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Attraction Property of Admissible Integral Manifolds and Applications to Fisher-Kolmogorov Model

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Abstract. In this paper we investigate the attraction property of an unstable manifold of admissible classes for solutions to the semi-linear evolution equation of the form $u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))\,d\xi$. These manifolds are constituted by trajectories of the solutions belonging to admissible function spaces which contain wide classes of function spaces like L_p-spaces, the Lorentz spaces $L_{p,q}$ and many other function spaces occurring in interpolation theory. We then apply our abstract results to study Fisher-Kolmogorov model with time-dependent environmental capacity.

1. Introduction and preliminaries

Consider the semi-linear differential equation

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad t \in \mathbb{R},\ x \in X \tag{1.1}$$

where $A(t)$ is a (possibly unbounded) linear operator on a Banach space X for every fixed t, and $f\colon \mathbb{R} \times X \to X$ is a nonlinear operator. When the linear part has an exponential dichotomy (or trichotomy), one tries to find conditions on the nonlinear forcing term f such that Equation (1.1) has an integral manifold (e.g., a stable, unstable or center manifold).

The problem of the existence of the integral manifolds is a matter of great interest of many authors since, on the one hand, it brings out the geometric structures of the solutions to semi-linear differential equations, and on the other hand it allows to implement the reduction principle to deduce the complicated equations to the simpler ones on such manifolds thanks to the attraction properties of that manifolds. To obtain such existence, one needs the characterizations of the exponential dichotomies (or trichotomies) of the linear part in some function spaces. Such characterizations have been used to construct the forms of operators determining the manifolds. We refer the reader to [2, 3, 7, 19] for

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recent contributions to the theory of exponential dichotomy and trichotomy of evolution equations. On the other hand, one needs to impose some conditions on the nonlinear term, and the most popular condition regarding nonlinear term f for the existence of such manifolds is its uniform Lipchitz continuity with a sufficiently small Lipschitz constant (i.e., $\|f(t,x)-f(t,y)\| \leq q\,\|x-y\|$ for q being small enough). We refer the reader to [1, 5, 6, 20, 22] and references therein for detailed information on the matter. However, for equations arising in complicated reaction-diffusion processes (see e.g., [14,15]), the Lipschitz coefficients may depend on time and may not be small in classical sense. Therefore, one tries to extend the conditions on nonlinear parts such that they describe more exactly such reaction-diffusion processes.

Recently, using Lyapunov-Perron method and the admissibility of function spaces, we have given a more general condition on f for the existence of integral manifolds (see [8, 10, 11]), that is the non-uniform Lipschitz continuity (or the φ-Lipschitz property) of f, i.e., $\|f(t,x)-f(t,y)\| \leq \varphi(t)\,\|x-y\|$ for φ being a real and positive function which belongs to admissible function spaces defined in Definition 1.3 below. Furthermore, the existence of a new type of invariant manifolds has been proved in [9], namely the invariant-stable manifolds of *admissible classes* which are constituted by trajectories of solutions belonging to the Banach space $\mathcal{E}$ which can be a space of L_p type ($1 \leq p \leq \infty$) or a Lorentz space $L_{p,q}$ or some function space occurring in interpolation theory. Then, the result in [9] has been extended to the case of an invariant-unstable manifold of admissible class in [12]. However, the attraction property of such a manifold was left unsolved in that paper. And some real-world applications were not given.

In the present paper, we will prove the attraction property of such an invariant-unstable manifold of admissible class obtained in [12]. Our method is based on the Lyapunov-Perron equations and the choice of induced solutions lying in the manifold and belonging to some admissible space on which we can implement some procedures of functional analysis. Our main results are contained in Theorem 2.9. Moreover, we apply the obtained results to consider Fisher-Kolmogorov model with time-dependent environmental capacity.

We first recall some notions.

Definition 1.1. A family of bounded linear operators $\mathcal{U} = (U(t,s))_{t\geq s}$ on a Banach space X is a *(strongly continuous, exponentially bounded) evolution family* on the whole line $\mathbb{R}$ if

(i) $U(t,t) = \mathrm{Id}$ and $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s$,

(ii) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$,

(iii) there are constants $K \geq 1$ and $\alpha \in \mathbb{R}$ such that $\|U(t,s)\| \leq Ke^{\alpha(t-s)}$ for $t \geq s$.

This notion arises naturally from well-posed evolution equations of the form

$$\begin{cases} \dfrac{du(t)}{dt} = A(t)u(t), \quad t \geq s, \\ u(s) = x_s \in X, \end{cases} \tag{1.2}$$

where $A(t)$ is (in general) an unbounded linear operator on X for every fixed t. Roughly speaking, when the Cauchy problem (1.2) is well-posed, there exists an evolution family $\mathcal{U} = (U(t,s))_{t\geq s}$ solving (1.2), i.e., the solution of (1.2) is given by $u(t) := U(t,s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer to Pazy [17], Nagel and Nickel [16].

We recall some notions on function spaces and refer to Massera and Schäffer [13], Räbiger and Schnaubelt [18] for concrete applications (see also [7, 21]).

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}$. The space $L_{1,\mathrm{loc}}(\mathbb{R})$ of real-valued locally integrable functions on $\mathbb{R}$ (modulo λ-nullfunctions) becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n} |f(t)|\, dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{Z}$ (see [13, Chapt. 2, § 20]).

We can now define Banach function spaces as follows.

Definition 1.2. A vector space $E_{\mathbb{R}}$ of real-valued Borel-measurable functions on $\mathbb{R}$ (modulo λ-nullfunctions) is called *a Banach function space* (over $(\mathbb{R}, \mathcal{B}, \lambda)$) if

(1) $E_{\mathbb{R}}$ is Banach lattice with respect to a norm $\|\cdot\|_{E_{\mathbb{R}}}$, i.e., $(E_{\mathbb{R}}, \|\cdot\|_{E_{\mathbb{R}}})$ is a Banach space, and if $\varphi \in E_{\mathbb{R}}$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leq |\varphi(\cdot)|$, λ-a.e., then $\psi \in E_{\mathbb{R}}$ and $\|\psi\|_{E_{\mathbb{R}}} \leq \|\varphi\|_{E_{\mathbb{R}}}$,

(2) the characteristic functions χ_A belong to $E_{\mathbb{R}}$ for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\in\mathbb{R}} \left\|\chi_{[t,t+1]}\right\|_{E_{\mathbb{R}}} < \infty$ and $\inf_{t\in\mathbb{R}} \left\|\chi_{[t,t+1]}\right\|_{E_{\mathbb{R}}} > 0$,

(3) $E_{\mathbb{R}} \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R})$, i.e., for each seminorm p_n of $L_{1,\mathrm{loc}}(\mathbb{R})$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leq \beta_{p_n} \|f\|_{E_{\mathbb{R}}}$ for all $f \in E_{\mathbb{R}}$.

We remark that condition (3) in the above definition means that for each compact interval $J \subset \mathbb{R}$ there exists a number $\beta_J \geq 0$ such that $\int_J |f(t)|\, dt \leq \beta_J \|f\|_{E_{\mathbb{R}}}$ for all $f \in E_{\mathbb{R}}$.

