$$

in $l_\rho^2\times l_\rho^2$, and $W^{(m)}(t,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)},u_\tau,v_\tau)=(U^{(m)}(t,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)},u_\tau,v_\tau),V^{(m)}(t,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)},u_\tau,v_\tau))$ satisfies

$$\begin{cases}\dfrac{dU^{(m)}}{dt}=\mathbb{B}_m(t,\omega)U^{(m)}+e^{-\epsilon z(\theta_t\omega)}f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(\epsilon,m)})-f(u^{(m)})-V^{(m)}\\ \qquad\qquad +e^{-\epsilon z(\theta_t\omega)}g(t)-g(t)+\epsilon z(\theta_t\omega)\widetilde{u}^{(\epsilon,m)},\\ \dfrac{dV^{(m)}}{dt}=\sigma U^{(m)}-\delta V^{(m)}+e^{-\epsilon z(\theta_t\omega)}h(t)-h(t)+\epsilon z(\theta_t\omega)\widetilde{v}^{(\epsilon,m)},\\ U^{(m)}(\tau)=\widetilde{u}_\tau^{(\epsilon)}-u_\tau,\quad V^{(m)}(\tau)=\widetilde{v}_\tau^{(\epsilon)}-v_\tau.\end{cases}\tag{3.35}$$

Taking the inner product of (3.35) with $(U^{(m)},V^{(m)})$ in $l_\rho^2\times l_\rho^2$, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|U^{(m)}\right\|_\rho^2+\frac{1}{\sigma}\left\|V^{(m)}\right\|_\rho^2\right)\\ &=\left(\mathbb{B}_m(t,\omega)U^{(m)},U^{(m)}\right)_\rho+\left(e^{-\epsilon z(\theta_t\omega)}f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(\epsilon,m)})-f(u^{(m)}),U^{(m)}\right)_\rho\\ &\quad+\left(e^{-\epsilon z(\theta_t\omega)}g(t)-g(t),U^{(m)}\right)_\rho+\left(\epsilon z(\theta_t\omega)\widetilde{u}^{(\epsilon,m)},U^{(m)}\right)_\rho\\ &\quad-\frac{\delta}{\sigma}\left\|V^{(m)}\right\|_\rho^2+\frac{1}{\sigma}\left(e^{-\epsilon z(\theta_t\omega)}h(t)-h(t),V^{(m)}\right)_\rho+\left(\frac{\epsilon}{\sigma}z(\theta_t\omega)\widetilde{v}^{(\epsilon,m)},V^{(m)}\right)_\rho.\end{aligned}\tag{3.36}$$

We now estimate the terms in (3.36):

$$\begin{aligned}\left(\mathbb{B}_m(t,\omega)U^{(m)},U^{(m)}\right)_\rho &= \sum_{i\in\mathbb{Z}}\left(\rho_i U_i^{(m)}\sum_{j=-q}^{q}\eta_{i,j}^{(m)}(t,\omega)U_{i+j}^{(m)}\right)\\ &\le (1+q+\widetilde{q})\widehat{\beta}(\theta_t\omega)\left\|U^{(m)}\right\|_\rho^2,\end{aligned}\tag{3.37}$$

$$\left(e^{-\epsilon z(\theta_t\omega)}g(t)-g(t),U^{(m)}\right)_\rho \le \frac{2(e^{-\epsilon z(\theta_t\omega)}-1)^2}{\kappa}\|g(t)\|_\rho^2+\frac{\kappa}{8}\left\|U^{(m)}\right\|_\rho^2,\tag{3.38}$$

$$\frac{1}{\sigma}\left(e^{-\epsilon z(\theta_t\omega)}h(t)-h(t),V^{(m)}\right)_\rho \le \frac{(e^{-\epsilon z(\theta_t\omega)}-1)^2}{2\delta\sigma}\|h(t)\|_\rho^2+\frac{\delta}{2\sigma}\left\|V^{(m)}\right\|_\rho^2,\tag{3.39}$$

