TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 3, pp. 509–521, June 2016
DOI: 10.11650/tjm.20.2016.6422
This paper is available online at `http://journal.tms.org.tw`

On Gated Sets in Graphs

Manoj Changat, Iztok Peterin* and Abisha Ramachandran

Abstract. A subset K of vertices of a graph G is gated if for every vertex $x \in V(G)$ there exists a gate $v \in K$ which is on a shortest path between x and any vertex u of K. We give a characterization of gated sets in an arbitrary graph G and several necessary conditions. This characterization yields very nice results in the case of weakly modular graphs, which are also presented. We also show that the trees are precisely the graphs which present a convex geometry with respect to the gated convexity.

1. Introduction

The notion of the gate in a graph, which plays an important role in metric graph theory, was first introduced by Goldman and Witzgall [16]. General properties of gated sets are that every gated set is also geodesic convex (see [14]), that a map which maps a vertex to its gate in a gated set is a weak retraction (see Lemma 16.2 in [17]), that the intersection of two gated sets yields a gated set again (see Lemma 16.3 in [17]) and that the family of gated sets has the Helly property (see Corollary 16.3 in [17]). However the majority of results about gated sets is concerned with special graph classes related to metric properties. The origin of these families of graphs is the well-known class of median graphs, where gated sets coincide with geodesic convex sets (see Lemma 12.5 in [17]). The same holds also for modular graphs which are a bipartite generalization of median graphs. However, all other generalizations of median graphs connected with gated sets are non-bipartite. They are quasi-median graphs [5, 19], pseudo-median graphs [4], weakly median graphs [3], pre-median graphs [12], fiber-complemented graphs [12], weakly modular graphs [6, 13], cage-amalgamation graphs [9], absolute C-median graphs [6] and bucolic graphs [7].

The structure of all these classes has a strong relation with gated sets. All of them can be built either by expansion procedures or amalgamation, with the exception of absolute C-median graphs (for which this is a conjecture). In an expansion procedure every graph of the class under observation can be built if we start in a graph which is elementary (i.e., its only gated sets are singletons and the whole vertex set) and then expand it with some

Received June 11, 2015, accepted December 14, 2015.
Communicated by Xuding Zhu.
2010 *Mathematics Subject Classification.* 05C75, 52A01.
Key words and phrases. Gated set, Weakly modular graph, Convex geometry.
*Corresponding author.

rules over a gated set. These rules and elementary graphs are different with respect to the graph class with which we are dealing. Similarly we can build every graph of the class under observation from the gated amalgamations of Cartesian products of its elementary graphs.

In this work, we present a characterization of gated sets for general graphs together with several necessary conditions (see Section 3). The tools for this characterization are related to the triangle and the quadrangle property which define the class of weakly modular graphs. Hence it is no surprise that gated sets behave very nicely in weakly modular graphs (see Section 4) and also in pre-median graphs, weakly median graphs, quasi-median graphs and pseudo-median graphs, which all have weakly modular graphs as their superclass. In particular, two conditions, each of them characteristic for gated sets in weakly modular graphs, are the same as conditions which are characteristic for convex sets in partial cubes (another bipartite generalization of median graphs) and one of them even for all bipartite graphs (see the Convexity lemma in [18]).

In [15] Farber and Jamison study the problem from the abstract convexity theory, which is sometimes referred to as Minkowski-Krein-Milman property or convex geometry property. In the case of monophonic convexity exactly chordal graphs are convex geometries, while in the geodesic convexity these are precisely Ptolemaic graphs (i.e., distance-hereditary chordal graphs). In a similar way, totally balanced hypergraphs and strongly chordal graphs have been characterized as convex geometries of some particular (hyper)graph convexities [15]. For the Steiner convexity it was shown that precisely 3-fan-free chordal graphs are convex geometry with respect to this convexity [10]. Recently, in [2], the so-called toll convexity was introduced on a special kind of walks and it was shown that exactly interval graphs are convex geometries with respect to the toll convexity.

2. Preliminaries

In this paper, we consider only finite, connected, undirected graphs G without loops and multiple edges. A u,v-path path P of G is a *u,v-shortest path* or a *geodesic* if it is of the smallest length (where the length of a path $d_G(u,v)$ is the number of its edges). A *chord* of a path P is an edge joining two non-consecutive vertices of P. A path is *induced* (or chordless or monophonic) if it has no chord. If all pairs of vertices of a subgraph H of G which are adjacent in G are also adjacent in H, then H is an *induced subgraph.* For $K \subset V(G)$ we denote by $\langle K \rangle$ the subgraph of G induced by vertices of K. A subgraph H of G is *isometric* if $d_H(u,v) = d_G(u,v)$ for every pair $u,v \in V(H)$.