Let now $E_{\mathbb{R}}$ be a Banach function space and X be a Banach space. We set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}, X) := \{f \colon \mathbb{R} \to X : f \text{ is strongly measurable and } \|f(\cdot)\| \in E_{\mathbb{R}}\}$$

(modullo λ-nullfunctions) endowed with the norm

$$\|f\|_{\mathcal{E}} := \|\|f(\cdot)\|\|_{E_{\mathbb{R}}}\,.$$

Then $\mathcal{E}$ is a Banach space called *the Banach space corresponding to the Banach function space* $E_{\mathbb{R}}$.

Definition 1.3. The Banach function space $E_{\mathbb{R}}$ is called *admissible* if

(i) there is a constant $M \geq 1$ such that for every compact interval $[a,b] \subset \mathbb{R}$ we have

$$\int_a^b |\varphi(t)|\,dt \leq \frac{M(b-a)}{\left\|\chi_{[a,b]}\right\|_{E_{\mathbb{R}}}} \|\varphi\|_{E_{\mathbb{R}}} \quad \text{for all } \varphi \in E_{\mathbb{R}}, \tag{1.3}$$

(ii) for $\varphi \in E_{\mathbb{R}}$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1} \varphi(\tau)\,d\tau$ belongs to $E_{\mathbb{R}}$,

(iii) $E_{\mathbb{R}}$ is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined by

$$\begin{aligned} T_\tau^+\varphi(t) &:= \varphi(t-\tau) \quad \text{for } t \in \mathbb{R}, \\ T_\tau^-\varphi(t) &:= \varphi(t+\tau) \quad \text{for } t \in \mathbb{R}. \end{aligned} \tag{1.4}$$

Moreover, there are constants N_1, N_2 such that $\|T_\tau^+\| \leq N_1$, $\|T_\tau^-\| \leq N_2$ for all $\tau \in \mathbb{R}$.

Example 1.4. Besides the spaces $L_p(\mathbb{R})$, $1 \leq p \leq \infty$, and the space

$$\boldsymbol{M}(\mathbb{R}) := \left\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}) : \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(\tau)|\,d\tau < \infty \right\}$$

endowed with the norm $\|f\|_{\boldsymbol{M}} := \sup_{t\in\mathbb{R}} \int_t^{t+1} |f(\tau)|\,d\tau$, many other function spaces occurring in interpolation theory, e.g., the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 < q < \infty$ are admissible.

Remark 1.5. It can be easily seen that if $E_{\mathbb{R}}$ is an admissible Banach function space then $E_{\mathbb{R}} \hookrightarrow \boldsymbol{M}(\mathbb{R})$.

We now collect some properties of admissible Banach function spaces in the following proposition whose proof can be done in the same way as in [7, Proposition 2.6]).

Proposition 1.6. *Let $E_{\mathbb{R}}$ be an admissible Banach function space. Then the following assertions hold.*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R})$ such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E_{\mathbb{R}}$, where Λ_1 is defined as in Definition* 1.3(ii)*. For $\sigma > 0$ we define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by*

$$\begin{aligned} \Lambda'_\sigma\varphi(t) &= \int_{-\infty}^t e^{-\sigma(t-s)}\varphi(s)\,ds, \\ \Lambda''_\sigma\varphi(t) &= \int_t^\infty e^{-\sigma(s-t)}\varphi(s)\,ds. \end{aligned}$$

Then, $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to $E_\mathbb{R}$. In particular, if $\sup_{t\in\mathbb{R}}\int_t^{t+1}|\varphi(\tau)|\,d\tau < \infty$ (this will be satisfied if $\varphi \in E_\mathbb{R}$ (see Remark 1.5)) then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover,

$$\left\|\Lambda'_\sigma\varphi\right\|_{E_\mathbb{R}} \le \frac{N_1}{1-e^{-\sigma}}\left\|\Lambda_1\varphi\right\|_{E_\mathbb{R}} \quad \textit{and} \quad \left\|\Lambda''_\sigma\varphi\right\|_{E_\mathbb{R}} \le \frac{N_2}{1-e^{-\sigma}}\left\|\Lambda_1\varphi\right\|_{E_\mathbb{R}}.$$

(b) *$E_\mathbb{R}$ contains exponentially decaying functions $\psi(t) = e^{-\alpha|t|}$ for $t \in \mathbb{R}$ and any fixed constant $\alpha > 0$.*

(c) *$E_\mathbb{R}$ does not contain exponentially growing functions $f(t) = e^{b|t|}$ for $t \in \mathbb{R}$ and a constant $b > 0$.*

We next define the associate spaces of Banach function spaces as follows.

Definition 1.7. Let $E_\mathbb{R}$ be an admissible Banach function space and denote by $S(E_\mathbb{R})$ the unit sphere in $E_\mathbb{R}$. Recall that $L_1 = \left\{g\colon \mathbb{R}\to\mathbb{R} \mid g \text{ is mesurable and } \int_{-\infty}^{\infty}|g(t)|\,dt < \infty\right\}$. Then, we consider the set $E'_\mathbb{R}$ of all measurable real-valued functions ψ on $\mathbb{R}$ such that

$$\varphi\psi \in L_1, \quad \int_{-\infty}^{\infty}|\varphi(t)\psi(t)|\,dt \le k \quad \text{for all } \varphi \in S(E_\mathbb{R}),$$

where k depends only on ψ. Then, $E'_\mathbb{R}$ is a normed space with the norm given by (see [13, Chapt. 2, 22.M])

$$\|\psi\|_{E'_\mathbb{R}} := \sup\left\{\int_{-\infty}^{\infty}|\varphi(t)\psi(t)|\,dt : \varphi \in S(E_\mathbb{R})\right\} \quad \text{for } \psi \in E'_\mathbb{R}.$$

We call $E'_\mathbb{R}$ the *associate space* of $E_\mathbb{R}$.

Remark 1.8. Let $E_\mathbb{R}$ be an admissible Banach function space and $E'_\mathbb{R}$ be its associate space. Then, from [13, Chapt. 2, 22.M] we also have that the following "Hölder's inequality" holds:

$$\int_{-\infty}^{\infty}|\varphi(t)\psi(t)|\,dt \le \|\varphi\|_{E_\mathbb{R}}\|\psi\|_{E'_\mathbb{R}} \quad \text{for all } \varphi \in E_\mathbb{R},\ \psi \in E'_\mathbb{R}. \tag{1.5}$$

In order to show the existence and attractiveness of an $\mathcal{E}$-class-unstable manifold, we suppose the following hypothesis.

Hypothesis 1.9. We will consider the admissible Banach function space $E_\mathbb{R}$ such that its associate space $E'_\mathbb{R}$ is also an admissible Banach function space. Moreover, for such an admissible Banach function space $E_\mathbb{R}$ we suppose that $E'_\mathbb{R}$ contains an *exponentially $E_\mathbb{R}$-invariant* function, that is the function $\varphi \ge 0$ having the property that, for any fixed $\nu > 0$ the function h_ν defined by

$$h_\nu(t) := \left\|e^{-\nu|t-\cdot|}\varphi(\cdot)\right\|_{E'_\mathbb{R}} \quad \text{for } t \in \mathbb{R}$$

belongs to $E_\mathbb{R}$.