$$\begin{aligned}\left(\epsilon z(\theta_t\omega)\widetilde{u}^{(\epsilon,m)},U^{(m)}\right)_\rho &= \sum_{i\in\mathbb{Z}}\epsilon\rho_i z(\theta_t\omega)\widetilde{u}_i^{(\epsilon,m)}\cdot U_i^{(m)}\\ &= \sum_{i\in\mathbb{Z}}\epsilon\rho_i z(\theta_t\omega)\left(U_i^{(m)}-u_i^{(m)}\right)\cdot U_i^{(m)}\\ &= \sum_{i\in\mathbb{Z}}\epsilon\rho_i z(\theta_t\omega)\left(\left|U_i^{(m)}\right|^2-u_i^{(m)}\cdot U_i^{(m)}\right)\\ &\le \left(\epsilon z(\theta_t\omega)+\frac{\kappa}{8}\right)\left\|U^{(m)}\right\|_\rho^2+\frac{2|\epsilon z(\theta_t\omega)|^2}{\kappa}\left\|u^{(m)}\right\|_\rho^2,\end{aligned}\tag{3.40}$$

$$\begin{aligned}\left(\frac{\epsilon}{\sigma}z(\theta_t\omega)\widetilde{v}^{(\epsilon,m)},V^{(m)}\right)_\rho &= \sum_{i\in\mathbb{Z}}\frac{\epsilon}{\sigma}\rho_i z(\theta_t\omega)\widetilde{v}_i^{(\epsilon,m)}\cdot V_i^{(m)}\\ &= \sum_{i\in\mathbb{Z}}\frac{\epsilon}{\sigma}\rho_i z(\theta_t\omega)\left(V_i^{(m)}-v_i^{(m)}\right)\cdot V_i^{(m)}\\ &= \sum_{i\in\mathbb{Z}}\frac{\epsilon}{\sigma}\rho_i z(\theta_t\omega)\left(\left|V_i^{(m)}\right|^2-v_i^{(m)}\cdot V_i^{(m)}\right)\\ &\le \frac{1}{\sigma}\left(\epsilon z(\theta_t\omega)+\frac{\delta}{2}\right)\left\|V^{(m)}\right\|_\rho^2+\frac{|\epsilon z(\theta_t\omega)|^2}{2\delta\sigma}\left\|v^{(m)}\right\|_\rho^2,\end{aligned}\tag{3.41}$$

$$\begin{aligned}&\left(e^{-\epsilon z(\theta_t\omega)}f(e^{\epsilon z(\theta_t\omega)}\widetilde{u}^{(\epsilon,m)})-f(u^{(m)}),U^{(m)}\right)_\rho\\ &= \sum_{i\in\mathbb{Z}}\rho_i\left(e^{-\epsilon z(\theta_t\omega)}f_i(e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(\epsilon,m)})-f_i(u_i^{(m)})\right)\cdot U_i^{(m)}\\ &\le \sum_{i\in\mathbb{Z}}\rho_i e^{-\epsilon z(\theta_t\omega)}f_i(e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(\epsilon,m)})\cdot U_i^{(m)}-\sum_{i\in\mathbb{Z}}\rho_i e^{-\epsilon z(\theta_t\omega)}f_i(e^{\epsilon z(\theta_t\omega)}u_i^{(m)})\cdot U_i^{(m)}\\ &\quad -\sum_{i\in\mathbb{Z}}\rho_i f_i(u_i^{(m)})\cdot U_i^{(m)}+\sum_{i\in\mathbb{Z}}\rho_i e^{-\epsilon z(\theta_t\omega)}f_i(u_i^{(m)})\cdot U_i^{(m)}\\ &\quad +\sum_{i\in\mathbb{Z}}\rho_i e^{-\epsilon z(\theta_t\omega)}f_i(e^{\epsilon z(\theta_t\omega)}u_i^{(m)})\cdot U_i^{(m)}-\sum_{i\in\mathbb{Z}}\rho_i e^{-\epsilon z(\theta_t\omega)}f_i(u_i^{(m)})\cdot U_i^{(m)}\\ &\le \sum_{i\in\mathbb{Z}}\rho_i e^{-\epsilon z(\theta_t\omega)}\left|f_i(e^{\epsilon z(\theta_t\omega)}\widetilde{u}_i^{(\epsilon,m)})-f_i(e^{\epsilon z(\theta_t\omega)}u_i^{(m)})\right|\cdot\left|U_i^{(m)}\right|\end{aligned}$$