The *geodesic interval* between vertices u and v in G, denoted as $I(u,v)$, is the set of all vertices lying on a shortest path between u and v. A *geodesic convex* set in G is a set which contains $I(u,v)$ for each pair $u,v \in V(G)$. Let K be a subset of vertices in G, and let u be a vertex of G. A *gate* for u in K is a vertex x in K such that x lies in $I(u,w)$, for each vertex w in K. Clearly, if u has a gate in K, then it is unique and it is the vertex in K closest to u. Trivially, a vertex in K is its own gate. A subset K of $V(G)$ is called *gated*, if every vertex v of G has the gate $p_K(v)$ in K. For a gated set K we call the map $p_K \colon V(G) \to K$ which maps a vertex to its gate in K the *gate map*. The gate map is a weak retraction, see Lemma 16.2 in [17]. In particular this means that $d_{\langle K\rangle}(p_k(v), p_k(u)) \leq d_G(u,v)$ for any pair $u,v \in V(G)$. Tardif [20] proposed the term *prefiber* for gated sets because of their role in the Cartesian product of graphs.

A *convexity* on a non-empty set X is a family Γ of subsets of X containing X as well as the empty set $\emptyset$ such that Γ is closed under arbitrary intersections and nested unions. The members of Γ are called *convex sets*. If X is finite, then the nested union property follows trivially and the definition of convexity just reduces to the condition that the intersection of convex sets is convex and both X and $\emptyset$ are convex. As already mentioned in the introduction, this holds for gated sets. Hence the family of gated sets in a graph G forms a convexity on $V(G)$ called the *gated convexity*. An interval function $R\colon X \times X \to 2^X$ on a non-empty set X has the property that $x,y \in R(x,y)$ and $R(x,y) = R(y,x)$, and R-convex sets are defined as the sets S such that $R(x,y) \subseteq S$ for any $x,y \in S$. The family of R convex sets form a convexity on X known as the interval convexity, see Calder [11]. Hence the family of all geodesic convex sets in a graph also forms an interval convexity. Geodesic convex sets form probably the most natural convexity and therefore we will call them simply convex sets. It is interesting to note that the gated convexity is also an interval convexity generated by the pre-fiber interval function defined by $F_I(u,v) = \{w : I(u,w) \cap I(w,v) = \{w\}\}$. For more about general convexities see the book [21].

A vertex s from a convex set S is an *extreme vertex* of S, if $S - \{s\}$ is also convex. A graph G is called a *convex geometry* with respect to a given convexity, if any convex set of G is the convex hull of its extreme vertices. An alternative definition of convex geometries, using the so-called anti-exchange axiom is also often used (cf. [1], where convex geometries are studied in the context of lattices).

A graph G satisfies the *triangle property* if for any edge uv and any vertex x with $d_G(u,x) = d_G(v,x) = \ell \geq 2$, there exists a common neighbor y of u and v with $d_G(x,y) = \ell - 1$. See the left part of Figure 2.1 for the triangle property. A graph G satisfies the *quadrangle property* if, for any vertices u,v,w and x, where $d_G(u,x) = d_G(v,x) = \ell = d_G(x,w) - 1$ and w is a common neighbor of u and v, there exists a common neighbor y of u and v with $d_G(x,y) = \ell - 1$. See the right part of Figure 2.1 for the quadrangle property. A graph G is *weakly modular* if it satisfies both the triangle and the quadrangle property.

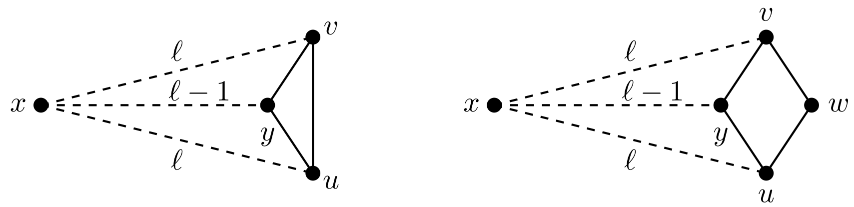

Figure 2.1: Triangle property (left) and quadrangle property (right).

3. Gated sets in graphs

We start with a characterization of gated sets in an arbitrary graph G. For this we need two new tools which are in a way generalizations of the triangle and the quadrangle property. Let G be a graph and K a subset of $V(G)$. We say that the *odd cycle property is satisfied for* K if for every $u, v \in K$ such that $uv \in E(G)$ and for every $x \in V(G) - K$ for which $d_G(u,x) = d_G(v,x) = \ell$, there exists a $y \in K$ such that $d_G(x,y) < \ell$ and y is on a shortest x,u-path and on a shortest x,v-path. See the left part of Figure 3.1 for the odd cycle property. Note that a shortest y,u-path in G, a shortest y,v-path in G and uv contains an odd cycle, which justifies the name. Also y need not be the gate for x in K, but the gate of x in K, if it exists, always witnesses the odd cycle property. Similarly we say that the *even cycle property is satisfied for* K if for every $u, v \in K$ such that $d_{\langle K\rangle}(u,v) = 2$ with a common neighbor $w \in K$ and for every $x \in V(G) - K$ for which $d_G(u,x) = d_G(v,x) = \ell$ and $d_G(x,w) = \ell + 1$, there exists a $y \in K$ such that $d_G(x,y) < \ell$ and y is on a shortest x,u-path and on a shortest x,v-path. See the right part of Figure 3.1 for the even cycle property. Similar as in the case of the odd cycle property a shortest y,u-path in G, a shortest y,v-path in G and the path uwv in $\langle K\rangle$ contains an even cycle. Again y need not be the gate for x in K, but the gate of x in K, if it exists, always witnesses the even cycle property.