We also give here some examples of admissible Banach function spaces and their associate spaces which satisfy the above Hypothesis with an exponentially $E_{\mathbb{R}}$-invariant function $\varphi(t) = ce^{-\alpha|t|}$ for $t \in \mathbb{R}$ and any fixed constants $c, \alpha > 0$.

Example 1.10. The associate space of L_p is $L'_p = L_q$ for $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p \le \infty$, here as usual we take the conventions that $q = \infty$ if $p = 1$, and $q = 1$ if $p = \infty$.

Besides the functions of the form $\varphi(t) = ce^{-\alpha|t|}$, one can see that the functions of the form $\varphi = c\chi_{[a,b]}$ for any fixed constant $c > 0$ and any finite interval $[a,b] \subset \mathbb{R}$ are also exponentially L_p-invariant functions.

2. Unstable manifolds of $\mathcal{E}$-class and their attraction property

In this section we consider the existence and attraction property of an unstable manifold of $\mathcal{E}$-class for evolution equations defined on the whole line $\mathbb{R}$ under the conditions that the evolution family $(U(t,s))_{t\ge s}$ has an exponential dichotomy on the whole line and the nonlinear term $f(t,x)$ is φ-Lipschitz. To this purpose, we first recall the concept of exponential dichotomy and some other notions defined on the whole line.

Definition 2.1. An evolution family $(U(t,s))_{t\ge s}$ on the Banach space X is said to have an *exponential dichotomy* on $\mathbb{R}$ if there exist bounded linear projections $P(t)$, $t \in \mathbb{R}$, on X and positive constants N, ν such that

(a) $U(t,s)P(s) = P(t)U(t,s)$, $t \ge s$,

(b) the restriction $U(t,s)_|\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$, $t \ge s$, is an isomorphism (and we denote its inverse by $(U(t,s)_|)^{-1} = U(s,t)_|$ for $t \ge s$),

(c) $\|U(t,s)x\| \le Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X$, $t \ge s$,

(d) $\left\|U(s,t)_|x\right\| \le Ne^{-\nu(t-s)}\|x\|$ for $x \in \operatorname{Ker} P(t)$, $t \ge s$.

For an evolution family $(U(t,s))_{t\ge s}$ having an exponential dichotomy on the whole line, we can define the Green's function on $\mathbb{R}$ as follows:

$$\mathcal{G}(t,\tau) = \begin{cases} P(t)U(t,\tau) & \text{for } t \ge \tau, \\ -U(t,\tau)_|(I - P(\tau)) & \text{for } t < \tau. \end{cases} \tag{2.1}$$

Thus, we have

$$\|\mathcal{G}(t,\tau)\| \le N(1+H)e^{-\nu|t-\tau|} \quad \text{for all } t \ne \tau$$

where $H := \sup_{t\in\mathbb{R}} \|P(t)\| < \infty$.

We also need the following notion of φ-Lipschitz functions.

Definition 2.2. Let $E_{\mathbb{R}}$ be an admissible Banach function space and φ be a positive function belonging to $E_{\mathbb{R}}$. A function $f\colon \mathbb{R}\times X \to X$ is said to be *φ-Lipschitz* if f satisfies

(i) $f(t,0) = 0$ for a.e. $t \in \mathbb{R}$,

(ii) $\|f(t,x_1) - f(t,x_2)\| \le \varphi(t)\,\|x_1 - x_2\|$ for a.e. $t \in \mathbb{R}$ and all x_1, $x_2 \in X$.

Note that if $f\colon \mathbb{R}\times X \to X$ (with $f(t,0) = 0$) satisfies (ii) for x_1, x_2 belonging to a ball $B_\rho := \{x \in X : \|x\| \le \rho\}$ for some fixed $\rho > 0$, then f is called *locally φ-Lipschitz*.

In this section, we consider the mild solutions of Eq. (1.1), that is the solutions to the following integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))\,d\xi \quad \text{for } t \ge s. \tag{2.2}$$

We now give the definition of an unstable manifold of $\mathcal{E}$-class.

Definition 2.3. A set $\boldsymbol{U} \subset \mathbb{R}\times X$ is said to be an *invariant unstable manifold of $\mathcal{E}$-class* (or *$\mathcal{E}$-class-unstable manifold*) for the solutions to Eq. (2.2) if for every $t \in \mathbb{R}$ the phase spaces X splits into a direct sum $X = X_0(t) \oplus X_1(t)$ such that

$$\inf_{t\in\mathbb{R}} Sn(X_0(t),X_1(t)) := \inf_{t\in\mathbb{R}}\inf_{i=0,1}\{\|x_0 + x_1\| : x_i \in X_i(t),\ \|x_i\| = 1\} > 0$$

and there exists a family of Lipschitz continuous mappings

$$g_t\colon X_1(t) \to X_0(t), \quad t \in \mathbb{R},$$

with the Lipschitz constants being independent of t such that

(i) $\boldsymbol{U} = \{(t, x + g_t(x)) \in \mathbb{R}\times(X_1(t)\oplus X_0(t)) \mid t \in \mathbb{R},\, x \in X_1(t)\}$, and we denote by $\boldsymbol{U}_t = \{x + g_t(x) : (t, x + g_t(x)) \in \boldsymbol{U}\}$ called the surface of the manifold $\boldsymbol{U}$ at time t.

(ii) $\boldsymbol{U}_t$ is homemorphic to $X_1(t)$ for all $t \in \mathbb{R}$.

(iii) To each $x_0 \in \boldsymbol{U}_{t_0}$ there corresponds one and only one solution $u(\cdot)$ to Eq. (2.2) on $(-\infty, t_0]$ satisfying conditions that $u(t_0) = x_0$ and the function $\chi_{(-\infty,t_0]}u$ (i.e., $(\chi_{(-\infty,t_0]}u)(t) = \chi_{(-\infty,t_0]}(t)u(t)$ for all $t \in \mathbb{R}$) belongs to $\mathcal{E}$.

(iv) $\boldsymbol{U}$ is invariant under Eq. (2.2) in the sense that, if $u(\cdot) \in \mathcal{E}$ is a solution of Eq. (2.2) satisfying condition $u(t_0) = x_0 \in \boldsymbol{U}_{t_0}$ for some $t_0 \in \mathbb{R}$ then $u(t) \in \boldsymbol{U}_t$ for all $t \in \mathbb{R}$.

The existence of an invariant unstable manifold of $\mathcal{E}$-class for the solutions of Eq. (2.2) has been essentially proved in [12, Theorem 6.11]. We give the proof here for sake of the completeness. To do this, we need the following lemma (whose proof can be found in [12]) giving the structure of solutions to Eq. (2.2) which belong to $\mathcal{E}$.