$$
\begin{aligned}
(3.42)\qquad & + \sum_{i\in\mathbb{Z}} \rho_i \left| f_i(u_i^{(m)}) - e^{-\epsilon z(\theta_t\omega)} f_i(u_i^{(m)}) \right| \cdot \left| U_i^{(m)} \right| \\
& + \sum_{i\in\mathbb{Z}} \rho_i e^{-\epsilon z(\theta_t\omega)} \left| f_i(e^{\epsilon z(\theta_t\omega)} u_i^{(m)}) - f_i(u_i^{(m)}) \right| \cdot \left| U_i^{(m)} \right| \\
\leq & \sum_{i\in\mathbb{Z}} \kappa\rho_i \left| \widetilde{u}_i^{(\epsilon,m)} - u_i^{(m)} \right| \cdot \left| U_i^{(m)} \right| \\
& + \sum_{i\in\mathbb{Z}} \rho_i \left| 1 - e^{-\epsilon z(\theta_t\omega)} \right| \cdot \left| f_i(u_i^{(m)}) - f_i(0) \right| \cdot \left| U_i^{(m)} \right| \\
& + \sum_{i\in\mathbb{Z}} \kappa\rho_i e^{-\epsilon z(\theta_t\omega)} \left| 1 - e^{\epsilon z(\theta_t\omega)} \right| \cdot \left| u_i^{(m)} \right| \cdot \left| U_i^{(m)} \right| \\
\leq & \sum_{i\in\mathbb{Z}} \kappa\rho_i \left| U_i^{(m)} \right|^2 + \sum_{i\in\mathbb{Z}} \kappa\rho_i (1 + e^{-\epsilon z(\theta_t\omega)}) \left| 1 - e^{\epsilon z(\theta_t\omega)} \right| \cdot \left| u_i^{(m)} \right| \cdot \left| U_i^{(m)} \right| \\
\leq & \frac{5\kappa}{4} \left\| U^{(m)} \right\|_\rho^2 + \kappa(1 + e^{-\epsilon z(\theta_t\omega)})^2 \left| 1 - e^{\epsilon z(\theta_t\omega)} \right|^2 \cdot \left\| u^{(m)} \right\|_\rho^2 .
\end{aligned}
$$

Then it follows from (3.37)–(3.42) that

$$
\begin{aligned}
& \frac{d}{dt} \left(\left\| U^{(m)} \right\|_\rho^2 + \frac{1}{\sigma} \left\| V^{(m)} \right\|_\rho^2 \right) \\
& \leq \left(2\epsilon z(\theta_t\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_t\omega) + 3\kappa \right) \left\| U^{(m)} \right\|_\rho^2 + \frac{2\epsilon z(\theta_t\omega)}{\sigma} \left\| V^{(m)} \right\|_\rho^2 \\
(3.43)\qquad & \quad + \frac{|\epsilon z(\theta_t\omega)|^2}{\delta\sigma} \left\| v^{(m)} \right\|_\rho^2 + \left(e^{-\epsilon z(\theta_t\omega)} - 1 \right)^2 \left(\frac{4}{\kappa} \|g(t)\|_\rho^2 + \frac{1}{\delta\sigma} \|h(t)\|_\rho^2 \right) \\
& \quad + \left(\frac{4|\epsilon z(\theta_t\omega)|^2}{\kappa} + \kappa \left(1 + e^{-\epsilon z(\theta_t\omega)} \right)^2 \left| 1 - e^{\epsilon z(\theta_t\omega)} \right|^2 \right) \left\| u^{(m)} \right\|_\rho^2 .
\end{aligned}
$$