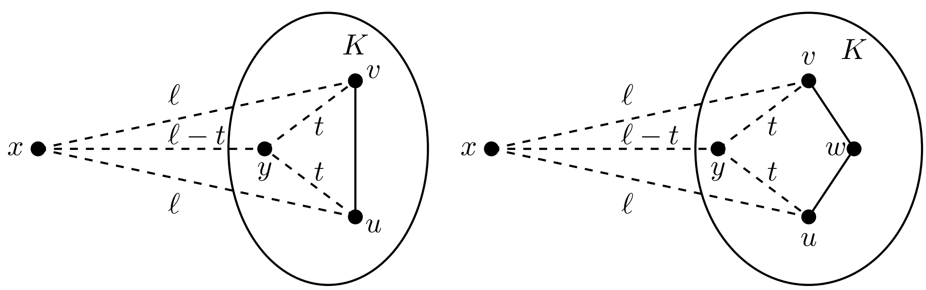

Figure 3.1: The odd cycle property (left) and the even cycle property (right).

These two definitions are not direct generalizations of the triangle property and the quadrangle property. To obtain such generalizations we need to set $K = V(G)$ in the definitions of both odd and even cycle properties. In addition we need a demand that $y \neq x$, otherwise everything is satisfied by nothing. While graphs which witnesses both such properties may be worthy of independent treatment, we need here the existence of such an odd and an even cycle in $\langle K\rangle$ to ensure that K is gated as shown by Theorem 3.2. Before this theorem we need additional result, which may be of independent interest.

Lemma 3.1. *Let G be a graph and let $K \subseteq V(G)$ induce a connected subgraph of G. If the odd cycle property and the even cycle property holds for K, then $\langle K\rangle$ is an isometric subgraph of G.*

Proof. Suppose that $\langle K\rangle$ is not an isometric subgraph of G. Let $a, b \in K$ be such that $d_G(a,b) < d_{\langle K\rangle}(a,b)$. (Notice that $d_{\langle K\rangle}(a,b)$ exists, since $\langle K\rangle$ is connected.) Among all such pairs of vertices we may choose a and b such that $d_G(a,b) + d_{\langle K\rangle}(a,b)$ is minimum. Let $P = x_0x_1\cdots x_s$, $x_0 = a$ and $x_s = b$, be a shortest a,b-path in G and let $Q = y_0y_1\cdots y_t$, $y_0 = b$ and $y_t = a$ be a shortest a,b-path in $\langle K\rangle$. Clearly $1 < s < t$ and we can choose such a notation that a and b are the only vertices of K on P. In particular $x_1 \notin K$.

Suppose first that there exists an index ℓ, $0 < \ell < t$, such that $d_G(x_1, y_\ell) = d_G(x_1, y_{\ell+1})$. If the odd cycle property holds for x_1, y_ℓ and $y_{\ell+1}$, then there exists a $u \in K$ which is on a shortest x_1, y_ℓ-path R and on a shortest $x_1, y_{\ell+1}$-path R' in G. Suppose that R is of length r. Clearly, also R' is of length r. We have

$$\begin{aligned} d_G(a,u) + d_{\langle K\rangle}(a,u) &\leq r + 1 + t - \ell - 1 \leq (s - 1 + \ell) + t - \ell \\ &= s + t - 1 < d_G(a,b) + d_{\langle K\rangle}(a,b). \end{aligned}$$

By the choice of a and b this implies that $d_{\langle K\rangle}(a,u) = d_G(a,u)$.

Now we may assume that $d_G(x_1, y_\ell) \neq d_G(x_1, y_{\ell+1})$ for every ℓ, $0 < \ell < t$. Since x_1 and y_t are adjacent and $t > s$, there exists an index ℓ, $0 < \ell < t$, such that $d_G(x_1, y_{\ell-1}) = d_G(x_1, y_{\ell+1}) = d_G(x_1, y_\ell) - 1$. If the even cycle property holds for $x_1, y_{\ell-1}, y_{\ell+1}$ and y_ℓ, then there exists a $u \in K$ which is on a shortest $x_1, y_{\ell-1}$-path R and on a shortest $x_1, y_{\ell+1}$-path R' in G. Suppose that R (and then also R') is of length r. We have

$$\begin{aligned} d_G(a,u) + d_{\langle K\rangle}(a,u) &\leq r + 1 + t - \ell - 1 \leq (s - 1 + \ell - 1) + t - \ell \\ &= s + t - 2 < d_G(a,b) + d_{\langle K\rangle}(a,b). \end{aligned}$$

By the choice of a and b this implies that $d_{\langle K\rangle}(a,u) = d_G(a,u)$.