Lemma 2.4. [12, Lemm. 6.8] *Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding constants N, ν and projections $(P(t))_{t\in\mathbb{R}}$. Let $E_{\mathbb{R}}$ and $E'_{\mathbb{R}}$ be respectively an admissible Banach function space and its associate space. Suppose that $\varphi \in E'_{\mathbb{R}}$ be an exponentially $E_{\mathbb{R}}$-invariant function defined as in Hypothesis* 1.9. *Let $f\colon \mathbb{R}\times X \to X$ be φ-Lipschitz. Let $u(\cdot)$ be a solution to Eq.* (2.2) *such that for a fixed t_0 the function $\chi_{(-\infty,t_0]}u$ belongs to $\mathcal{E}$. Then, for $t \leq t_0$ we have that $u(\cdot)$ can be rewritten in the form*

$$u(t) = U(t,t_0)_{|}v_1 + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau) f(\tau,u(\tau))\,d\tau \tag{2.3}$$

where $v_1 \in X_1(t_0) = (I - P(t_0))X$ and $\mathcal{G}(t,\tau)$ is the Green's function defined by Formula (2.1).

Remark 2.5. By computing directly, we can see that the converse of Lemma 2.4 is also true. It means that all solutions of Eq. (2.3) also satisfy Eq. (2.2) for all $t \leq t_0$.

We also need the following theorem for the construction of an unstable manifold of $\mathcal{E}$-class.

Theorem 2.6. [12, Lemm. 6.10] *Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding projections $(P(t))_{t\in\mathbb{R}}$ and the dichotomy constants N, $\nu > 0$. Let $E_{\mathbb{R}}$ and $E'_{\mathbb{R}}$ be respectively an admissible Banach function space and its associate space. Suppose that $\varphi \in E'_{\mathbb{R}}$ be an exponentially $E_{\mathbb{R}}$-invariant function defined as in Hypothesis* 1.9. *Define the function h_ν by $h_\nu(t) := \left\| e^{-\nu|t-\cdot|}\varphi(\cdot)\right\|_{E'_{\mathbb{R}}}$ for $t \in \mathbb{R}$. Then, if the function f is φ-Lipschitz for φ satisfying $N(1+H)\left\|h_\nu\right\|_{E_{\mathbb{R}}} < 1$ then there corresponds to each $v_1 \in X_1(t_0)$ one and only one solution $u(\cdot)$ of Eq.* (2.2) *on $(-\infty,t_0]$ satisfying $(I-P(t_0))u(t_0) = v_1$ and $\chi_{(-\infty,t_0]}u \in \mathcal{E}$. Moreover, for any two solutions $u_1(\cdot)$ and $u_2(\cdot)$ corresponding to $v_1, v_2 \in X_1(t_0)$ we have estimate*

$$\|u_1(t) - u_2(t)\| \leq C_\mu e^{-\mu(t_0-t)}\|v_1 - v_2\| \quad \text{for } t \leq t_0, \tag{2.4}$$

where μ and C_μ are positive constants independent of t_0, u_1 and u_2.

We then state the theorem on the existence of an unstable manifold of $\mathcal{E}$-class.

Theorem 2.7. *Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding projections $(P(t))_{t\in\mathbb{R}}$ and the dichotomy constants N, $\nu > 0$. Let $E_{\mathbb{R}}$ and $E'_{\mathbb{R}}$ be respectively an admissible Banach function space and its associate space. Suppose that $\varphi \in E'_{\mathbb{R}}$ be an exponentially $E_{\mathbb{R}}$-invariant function defined as in Hypothesis* 1.9. *Define the functions e_ν and h_ν by $e_\nu(t) := e^{-\nu|t|}$ and $h_\nu(t) := \left\| e^{-\nu|t-\cdot|}\varphi(\cdot)\right\|_{E'_{\mathbb{R}}}$ for $t \in \mathbb{R}$. Then, if the function f is φ-Lipschitz for φ satisfying*

$$N^2 N_1(1+H)\left\|e_\nu\right\|_{E_{\mathbb{R}}}\left\|\varphi\right\|_{E'_{\mathbb{R}}} + N(1+H)\left\|h_\nu\right\|_{E_{\mathbb{R}}} < 1,$$

then there exists an invariant unstable manifold $\boldsymbol{U}$ *of* $\mathcal{E}$*-class for the solutions of Eq.* (2.2)*. Moreover, every two solutions* $u_1(\cdot), u_2(\cdot)$ *on the manifold* $\boldsymbol{U}$ *attract each other backwardly and exponentially in the sense that they satisfy the following estimate*

$$\|u_1(t) - u_2(t)\| \le C_\mu e^{-\mu(t_0-t)} \|(I - P(t_0))(u_1(t_0) - u_2(t_0))\| \quad \textit{for } t \le t_0 \tag{2.5}$$

where μ, C_μ *are positive constants independent of* t_0 *and* u_1, u_2*.*

Proof. The proof of this theorem has been essentially done in [12, Theorem 6.11]. We give it here for sake of the completeness.

Since the evolution family $(U(t,s))_{t\ge s}$ has an exponential dichotomy, for each $t \in \mathbb{R}$ we have that the phase space X splits into the direct sum $X = X_0(t) \oplus X_1(t)$, where $X_0(t) = P(t)X$ and $X_1(t) = \operatorname{Ker} P(t)$. Moreover, since $\sup_{t\in\mathbb{R}} \|P(t)\| < \infty$ we obtain that

$$\inf_{t\in\mathbb{R}} Sn(X_0(t), X_1(t)) := \inf_{t\in\mathbb{R}} \inf_{i=0,1} \{\|x_0 + x_1\| : x_i \in X_i(t),\ \|x_i\| = 1\} > 0.$$

We now construct the family of Lipschitz mappings $(g_t)_{t\in\mathbb{R}}$ satisfying the conditions of Definition 2.3. For each $t_0 \in \mathbb{R}$, we define $g_{t_0}\colon X_1(t_0) \to X_0(t_0)$ as follows:

$$g_{t_0}(y) = \int_{-\infty}^{t_0} \mathcal{G}(t_0,\tau) f(\tau, x(\tau))\, d\tau,$$

where $x(\cdot)$ is the unique solution on $(-\infty, t_0]$ of Eq. (2.2) such that $\chi_{(-\infty,t_0]}x \in \mathcal{E}$ and $(\operatorname{Id} - P(t_0))x(t_0) = y$. The existence of the solution $x(\cdot)$ is asserted in Theorem 2.6. By definition of Green's function (see (2.1)), we obtain that $g_{t_0}(y) \in X_0(t_0)$. Next, we prove g_{t_0} is Lipschitz mapping. In fact, we have

$$\begin{aligned}
\|g_{t_0}(y_1) - g_{t_0}(y_2)\| &\le \int_{-\infty}^{t_0} \|\mathcal{G}(t_0,\tau)\| \, \|f(\tau,x_1(\tau)) - f(\tau, x_2(\tau))\|\, d\tau \\
&\le N(1+H) \int_{-\infty}^{t_0} e^{-\nu|t-\tau|} \varphi(\tau) \|x_1(\tau) - x_2(\tau)\|\, d\tau \\
&\le N(1+H) \int_{-\infty}^{t_0} \varphi(\tau) \|x_1(\tau) - x_2(\tau)\|\, d\tau \\
&\overset{\text{by (1.5)}}{\le} N(1+H) \|\varphi\|_{E'_{\mathbb{R}}} \|x_1 - x_2\|_{E_{\mathbb{R}}}.
\end{aligned} \tag{2.6}$$