By applying Gronwall's inequality to (3.43) from τ to $t+\tau$, we have

$$
\begin{aligned}
& \left\| U^{(m)}(t+\tau,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)},u_\tau,v_\tau) \right\|_\rho^2 + \frac{1}{\sigma} \left\| V^{(m)}(t+\tau,\tau,\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)},u_\tau,v_\tau) \right\|_\rho^2 \\
\leq & e^{\int_\tau^{t+\tau} \left(2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) + 3\kappa \right) dr} \left(\left\| \widetilde{u}_\tau^{(\epsilon)} - u_\tau \right\|_\rho^2 + \frac{1}{\sigma} \left\| \widetilde{v}_\tau^{(\epsilon)} - v_\tau \right\|_\rho^2 \right) \\
& + \int_\tau^{t+\tau} e^{\int_s^{t+\tau} \left(2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) + 3\kappa \right) dr} \frac{|\epsilon z(\theta_s\omega)|^2}{\delta\sigma} \left\| v^{(m)}(s,\tau,\omega,u_\tau,v_\tau) \right\|_\rho^2 ds \\
& + \int_\tau^{t+\tau} e^{\int_s^{t+\tau} \left(2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) + 3\kappa \right) dr} \left(e^{-\epsilon z(\theta_s\omega)} - 1 \right)^2 \\
& \quad \times \left(\frac{4}{\kappa} \|g(s)\|_\rho^2 + \frac{1}{\delta\sigma} \|h(s)\|_\rho^2 \right) ds \\
& + \int_\tau^{t+\tau} e^{\int_s^{t+\tau} \left(2\epsilon z(\theta_r\omega) + 2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega) + 3\kappa \right) dr} \\
& \quad \times \left(\frac{4|\epsilon z(\theta_s\omega)|^2}{\kappa} + \kappa(1 + e^{-\epsilon z(\theta_s\omega)})^2 \left| 1 - e^{\epsilon z(\theta_s\omega)} \right|^2 \right) \left\| u^{(m)}(s,\tau,\omega,u_\tau,v_\tau) \right\|_\rho^2 ds.
\end{aligned}
$$

We now replace ω in the above by $\theta_{-\tau}\omega$ to yield

(3.44)

$$\begin{aligned}
&\left\|U^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)},u_\tau,v_\tau)\right\|_\rho^2+\frac{1}{\sigma}\left\|V^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)},u_\tau,v_\tau)\right\|_\rho^2\\
&\le e^{\int_0^t\left(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)+3\kappa\right)dr}\left(\left\|\widetilde{u}_\tau^{(\epsilon)}-u_\tau\right\|_\rho^2+\frac{1}{\sigma}\left\|\widetilde{v}_\tau^{(\epsilon)}-v_\tau\right\|_\rho^2\right)\\
&\quad+\int_0^t e^{\int_s^t\left(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)+3\kappa\right)dr}\frac{|\epsilon z(\theta_s\omega)|^2}{\delta\sigma}\left\|v^{(m)}(s+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\right\|_\rho^2ds\\
&\quad+\int_0^t e^{\int_s^t\left(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)+3\kappa\right)dr}\left(e^{-\epsilon z(\theta_s\omega)}-1\right)^2\\
&\quad\times\left(\frac{4}{\alpha}\|g(s+\tau)\|_\rho^2+\frac{1}{\delta\sigma}\|h(s+\tau)\|_\rho^2\right)ds\\
&\quad+\int_0^t e^{\int_s^t\left(2\epsilon z(\theta_r\omega)+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)+3\kappa\right)dr}\\
&\quad\times\left(\frac{4|\epsilon z(\theta_s\omega)|^2}{\kappa}+\kappa\left(1+e^{-\epsilon z(\theta_s\omega)}\right)^2\left|1-e^{\epsilon z(\theta_s\omega)}\right|^2\right)\left\|u^{(m)}(s+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\right\|_\rho^2ds.
\end{aligned}$$