In both cases we may choose u so that it is closest to x_1. Since x_1 is adjacent to a, we have $d_G(a,u) - 1 \leq d_G(x_1,u) \leq d_G(a,u) + 1$. If $d_G(x_1,u) = d_G(a,u) + 1$, then a shortest x_1, u-path contains a and we have $d_{\langle K\rangle}(a, y_\ell) = d_{\langle K\rangle}(a, y_{\ell+1})$ or $d_{\langle K\rangle}(a, y_{\ell-1}) = d_{\langle K\rangle}(a, y_{\ell+1})$, respectively, which is not possible. So let $d_G(a,u) - 1 \leq d_G(x_1,u) \leq d_G(a,u)$ and let $R = z_0z_1\cdots z_k$, $a = z_0$ and $u = z_k$, be a shortest a,u-path in $\langle K\rangle$. For $1 \leq i \leq k$

there must exists two adjacent vertices z_i and z_{i-1} on R with $d_G(x_1, z_i) = d_G(x_1, z_{i+1})$ or two vertices z_{i-1} and z_{i+1} on R with $d_G(x_1, z_{i-1}) = d_G(x_1, z_{i+1})$. We choose i to be the biggest integer when one of these two options occurs. In the first case there exists $u' \in K$ by the odd cycle property for x_1, z_i and z_{i-1}. In the second case we have $d_G(x_1, z_{i+1}) - 1 \leq d_G(x_1, z_i) \leq d_G(x_1, z_{i+1}) + 1$. If $d_G(x_1, z_i) = d_G(x_1, z_{i+1}) + 1$, then set $u' = z_i$. If $d_G(x_1, z_i) = d_G(x_1, z_{i+1})$, then there exists u' by the odd cycle property for x_1, z_i and z_{i+1}. If $d_G(x_1, z_i) = d_G(x_1, z_{i+1}) + 1$, then there exists u' by the even cycle property for x_1, z_{i+1} and z_{i-1}. By the same reasons as above we have that $d_{\langle K \rangle}(a, u') = d_G(a, u')$ and that $d_G(a, u') \leq d_G(x_1, u') \leq d_G(a, u') + 1$. This yields the final contradiction with the choice of u, since by the choice of i vertex u' satisfies the odd cycle property for x_1, y_ℓ and $y_{\ell+1}$ or the even cycle property for $x_1, y_{\ell-1}, y_{\ell+1}$ and y_ℓ, respectively, and the proof is complete. $\square$

Theorem 3.2. *Let G be a graph and let $K \subseteq V(G)$ induce a connected subgraph of G. A set K is gated in G if and only if the odd cycle property and the even cycle property holds for K.*

Proof. Let K be gated in G. If the odd cycle property or the even cycle property does not hold for some $u, v \in K$ and some $x \in V(G) - K$, then x has no gate in K, which is not possible for a gated set K.

Suppose that K is not gated in G. Then there exists a vertex $x \in V(G) - K$ without a gate in K. This means that for every $a \in K$ we have that

$$d_G(x, b) \neq d_G(x, a) + d_{\langle K \rangle}(a, b) \tag{3.1}$$

for some $b \in K$. Among all such vertices we may choose x to be as close to K as possible and let a be such that $d_G(x, K) = d_G(x, a)$. Moreover, let b be a vertex closest to a in $\langle K \rangle$ and among all such, let in addition b be closest to x also. If $\langle K \rangle$ is not an isometric subgraph of G, then the odd cycle property or the even cycle property does not hold by Lemma 3.1. Hence we may assume that $\langle K \rangle$ is an isometric subgraph of G.

If $ab \in E(G)$, then $d_G(x, b) = d_G(x, a) = \ell$ by (3.1) and the choice of a and there exists no vertex $c \in K$ such that c is on a shortest x, a-path and on a shortest x, b-path and $d_G(x, c) < \ell$ by the choice of a. Hence the odd cycle property does not hold. If $d_{\langle K \rangle}(a, b) = 2$, then $d_G(x, a) \leq d_G(x, b) \leq d_G(x, a) + 1$ by (3.1) and the choice of a. Let first $d_G(x, b) = d_G(x, a) + 1$ and let d be a common neighbor of a and b in K. We have $d_G(x, d) = d_G(x, a) + 1$ by (3.1) and the choice of b. If the odd cycle property does not hold for b, d and x for K, then we are done. Otherwise there exists $c \in K$ which is on a shortest x, b-path and on a shortest x, d-path with $d_G(x, c) < \ell + 1$. By the choice of a we have that $d_G(x, c) = \ell$. In particular this means that c is adjacent to d. If $ac \in E(G)$, then the odd cycle property does not hold for a, c and x by $d_G(x, K) = d_G(x, a)$. If $ac \notin E(G)$, then c contradicts the choice of b. If $d_G(x, a) = d_G(x, b) = \ell$, then there exist no vertex $c \in K$ such that c is on a shortest x, a-path and on a shortest x, b-path and $d_G(x, c) < \ell$ by the choice of a. Therefore the even cycle property does not hold for a, b and x for K since $d_G(x, d) = \ell + 1$.