On the other hand, we also have

$$\begin{aligned}
\|x_1(t) - x_2(t)\| &= \left\| U(t,t_0)_|(y_1 - y_2) + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)(f(\tau,x_1(\tau)) - f(\tau,x_2(\tau)))\, d\tau \right\| \\
&\le N e^{-\nu(t_0-t)} \|y_1 - y_2\| + N(1+H) \int_{-\infty}^{t_0} e^{-\nu|t-\tau|} \varphi(\tau) \|x_1(\tau) - x_2(\tau)\|\, d\tau
\end{aligned}$$

for $t \le t_0$. This yields that

$$\|x_1(t) - x_2(t)\| \le N(T^+_{t_0} e_\nu)(t) \|y_1 - y_2\| + N(1+H) h_\nu(t) \|x_1 - x_2\|_{E_{\mathbb{R}}}.$$

Hence,

$$\|x_1 - x_2\|_{E_{\mathbb{R}}} \le NN_1 \|e_\nu\|_{E_{\mathbb{R}}} \|y_1 - y_2\| + N(1+H) \|h_\nu\|_{E_{\mathbb{R}}} \|x_1 - x_2\|_{E_{\mathbb{R}}}.$$

Therefore,

$$\|x_1 - x_2\|_{E_{\mathbb{R}}} \le \frac{NN_1 \|e_\nu\|_{E_{\mathbb{R}}}}{1 - N(1+H) \|h_\nu\|_{E_{\mathbb{R}}}} \|y_1 - y_2\|.$$

Substituting this inequality into (2.6) we obtain that

$$\|g_{t_0}(y_1) - g_{t_0}(y_2)\| \le \frac{N^2 N_1 (1+H) \|e_\nu\|_{E_{\mathbb{R}}} \|\varphi\|_{E'_{\mathbb{R}}}}{1 - N(1+H) \|h_\nu\|_{E_{\mathbb{R}}}} \|y_1 - y_2\|.$$

Thus, g_{t_0} is Lipschitz mapping with the Lipschitz constant $\frac{N^2 N_1 (1+H) \|e_\nu\|_{E_{\mathbb{R}}} \|\varphi\|_{E'_{\mathbb{R}}}}{1 - N(1+H) \|h_\nu\|_{E_{\mathbb{R}}}}$ independent of t_0.

Put $\boldsymbol{U} = \{(t, y + g_t(y)) \in \mathbb{R} \times (X_0(t) \oplus X_1(t)) \mid t \in \mathbb{R},\ y \in X_1(t)\}$. Since the Lipschitz constant $\frac{N^2 N_1 (1+H) \|e_\nu\|_{E_{\mathbb{R}}} \|\varphi\|_{E'_{\mathbb{R}}}}{1 - N(1+H) \|h_\nu\|_{E_{\mathbb{R}}}} < 1$ (or equivalently $N^2 N_1 (1+H) \|e_\nu\|_{E_{\mathbb{R}}} \|\varphi\|_{E'_{\mathbb{R}}} + N(1+H) \|h_\nu\|_{E_{\mathbb{R}}} < 1$), we obtain that $\boldsymbol{U}_t$ is homeomorphic to $X_1(t)$. The condition (iii) in Definition 2.3 now follows from Theorem 2.6. Next, we show that $\boldsymbol{U}$ is invariant. Let $x(\cdot)$ be a solution in $\mathcal{E}$ of Eq. (2.2) such that $x(t_0) = x_0 \in \boldsymbol{U}_{t_0}$. We will prove that $x(s) \in \boldsymbol{U}_s$ for all $s \in \mathbb{R}$.

Firstly, for $s \le t_0$, by Lemma 2.4 we have that

$$x(s) = U(s, t_0)_| v_1 + \int_{-\infty}^{t_0} \mathcal{G}(s, \tau) f(\tau, x(\tau))\, d\tau.$$

Put $w_s = U(s, t_0)_| v_1 + \int_s^{t_0} \mathcal{G}(s, \tau) f(\tau, x(\tau))\, d\tau$. We obtain that $w_s \in \operatorname{Ker} P(s)$ and

$$x(s) = w_s + \int_{-\infty}^{s} \mathcal{G}(s, \tau) f(\tau, x(\tau))\, d\tau.$$

On the other hand, for $t \le s$ we have that

$$\begin{aligned}
&U(t, s)_| w_s + \int_{-\infty}^{s} \mathcal{G}(t, \tau) f(\tau, x(\tau))\, d\tau \\
&= U(t, s)_| U(s, t_0)_| v_1 + U(t, s)_| \int_s^{t_0} \mathcal{G}(s, \tau) f(\tau, x(\tau))\, d\tau + \int_{-\infty}^{s} \mathcal{G}(t, \tau) f(\tau, x(\tau))\, d\tau \\
&= U(t, t_0)_| v_1 + \int_s^{t_0} \mathcal{G}(t, \tau) f(\tau, x(\tau))\, d\tau + \int_{-\infty}^{s} \mathcal{G}(t, \tau) f(\tau, x(\tau))\, d\tau \\
&= U(t, t_0)_| v_1 + \int_{-\infty}^{t_0} \mathcal{G}(t, \tau) f(\tau, x(\tau))\, d\tau = x(t).
\end{aligned}$$

Thus, $x(s) = w_s + g_s(w_s)$. This leads to $x(s) \in \boldsymbol{U}_s$ for all $s \le t_0$.

Secondly, for $s > t_0$, we assume that $v(\cdot)$ is solution of Eq. (2.2) on $[t_0, s]$ such that $v(t_0) = x(t_0)$. We put

$$w(t) = \begin{cases} x(t) & \text{if } t \le t_0, \\ v(t) & \text{if } t \in [t_0, s]. \end{cases}$$

For $t \in [t_0, s]$ we have

$$\begin{aligned} w(t) = v(t) &= U(t,t_0)v(t_0) + \int_{t_0}^{t} U(t,\tau)f(\tau, v(\tau))\,d\tau \\ &= U(t,t_0)\left(v_1 + \int_{-\infty}^{t_0} \mathcal{G}(t_0,\tau)f(\tau,x(\tau))\,d\tau\right) + \int_{t_0}^{t} U(t,\tau)f(\tau,v(\tau))\,d\tau \\ &= U(t,t_0)v_1 + \int_{-\infty}^{t_0} U(t,\tau)P(\tau)f(\tau,x(\tau))\,d\tau + \int_{t_0}^{t} U(t,\tau)P(\tau)f(\tau,v(\tau))\,d\tau \\ &\quad + \int_{t_0}^{t} U(t,\tau)(I-P(\tau))f(\tau,v(\tau))\,d\tau \\ &= U(t,t_0)v_1 + \int_{t_0}^{t} U(t,\tau)(I-P(\tau))f(\tau,w(\tau))\,d\tau + \int_{-\infty}^{t} \mathcal{G}(t,\tau)f(\tau,w(\tau))\,d\tau. \end{aligned}$$

Putting $\nu_2 = U(s,t_0)v_1 + \displaystyle\int_{t_0}^{s} U(s,\tau)(I-P(\tau))f(\tau,w(\tau))\,d\tau \in \operatorname{Ker} P(s)$ we obtain

$$\begin{aligned} U(t,s)_{|}\nu_2 = U(t,t_0)v_1 &+ \int_{t_0}^{t} U(t,\tau)(I-P(\tau))f(\tau,w(\tau))\,d\tau \\ &+ \int_{t}^{s} U(t,\tau)_{|}(I-P(\tau))f(\tau,w(\tau))\,d\tau. \end{aligned}$$