By (3.34), we have that

$$\begin{aligned}
&\left\|u^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\right\|_\rho^2+\left\|v^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\right\|_\rho^2\\
&\le c+c\int_0^t e^{\int_s^t\left(-\lambda+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)\right)dr}\left(\|g(s+\tau)\|_\rho^2+\|h(s+\tau)\|_\rho^2\right)ds\\
&\quad+c\sum_{i\in\mathbb{Z}}\rho_i\int_0^t e^{\int_s^t\left(-\lambda+2(1+q+\widetilde{q})\widehat{\beta}(\theta_r\omega)\right)dr}ds\\
&\le\mathcal{I}(t,\omega),
\end{aligned}\tag{3.45}$$

where $\mathcal{I}(t,\omega)$ is a positive-valued and continuous in t but independent of ϵ.

From (3.44) and (3.45), we see that for $\tau\in\mathbb{R}$, $t\in\mathbb{R}^+$, $\omega\in\Omega$, $\epsilon\to 0$ and $(\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)})$, $(u_\tau,v_\tau)\in l_\rho^2\times l_\rho^2$ with $(\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)})\to(u_\tau,v_\tau)$,

$$\lim_{\epsilon\to 0}\widetilde{\varphi}^{(\epsilon,m)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)})=\varphi^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\quad\text{in } l_\rho^2\times l_\rho^2.$$

Let $\{\epsilon_n\}\subset[-1,1]$ be a sequence of numbers with $\epsilon_n\to 0$ when $n\to+\infty$. Thus $\widetilde{\varphi}^{(\epsilon)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)})$ and $\varphi(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)$ being limit functions of subsequences of $\left\{\widetilde{\varphi}^{(\epsilon,m)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(\epsilon)},\widetilde{v}_\tau^{(\epsilon)})\right\}$ and $\left\{\varphi^{(m)}(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\right\}$ in $l_\rho^2\times l_\rho^2$ imply that for $\tau\in\mathbb{R}$, $t\in\mathbb{R}^+$, $\omega\in\Omega$, $\epsilon_n\to 0$ and $(\widetilde{u}_\tau^{(\epsilon_n)},\widetilde{v}_\tau^{(\epsilon_n)}),(u_\tau,v_\tau)\in l_\rho^2\times l_\rho^2$ with $(\widetilde{u}_\tau^{(\epsilon_n)},\widetilde{v}_\tau^{(\epsilon_n)})\to(u_\tau,v_\tau)$, the following holds:

$$\lim_{n\to\infty}\widetilde{\varphi}^{(\epsilon_n)}(t+\tau,\tau,\theta_{-\tau}\omega,\widetilde{u}_\tau^{(\epsilon_n)},\widetilde{v}_\tau^{(\epsilon_n)})=\varphi(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau,v_\tau)\quad\text{in } l_\rho^2\times l_\rho^2.$$

The proof is completed. □

4. Remarks

If the coupled linear operator $\mathbb{B}(\omega)$ in (3.4) is a constant operator $\mathbb{B}$ satisfying $-\mathbb{B} = D^*D = DD^*$, where $(Du)_i = \sum_{j=-m}^{j=m} d_j u_{i+j}$, $\forall\, u = (u_i)_{i\in\mathbb{Z}}$, $|d_j| \le c$ (constant), $-m \le j \le m$ (constant), and D^* is the adjoint of D, then the condition (A1) can be removed.

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