Let now $d_{\langle K\rangle}(a, b) > 2$. Let P be a shortest a, b-path in $\langle K\rangle$. Denote the vertices of P by $a_0, a_1, \ldots, a_k, a_{k+1}$ where $a = a_0$ and $b = a_{k+1}$ and additional $u = a_k$ and $v = a_{k-1}$. By the choice of a and b we have that $d_G(x, a_i) = d_G(x, a) + i$ for $1 \le i \le k$. (Otherwise, the first a_j for which this does not hold yields a contradiction with b being the closest vertex to a violating (3.1).) In particular the equality $d_G(x, u) = d_G(x, a) + k$ holds. Since $ub \in E(G)$, we have that $d_G(x, b) \le d_G(x, u) \le d_G(x, b) + 1$. Suppose first that $d_G(x, b) = d_G(x, u)$. If the odd cycle property does not hold for x, b and u, then we are done. Otherwise there exists $y \in K$ such that $d_G(x, u) = d_G(x, y) + d_{\langle K\rangle}(y, u)$ and $d_G(x, b) = d_G(x, y) + d_{\langle K\rangle}(y, b)$. In particular we have that $d_{\langle K\rangle}(y, b) = d_{\langle K\rangle}(y, u)$. If $d_{\langle K\rangle}(y, a) > d_{\langle K\rangle}(a, b)$, then there exists a vertex c on a shortest u, y-path at the distance $k + 1$ to a in $\langle K\rangle$. Clearly c contradicts the choice of b since it is closer to x than b. So let $d_{\langle K\rangle}(y, a) \le d_{\langle K\rangle}(a, b)$. By the choice of b, we have by (3.1) that

$$\begin{aligned} d_G(x, y) &= d_G(x, a) + d_{\langle K\rangle}(a, y) \\ &= d_G(x, b) - k + d_{\langle K\rangle}(a, y) \\ &= d_G(x, y) + d_{\langle K\rangle}(y, b) - k + d_{\langle K\rangle}(a, y). \end{aligned}$$

This computation implies that $d_{\langle K\rangle}(y, b) + d_{\langle K\rangle}(a, y) = k = d_{\langle K\rangle}(y, u) + d_{\langle K\rangle}(a, y)$. In other words, y is on a shortest a, u-path in $\langle K\rangle$, which is a contradiction with $d_{\langle K\rangle}(a, b) = k + 1$ by $d_{\langle K\rangle}(y, b) = d_{\langle K\rangle}(y, u)$.

Therefore we may assume that $d_G(x, u) = d_G(x, b) + 1$. With this b is on a shortest u, x-path. Also v is on a shortest u, x-path and both are neighbors of u. Hence we have that $d_G(x, b) = d_G(x, v)$. In addition, we have that $d_{\langle K\rangle}(v, b) = 2$, since v is on a shortest a, b-path that contains u. If the even cycle property does not hold for x, b, v and u, then we are done. Otherwise there exists a vertex $y \in K$ such that $d_G(x, v) = d_G(x, y) + d_{\langle K\rangle}(y, v)$ and $d_G(x, b) = d_G(x, y) + d_{\langle K\rangle}(y, b)$. Again we have that $d_{\langle K\rangle}(y, b) = d_{\langle K\rangle}(y, v)$. Similar as before, $d_{\langle K\rangle}(y, a) > d_{\langle K\rangle}(a, b)$ yields a contradiction with the choice of b. So let $d_{\langle K\rangle}(y, a) \le d_{\langle K\rangle}(a, b)$. By the choice of b, we have that

$$\begin{aligned} d_G(x, y) &= d_G(x, a) + d_{\langle K\rangle}(a, y) \\ &= d_G(x, b) - k + 1 + d_{\langle K\rangle}(a, y) \\ &= d_G(x, y) + d_{\langle K\rangle}(y, b) - k + 1 + d_{\langle K\rangle}(a, y). \end{aligned}$$

This implies that $d_{\langle K\rangle}(y, b) + d_{\langle K\rangle}(a, y) = k - 1 = d_{\langle K\rangle}(y, v) + d_{\langle K\rangle}(a, y)$. In other words, y

is on a shortest a, v-path in $\langle K\rangle$. This gives the final contradiction with $d_{\langle K\rangle}(a,b) = k+1$ by $d_{\langle K\rangle}(y,b) = d_{\langle K\rangle}(y,v)$ and the proof is complete. □

Next we present three necessary conditions for gated sets, which will be handy later on when discussing gated sets in weakly modular graphs. For this we need the notion of boundary of a set. Let K be a subset of $V(G)$ of a graph G. Every edge with one end vertex in K and other end vertex outside of K is called a *boundary edge* of K. The set of all boundary edges of K, denoted ∂K_E, is called the *edge boundary set* of K and the *vertex boundary set* of K, denoted ∂K_V, contains all vertices of K which have a neighbor in $V(G) - K$. Furthermore, let $C_1, \ldots, C_k$ be components of $G - K$ and we split ∂K_V into subsets $\partial K_V/C_i$ called the *boundary set of K with respect to C_i*, $1 \leq i \leq k$, which contains all vertices of ∂K_V which have a neighbor in C_i. Note that a vertex $v \in \partial K_V$ can be in more than one $\partial K_V/C_i$ and hence $\partial K_V/C_i$ may not form a partition of ∂K_V.