Thus,

$$\begin{aligned} w(t) &= U(t,s)_{|}\nu_2 - \int_{t}^{s} U(t,\tau)_{|}(I-P(\tau))f(\tau,w(\tau))\,d\tau + \int_{-\infty}^{t} \mathcal{G}(t,\tau)f(\tau,w(\tau))\,d\tau \\ &= U(t,s)_{|}\nu_2 + \int_{-\infty}^{s} \mathcal{G}(t,\tau)f(\tau,w(\tau))\,d\tau. \end{aligned}$$

For $t \le t_0$ we arrive at

$$\begin{aligned} w(t) = x(t) &= U(t,t_0)_{|}v_1 + \int_{-\infty}^{t_0} \mathcal{G}(t,\tau)f(\tau,x(\tau))\,d\tau \\ &= U(t,t_0)_{|}v_1 + \int_{-\infty}^{t} U(t,\tau)P(\tau)f(\tau,x(\tau))\,d\tau - \int_{t}^{t_0} U(t,\tau)_{|}(I-P(\tau))f(\tau,x(\tau))\,d\tau \\ &= U(t,t_0)_{|}v_1 - \int_{t}^{t_0} U(t,\tau)_{|}(I-P(\tau))f(\tau,w(\tau))\,d\tau + \int_{-\infty}^{t} \mathcal{G}(t,\tau)f(\tau,w(\tau))\,d\tau \\ &= U(t,s)_{|}\nu_2 - \int_{t}^{s} U(t,\tau)_{|}(I-P(\tau))f(\tau,w(\tau))\,d\tau + \int_{-\infty}^{t} \mathcal{G}(t,\tau)f(\tau,w(\tau))\,d\tau \\ &= U(t,s)_{|}\nu_2 + \int_{-\infty}^{s} \mathcal{G}(t,\tau)f(\tau,w(\tau))\,d\tau. \end{aligned}$$

Therefore, for all $t \le s$ there exists $\nu_2 \in \operatorname{Ker} P(s)$ such that

$$w(t) = U(t,s)_{|}\nu_2 + \int_{-\infty}^{s} \mathcal{G}(t,\tau) f(\tau, w(\tau))\, d\tau.$$

This yields $w(s) \in \boldsymbol{U}_s$ and thus $x(s) = w(s) \in \boldsymbol{U}_s$ for all $s > t_0$. □

Remark 2.8. If the nonlinear term satisfies the local φ-Lipschitz condition in a ball $B_\rho := \{x \in X : \|x\| \le \rho\}$, then in a similar way as above, we obtain the existence of a local unstable manifold of $\mathcal{E}$-class for solutions staying in that ball as $t \to -\infty$ (see [12, Theorem 6.5]).

We now come to our main result of this section, that is the attraction property of an invariant unstable manifold of $\mathcal{E}$-class for the solutions to (2.2). Concretely, we will show that the $\mathcal{E}$-class-unstable manifold $\boldsymbol{U} = \{(t, \boldsymbol{U}_t)\}_{t\in\mathbb{R}}$ exponentially attracts all solutions to Eq. (2.2) in the sense that any solution $u(\cdot)$ to Eq. (2.2) is exponentially attracted to some induced trajectory $u^*(\cdot)$ lying in the $\mathcal{E}$-class-unstable manifold. Precisely, we will prove the following theorem.

Theorem 2.9. *Assume that conditions of Theorem* 2.7 *is satisfied. For each fixed* $\alpha > 0$ *with* $\alpha < \nu$ *we define the functions* $e_{\nu-\alpha}(t) = e^{-(\nu-\alpha)|t|}$, $h_{\nu-\alpha}(t) = \left\| e^{-(\nu-\alpha)|t-\cdot|}\varphi(\cdot)\right\|_{E_\mathbb{R}}$ *for* $t \in \mathbb{R}$. *Assume that* $l < 1$ *where*

$$l := N(1+H)\max\left\{ N_1 q \left\|e_{\nu-\alpha}\right\|_{E_\mathbb{R}} + \left\|h_{\nu-\alpha}\right\|_{E_\mathbb{R}}, q + \frac{(N_1+N_2)\left\|\Lambda_1\varphi\right\|_\infty}{1-e^{-(\nu-\alpha)}} \right\}$$

for $q := \frac{N^2 N_1 (1+H)\|e_\nu\|_{E_\mathbb{R}} \|\varphi\|_{E'_\mathbb{R}}}{1-N(1+H)\|h_\nu\|_{E_\mathbb{R}}}$. *Then, the* $\mathcal{E}$*-class-unstable manifold* $\boldsymbol{U} = \{\boldsymbol{U}_t\}_{t\in\mathbb{R}}$ *exponentially attracts all solutions of Eq.* (2.2) *in the sense that for any solution* $u(\cdot)$ *to Eq.* (2.2) *with the initial value* $u(\xi)$ *there exist a solution* $u^*(\cdot)$ *lying in* $\boldsymbol{U}$ *(i.e.,* $u^*(t) \in \boldsymbol{U}_t$ *for all* $t \in \mathbb{R}$*) such that*

$$\|u(t) - u^*(t)\| \le Ce^{-\alpha(t-\xi)} \left\| g_\xi((I-P(\xi))u(\xi)) - P(\xi)u(\xi)\right\| \quad \text{for a.e. } t \ge \xi.$$

Proof. For fixed $\xi \in \mathbb{R}$ we introduce the space

$$E_{\xi,\alpha} = \left\{ w \in \mathcal{E} \text{ such that } w(t) = 0 \text{ for } t < \xi \text{ and } e^{\alpha(t-\xi)}\|w(t)\| \in E_\mathbb{R} \cap L_\infty(\mathbb{R}) \right\}$$

which is a Banach space endowed with the norm $|w|_\alpha = \max\{\left\| e^{\alpha(t-\xi)}\|w(t)\|\right\|_{E_\mathbb{R}}, \|e^{\alpha(t-\xi)} w\|_\infty\}$ with $\left\|e^{\alpha(t-\xi)}w\right\|_\infty = \operatorname{ess\,sup}_{t\ge\xi} e^{\alpha(t-\xi)}\|w(t)\|$. We will find $u^*(\cdot)$ in the form $u^*(t) = u(t) + w(t)$ such that $w \in E_{\xi,\alpha}$.