Theorem 3.3. *Let G be a graph, let $K \subseteq V(G)$ and let $C_1, \ldots, C_k$ be components of $G - K$. If K is gated in G, then $\langle \partial K_V/C_i\rangle$ is connected for every $i \in \{1, 2, \ldots, k\}$.*

Proof. Let K be gated, let $C_1, \ldots, C_k$ be components of $G - K$ and let $\partial K_V/C_i$ be the boundary set of K with respect to C_i, $1 \leq i \leq k$. Let a_1 and a_2 be any two vertices of $\partial K_V/C_i$, $1 \leq i \leq k$, and let b_1 and b_2 be their neighbors in C_i. Let Q be a shortest b_1, b_2-path in C_i. Clearly, the gate map maps Q to $\partial K_V/C_i$. Since p_K is a weak retraction, $p_K(Q)$ yields a a_1, a_2-path in $\langle \partial K_V/C_i\rangle$, which is therefore connected and the proof is complete. □

For the next result we need some additional notation which is well known in metric graph theory. For an edge ab of a graph G the set W_{ab} contains all vertices which are closer to a than to b and similar are in W_{ba} all vertices of G which are closer to b than to a. While sets W_{ab} and W_{ba} form a partition of $V(G)$ in any bipartite graph, this is not so in an arbitrary graph. Hence in addition we have the set ${}_aW_b$ which contains all vertices of G which are at the same distance to a and to b. By W_{ab}^C we denote the complement of W_{ab}. Clearly $W_{ab}^C = W_{ba} \cup {}_aW_b$.

Theorem 3.4. *Let G be a graph and let $K \subseteq V(G)$. If K is gated in G, then $K = \bigcap_{ab\in\partial K_E} W_{ab} = \bigcap_{ab\in\partial K_E} W_{ba}^C$ where $a \in \partial K_V$.*

Proof. Let K be a gated set of G and suppose that $K \neq \bigcap_{ab\in\partial K_E} W_{ab} = A$ where $a \in \partial K_V$. Let first $x \in A$ and $x \notin K$. Thus $x \in W_{ab}$ for every edge $ab \in \partial K_E$ and with this $d_G(x,a) < d_G(x,b)$. Let v be the gate of x in K and let P be a shortest x, a-path that contains v. The neighbor u of v that is closer to x on P is clearly not in K. (Note that u can be equal to x.) The edge uv is in the boundary of K, but $x \in W_{uv}$ and $x \notin W_{vu}$, which is a contradiction with $x \in A$.

Let now $y \in K$ and $y \notin A$. Thus there exists an edge $ab \in \partial K_E$, such that $y \notin W_{ab}$ for $a \in \partial K_V$, and we have that $d_G(y,b) \leq d_G(y,a) = d_{\langle K \rangle}(y,a)$. Therefore a is not the gate of b in K. But then b has no gate in K since ab is an edge, which is a contradiction again. Hence $K = A$.

Finally let $B = \bigcap_{ab \in \partial K_E} W_{ba}^C$ where $a \in \partial K_V$. Clearly $A \subseteq B$ since $W_{ab} \subseteq W_{ba}^C$. Suppose that $A \neq B$ and there exists $u \in B - A$. Obvious $u \in {}_aW_b$ for some $ab \in \partial K_E$ where $a \in \partial K_V$ and therefore $u \notin K$. On a shortest a,u-path there exists an edge $vw \in \partial K_E$ with $v \in \partial K_V$. Clearly $u \in W_{wv}$ and with this $u \notin B$, which is a contradiction. Thus $B = A$ and the proof is complete. □

Edges $e = ab$ and $f = uv$ are in relation Θ, if $d_G(v,a) + d_G(u,b) \neq d_G(v,b) + d_G(u,a)$. Equivalent definition is that an edge f of a graph G is in relation Θ with an edge $e = ab$, if f joins two different components of the partition of $V(G)$ induced by W_{ab}, W_{ba} and ${}_aW_b$. The next lemma involves the relation Θ.

Lemma 3.5. *Let G be a graph and let $K \subseteq V(G)$ induce a connected subgraph of G. If $K = \bigcap_{ab \in \partial K_E} W_{ab}$ where $a \in \partial K_V$, then no edge of ∂K_E is in relation Θ to an edge in $\langle K \rangle$.*

Proof. Let $K = \bigcap_{ab \in \partial K_E} W_{ab}$ where $a \in \partial K_V$ and let uv be an arbitrary edge in $\langle K \rangle$. Thus $u, v \in W_{ab}$ for every $ab \in \partial K_E$ where $a \in \partial K_V$ and we have that $d_G(u,b) = d_G(u,a) + 1$ and $d_G(v,b) = d_G(v,a) + 1$. The following computation shows that uv is not in relation Θ with ab:

$$d_G(v,a) + d_G(u,b) = d_G(v,a) + d_G(u,a) + 1 = d_G(v,b) + d_G(u,a)$$

and the proof is complete. □

The following result gives another necessary condition for gated sets for arbitrary graphs and is a direct consequence of Theorem 3.4 and Lemma 3.5.

Theorem 3.6. *Let G be a graph and let $K \subseteq V(G)$. If K is gated in G, then no edge of ∂K_E is in relation Θ to an edge in $\langle K \rangle$.*

Note that the converse of this theorem does not hold. The simplest examples for K are adjacent vertices in an odd cycle of length at least five. For them no edge of ∂K_E is in relation Θ to an edge in $\langle K \rangle$, but K is clearly not gated.

4. Gated sets in weakly modular graphs

It is well known that gated sets coincide with geodesic convex sets in modular graphs and in particular, in median graphs. This is not so among weakly modular graphs. As shown by Chepoi [13] are gated sets in weakly modular graphs precisely those convex sets which are closed for triangles.

By Theorem 3.2 we have a strong connection between gated sets and the odd and the even cycle property. A special kind of the odd and the even cycle property, namely the triangle and the quadrangle property, respectively, guarantees the existence of appropriate vertices at the local range in weakly modular graphs. One only needs to include all such vertices in a set to be gated. We demonstrate this by two conditions on the boundary edge set of ∂K_E.

Theorem 4.1. *Let G be a weakly modular graph. The following statements are equivalent for $K \subseteq V(G)$ where $\langle K \rangle$ is connected.*

(i) *K is gated;*

(ii) *$K = \bigcap_{ab \in \partial K_E} W_{ab}$ where $a \in \partial K_V$;*

(iii) *No edge of ∂K_E is in relation Θ to an edge in $\langle K \rangle$.*

Proof. The implication from (i) to (ii) holds by Theorem 3.4 and implication from (ii) to (iii) holds by Lemma 3.5. It remains to prove the implication from (iii) to (i). For this suppose that K is not gated. By Theorem 3.2 the odd cycle property or the even cycle property does not hold. If the odd cycle property does not hold, then there exists an edge uv in $\langle K \rangle$, a vertex $x \in V(G) - K$ with $d_G(x,u) = d_G(x,v) = \ell$ and there exist no $y \in K$ with $d_G(x,y) < \ell$ and y is on a shortest x,u-path and on a shortest x,v-path. Since G is weakly modular, there exists $y \in V(G)$ such that $d_G(x,y) = \ell - 1$ and $d_G(y,u) = 1 = d_G(y,v)$. Thus $y \notin K$ and $yu, yv \in \partial K_E$. Vertices y, u, v form a triangle and hence $uv\Theta yu$ (as well as $uv\Theta yv$) and we have an edge from ∂K_E in relation Θ to an edge in $\langle K \rangle$.

If the even cycle property does not hold, then there exists $u, v \in K$ with $d_{\langle K \rangle}(u,v) = 2$ where w is a common neighbor of u and v in $\langle K \rangle$, a vertex $x \in V(G) - K$ with $d_G(x,u) = d_G(x,v) = \ell = d_G(x,w) - 1$ and there exist no $y \in K$ with $d_G(x,y) < \ell$ and y is on a shortest x,u-path and on a shortest x,v-path. Since G is weakly modular, there exists $y \in V(G)$ such that $d_G(x,y) = \ell - 1$ and $d_G(y,u) = 1 = d_G(y,v)$. Thus $y \notin K$ and $yu, yv \in \partial K_E$. Vertices y, u, w, v form an induced four cycle and hence $uw\Theta yv$ (as well as $vw\Theta yu$) and we have an edge from ∂K_E in relation Θ to an edge in $\langle K \rangle$ and the proof is complete. □

Recall that gated sets coincide with convex sets in median graphs. A generalization of convex sets in all bipartite graphs is known as the Convexity lemma, see [18].

Lemma 4.2 (Convexity lemma)**.** *A $K \subseteq V(G)$ of a bipartite graph G is convex if and only if no edge of ∂K_E is in relation Θ to an edge in $\langle K \rangle$.*

Hence the equivalence between (i) and (iii) of Theorem 4.1 is a result of the same type, but the generalization goes to nonbipartiteness. Clearly we cannot expect that gatedness and convexity coincide also for weakly modular graphs. The simplest counterexample for this is an edge in a triangle, which is convex but not gated. On the other hand, it would be interesting to know more about gated sets in partial cubes and classes in between them and median graphs (see [8] for a detailed description of these classes).

5. Gated convexity as convex geometry

We finish our discussion by showing that trees represents gated convex geometries. Recall that a vertex s from a convex set S is an extreme vertex of S, if $S - \{s\}$ is also convex. A graph G is called a convex geometry with respect to a given convexity, if any convex set of G is the convex hull of its extreme vertices.

Theorem 5.1. *Graph G is a gated convex geometry if and only if G is a tree.*

Proof. Let G be a gated convex geometry and let K be a gated set in G. Hence K is a convex hull of its extreme vertices. In particular this means that there exists an extreme vertex in K and moreover, there exists nontrivial gated sets as $V(G)$ is gated. We show that G is a tree by induction on the number of vertices n. This is obvious for $n = 1$ or $n = 2$. Suppose now that G has $n > 2$ vertices. If x is an extreme vertex, then $G - x$ is gated and x must be of degree one in G. By induction hypothesis $G - x$ is a tree and therefore G is a tree.

Let now G be a tree. A subset K of $V(G)$ is gated if and only if $\langle K \rangle$ is a subtree of G by Theorem 3.2. It is obvious that extreme vertices in a subtree are all its leaves and the gated convex hull of these leaves is the starting subtree and nothing more. Thus every gated convex set is a convex hull of its extreme vertices and G is a gated convex geometry. □

Acknowledgments

The first author was partially supported by NBHMDAE, Govt. of India under the research grant No. 2/48(9)/2014/NBHM (R.P)/R & DII/14364. The second author was supported by the the scholar in residence program under the Erudite Scheme of the Kerala State Higher Education Council, Govt. of Kerala, India. The work of IP was in part financed by ARRS Slovenia under the grant P1-0297. The third author was partially supported by UGC, Govt. of India under the research grant No. F-17-57/09 (SA-I).