We see that $u^*(\cdot)$ is a solution of Eq. (2.2) if and only if $w(\cdot)$ is a solution of the equation

$$w(t) = U(t,\xi)w(\xi) + \int_\xi^t U(t,\tau)\left[f(\tau, u(\tau)+w(\tau)) - f(\tau, u(\tau))\right] d\tau.$$

To simplify the representation we put $F(t,w(t)) = f(t,u(t)+w(t)) - f(t,u(t))$. Then, $F\colon \mathbb{R}\times X \to X$ is also φ-Lipschitz and $F(t,0)=0$. The equation for $w(t)$ can be rewritten as

$$w(t) = U(t,\xi)w(\xi) + \int_\xi^t U(t,\tau)F(\tau,w(\tau))\,d\tau. \tag{2.7}$$

By [9, Lemma 4.3] and [9, Remark 4.4], we observe that the solution $w(t)$ of Eq. (2.7), which is defined on $[\xi,\infty)$ (here $w(t)=0$ for $t<\xi$), belongs to $\mathcal{E}$ if and only if it satisfies

$$w(t) = U(t,\xi)\nu_0 + \int_\xi^\infty \mathcal{G}(t,\tau)F(\tau,w(\tau))\,d\tau \tag{2.8}$$

for some $\nu_0 \in \operatorname{Im} P(\xi)$ and $t\geq\xi$. We will choose $\nu_0\in\operatorname{Im} P(\xi)$ such that $u^*(\xi) = u(\xi)+w(\xi)\in \boldsymbol{U}_\xi$. This means

$$P(\xi)(u(\xi)+w(\xi)) = g_\xi((I-P(\xi))(u(\xi)+w(\xi))).$$

Hence,

$$\nu_0 = P(\xi)w(\xi) = -P(\xi)u(\xi) + g_\xi((I-P(\xi))(u(\xi)+w(\xi))). \tag{2.9}$$

Substituting (2.9) into (2.8) we obtain that

$$\begin{aligned} w(t) &= U(t,\xi)\left[-P(\xi)u(\xi) + g_\xi((I-P(\xi))(u(\xi)+w(\xi)))\right] \\ &\quad + \int_\xi^\infty \mathcal{G}(t,\tau)F(\tau,w(\tau))\,d\tau \end{aligned} \tag{2.10}$$

for $t\geq\xi$. Thus, $u^*(t)$ is solution of Eq. (2.2) and satisfies $u^*(\xi)\in\boldsymbol{U}_\xi$ if $w(t)$ is solution of Eq. (2.10).

Next, in order to prove the existence of $u^*(t)$ satisfying assertions of the theorem, we will find solution $w(t)$ of the Eq. (2.10) in the Banach space $E_{\xi,\alpha}$. To this purpose, for $w\in E_{\xi,\alpha}$ we define the mapping T as follows

$$(Tw)(t) = \begin{cases} U(t,\xi)\left[-P(\xi)u(\xi) + g_\xi((I-P(\xi))(u(\xi)+w(\xi)))\right] & \\ + \displaystyle\int_\xi^\infty \mathcal{G}(t,\tau)F(\tau,w(\tau))\,d\tau & \text{for } t\geq\xi, \\ 0 & \text{for } t<\xi. \end{cases}$$

Firstly, we show that $Tw\in E_{\xi,\alpha}$. Indeed, for $t\geq\xi$

$$\begin{aligned} e^{\alpha(t-\xi)}\left\|(Tw)(t)\right\| &\leq N\left\|\nu_0\right\| + N(1+H)e^{\alpha(t-\xi)}\int_\xi^\infty e^{-\nu|t-\tau|}\varphi(\tau)\left\|w(\tau)\right\|\,d\tau \\ &\leq N\left\|\nu_0\right\| + N(1+H)\int_\xi^\infty e^{-(\nu-\alpha)|t-\tau|}\varphi(\tau)e^{\alpha(\tau-\xi)}\left\|w(\tau)\right\|\,d\tau \\ &\leq N\left\|\nu_0\right\| + \frac{N(1+H)(N_1+N_2)\left\|\Lambda_1\varphi\right\|_\infty}{1-e^{-(\nu-\alpha)}}\left\|w\right\|_\infty. \end{aligned}$$

Therefore, $e^{\alpha(t-\xi)}\left\|(Tw)(t)\right\| \in L_\infty(\mathbb{R})$. On the other hand, we also have

$$\begin{aligned} e^{\alpha(t-\xi)}\left\|(Tw)(t)\right\| &\le Ne^{-(\nu-\alpha)(t-\xi)}\left\|\nu_0\right\| \\ &\quad + N(1+H)\int_\xi^\infty e^{-(\nu-\alpha)|t-\tau|}\varphi(\tau)e^{\alpha(\tau-\xi)}\left\|w(\tau)\right\| d\tau \\ &\le Ne^{-(\nu-\alpha)(t-\xi)}\left\|\nu_0\right\| + N(1+H)h_{\nu-\alpha}(t)\left\|e^{\alpha(\tau-\xi)}\left\|w(\tau)\right\|\right\|_{E_\mathbb{R}}. \end{aligned}$$

Thus, $e^{\alpha(t-\xi)}\left\|(Tw)(t)\right\| \in E_\mathbb{R}$ since the functions $e^{-(\nu-\alpha)(t-\xi)}, h_{\nu-\alpha}(t) \in E_\mathbb{R}$ and property of Banach lattice of $E_\mathbb{R}$. Hence, $e^{\alpha(t-\xi)}\left\|(Tw)(t)\right\| \in E_\mathbb{R}\cap L_\infty(\mathbb{R})$. This leads to $Tw \in E_{\xi,\alpha}$. By Lipschitz continuity of g_ξ we have

$$\begin{aligned} \left\|\nu_0\right\| &\le \left\|g_\xi((I-P(\xi))u(\xi)) - P(\xi)u(\xi)\right\| \\ &\quad + \left\|g_\xi((I-P(\xi))(u(\xi)+w(\xi))) - g_\xi((I-P(\xi))u(\xi))\right\| \\ &\le \left\|g_\xi((I-P(\xi))u(\xi)) - P(\xi)u(\xi)\right\| + q\left\|(I-P(\xi))w(\xi)\right\| \\ &\le \left\|g_\xi((I-P(\xi))u(\xi)) - P(\xi)u(\xi)\right\| + q(1+H)|w|_\alpha. \end{aligned}$$

Therefore,

$$|Tw|_\alpha \le \max\left\{N, NN_1\left\|e_{\nu-\alpha}\right\|_{E_\mathbb{R}}\right\}\left\|g_\xi((I-P(\xi))u(\xi)) - P(\xi)u(\xi)\right\| + l|w|_\alpha \tag{2.11}$$

for $l := \max\{NN_1q(1+H)\left\|e_{\nu-\alpha}\right\|_{E_\mathbb{R}} + N(1+H)\left\|h_{\nu-\alpha}\right\|_{E_\mathbb{R}}, Nq(1+H) + \frac{N(1+H)(N_1+N_2)\|\Lambda_1\varphi\|_\infty}{1-e^{-(\nu-\alpha)}}\}$.