References

[1] K. Adaricheva, *Representing finite convex geometries by relatively convex sets*, European J. Combin. **37** (2014), 68–78. http://dx.doi.org/10.1016/j.ejc.2013.07.012

[2] L. Alcón, B. Brešar, T. Gologranc, M. Gutierrez, T. Kraner Šumenjak, I. Peterin and A. Tepeh, *Toll convexity*, European J. Combin. **46** (2015), 161–175. http://dx.doi.org/10.1016/j.ejc.2015.01.002

[3] H.-J. Bandelt and V. Chepoi, *Decomposition and l_1-embedding of weakly median graphs*, European J. Combin. **21** (2000), no. 6, 701–714. http://dx.doi.org/10.1006/eujc.1999.0377

[4] H.-J. Bandelt and H. M. Mulder, *Pseudo-median graphs: decomposition via amalgamation and Cartesian multiplication*, Discrete Math. **94** (1991), no. 3, 161–180. http://dx.doi.org/10.1016/0012-365x(91)90022-t

[5] H.-J. Bandelt, H. M. Mulder and E. Wilkeit, *Quasi-median graphs and algebras*, J. Graph Theory **18** (1994), no. 7, 681–703. http://dx.doi.org/10.1002/jgt.3190180705

[6] B. Brešar, *On the natural imprint function of a graph*, European J. Combin. **23** (2002), no. 2, 149–161. http://dx.doi.org/10.1006/eujc.2001.0555

[7] B. Brešar, J. Chalopin, V. Chepoi, T. Gologranc and D. Osajda, *Bucolic complexes*, Adv. Math. **243** (2013), 127–167. http://dx.doi.org/10.1016/j.aim.2013.04.009

[8] B. Brešar, W. Imrich, S. Klavžar, H. M. Mulder and R. Škrekovski, *Tiled partial cubes*, J. Graph Theory **40** (2002), no. 2, 91–103. http://dx.doi.org/10.1002/jgt.10031

[9] B. Brešar and A. Tepeh Horvat, *Cage-amalgamation graphs, a common generalization of chordal and median graphs*, European J. Combin. **30** (2009), no. 5, 1071–1081. http://dx.doi.org/10.1016/j.ejc.2008.09.003

[10] J. Cáceres, A. Márquez and M. L. Puertas, *Steiner distance and convexity in graphs*, European J. Combin. **29** (2008), no. 3, 726–736. http://dx.doi.org/10.1016/j.ejc.2007.03.007

[11] J. R. Calder, *Some elementary properties of interval convexities*, J. London Math. Soc. (2) **3** (1971), 422–428. http://dx.doi.org/10.1112/jlms/s2-3.3.422

[12] M. Chastand, *Fiber-complemented graphs, I. Structure and invariant subgraphs*, Discrete Math. **226** (2001), no. 1-3, 107–141. http://dx.doi.org/10.1016/s0012-365x(00)00183-7

[13] V. D. Chepoĭ, *Classification of graphs by means of metric triangles*, (Russian) Metody Diskret. Analiz. **49** (1989), 75–93.

[14] A. W. M. Dress and R. Scharlau, *Gated sets in metric spaces*, Aequationes Math. **34** (1987), no. 1, 112–120. http://dx.doi.org/10.1007/bf01840131

[15] M. Farber and R. E. Jamison, *Convexity in graphs and hypergraphs*, SIAM J. Algebraic Discrete Methods **7** (1986), no. 3, 433–444. http://dx.doi.org/10.1137/0607049

[16] A. J. Goldman and C. J. Witzgall, *A localization theorem for optimal facility placement*, Transportation Sci. **4** (1970), 406–409. http://dx.doi.org/10.1287/trsc.4.4.406

[17] R. Hammack, W. Imrich and S. Klavžar, *Handbook of Product Graphs*, Second edition, CRC Press, Boca Raton, FL, 2011.

[18] W. Imrich and S. Klavžar, *A convexity lemma and expansion procedures for bipartite graphs*, European J. Combin. **19** (1998), no. 6, 677–685. http://dx.doi.org/10.1006/eujc.1998.0229

[19] H. M. Mulder, *The Interval Function of a Graph*, Mathematical Centre Tracts **132**, Mathematisch Centrum, Amsterdam, 1980.

[20] C. Tardif, *Prefibers and the Cartesian product of metric spaces*, Discrete Math. **109** (1992), no. 1-3, 283–288. http://dx.doi.org/10.1016/0012-365x(92)90298-t

[21] M. L. J. van de Vel, *Theory of Convex Structures*, North-Holland, Amsterdam, 1993.

Manoj Changat
Department of Futures Studies, University of Kerala, Trivandrum-695034, India
E-mail address: mchangat@gmail.com

Iztok Peterin
University of Maribor, FEECS, Smetanova 17, 2000 Maribor, Slovenia
and
Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana
E-mail address: iztok.peterin@um.si

Abisha Ramachandran
Department of Futures Studies, University of Kerala, Trivandrum-695034, India
E-mail address: cyanabisha@gmail.com