Next, we prove that T is a contraction. Indeed, let w, v belong to $E_{\xi,\alpha}$. Then

$$\begin{aligned} &e^{\alpha(t-\xi)}\left\|(Tw)(t)-(Tv)(t)\right\| \\ &\le Ne^{-(\nu-\alpha)(t-\xi)}\left\|\nu_0-\mu_0\right\| + N(1+H)e^{\alpha(t-\xi)}\int_\xi^\infty e^{-\nu|t-\tau|}\left\|F(\tau,w(\tau))-F(\tau,v(\tau))\right\| d\tau \\ &\le Ne^{-(\nu-\alpha)(t-\xi)}\left\|\nu_0-\mu_0\right\| + N(1+H)\int_\xi^\infty e^{-(\nu-\alpha)|t-\tau|}\varphi(\tau)e^{\alpha(\tau-\xi)}\left\|w(\tau)-v(\tau)\right\| d\tau. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left\|\nu_0-\mu_0\right\| &= \left\|g_\xi((I-P(\xi))(u(\xi)+w(\xi))) - g_\xi((I-P(\xi))(u(\xi)+v(\xi)))\right\| \\ &\le q\left\|(I-P(\xi))(w(\xi)-v(\xi))\right\| \\ &\le q(1+H)\left\|w(\xi)-v(\xi)\right\| \\ &\le q(1+H)|w-v|_\alpha. \end{aligned}$$

Thus,

$$\left\|e^{\alpha(t-\xi)}(Tw-Tv)\right\|_\infty \le Nq(1+H)|w-v|_\alpha + \frac{N(1+H)(N_1+N_2)\left\|\Lambda_1\varphi\right\|_\infty}{1-e^{-(\nu-\alpha)}}|w-v|_\alpha,$$

and

$$\left\| e^{\alpha(t-\xi)} \left\| (Tw)(t) - (Tv)(t) \right\| \right\|_{E_{\mathbb{R}}} \le NN_1 q(1+H) \left\| e_{\nu-\alpha} \right\|_{E_{\mathbb{R}}} |w-v|_{\alpha} + N(1+H) \left\| h_{\nu-\alpha} \right\|_{E_{\mathbb{R}}} |w-v|_{\alpha}.$$

Therefore,

$$|Tw - Tv|_{\alpha} \le l|w-v|_{\alpha}.$$

Since $l < 1$, we obtain that T is a contraction on Banach space $E_{\xi,\alpha}$. Then, the equations $Tw = w$ has a unique solution $w \in E_{\xi,\alpha}$. From (2.11) it follows that

$$|w|_{\alpha} \le \frac{\max\left\{N, NN_1 \left\| e_{\nu-\alpha} \right\|_{E_{\mathbb{R}}}\right\}}{1-l} \left\| g_{\xi}((I-P(\xi))u(\xi)) - P(\xi)u(\xi) \right\|.$$

We have completed proof of the existence of the solution $u^* = u + w$ for Eq. (2.2) satisfies $u^*(t) \in \boldsymbol{U}_t$ for $t \ge \xi$ and

$$\begin{aligned} \|u^*(t) - u(t)\| = \|w(t)\| &\le e^{-\alpha(t-\xi)}|w|_{\alpha} \\ &\le \frac{\max\left\{N, NN_1 \left\| e_{\nu-\alpha} \right\|_{E_{\mathbb{R}}}\right\}}{1-l} e^{-\alpha(t-\xi)} \left\| g_{\xi}((I-P(\xi))u(\xi)) - P(\xi)u(\xi) \right\| \end{aligned}$$

for a.e. $t \ge \xi$. □

3. Fisher-Kolmogorov model with time-dependent environmental capacity

In this section, we will apply the above-obtained results to Fisher-Kolmogorov model with the time-dependent environmental capacity. This model is used to describe the spread of an advantageous gene in a population (see [14, 15]). Precisely, we consider the following problem

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \dfrac{\partial^2}{\partial x^2}u(t,x) + ru(t,x) - \dfrac{r}{K(t)}u^2(t,x) & \text{for } t \ge s,\ x \in [0,\pi], \\ u_x'(t,0) = u_x'(t,\pi) = 0, & t \in \mathbb{R}, \\ u(s,x) = \phi(x), & 0 \le x \le \pi. \end{cases} \tag{3.1}$$

Here, $u(t,x)$ represents the population density at location x and time t; the constant $r > 0$ is the linear reproduction rate, and $K(t) > 0$ is the carrying capacity of the population at time t. We suppose that $r \ne n^2$ for all $n \in \mathbb{N}$.

We then choose the Banach space $X = C[0,\pi]$ and consider the operator $A\colon X \to X$ defined by

$$\begin{aligned} Au &= \frac{\partial^2}{\partial x^2}u + ru, \\ D(A) &= \left\{u \in C^2[0,\pi] : u'(0) = u'(\pi) = 0\right\}. \end{aligned}$$

Therefore, this problem can be rewritten as an abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t,\cdot) = Au(t,\cdot) + F(t,u(t,\cdot)) & \text{for } t \geq s, \\ u(s,\cdot) = \phi(\cdot) \in X \end{cases}$$

where $F\colon \mathbb{R} \times X \to X$ is defined by

$$F(t,\phi)(x) = -\frac{r}{K(t)}\phi^2(x), \quad x \in [0,\pi].$$

It can be seen (see [4, Chap. II, p. 68]) that A generates an analytic semigroup $(e^{tA})_{t\geq 0}$. Since the spectrum of A is of the form $\sigma(A) = \{r, -1^2 + r, -2^2 + r, \ldots, -n^2 + r, \ldots\}$ and $r \neq n^2$ for all $n \in \mathbb{N}$, we have $\sigma(A) \cap i\mathbb{R} = \emptyset$. Therefore, in this case, the spectral mapping theorem for analytic semigroups yields that $(e^{tA})_{t\geq 0}$ is hyperbolic. Hence, the evolution family $(U(t,s))_{t\geq s}$ corresponding to A (i.e., $U(t,s) = e^{(t-s)A}$) has an exponential dichotomy.

Let now u_0 be a bounded (on the whole $\mathbb{R}$) solution of Eq. (3.1). The existence of u_0 is guaranteed by [12, Corollary 6.6]. We shall show the existence of a local unstable manifold of $\mathcal{E}$-class around solution $u_0(t,\cdot)$. To do this, we transform to the case of the trivial solution by setting $v(t,\cdot) = u(t,\cdot) - u_0(t,\cdot)$. We then arrive at the following abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}v(t,\cdot) = \widetilde{A}(t)v(t,\cdot) + F(t,v(t,\cdot)) & \text{for } t \geq s, \\ v(s,\cdot) = \phi(\cdot) - u_0(s,\cdot) \in X, \end{cases} \tag{3.2}$$

where

$$\widetilde{A}(t)f = Af - \frac{2ru_0(t,\cdot)}{K(t)}f = Af + C(t)f \quad \text{and} \quad D(\widetilde{A}(t)) = D(A).$$

We shall prove the existence of a local $\mathcal{E}$-class-unstable manifold around the trivial solution 0 of Equation (3.2). This manifold is exactly the local $\mathcal{E}$-class unstable manifold around solution $u_0(t,\cdot)$ of Equation (3.1).

We note that if $K(t) = K$ is a constant independent of t, and $u_0 = K$ is an equilibrium point of Eq. (3.1), then $\widetilde{A}(t) = A - 2rI$ is independent of t. This operator has the spectrum lying on the left side of the imaginary axis on the complex plane, and it generates an analytic semigroup. Therefore, the evolution family $(U(t,s))_{t\geq s}$ corresponding to $A - 2rI$ is exponentially stable. Therefore, the trivial solution of (3.2) is stable by Lyapunov linearized stability principle. Hence, in this case the solution $u_0 = K$ of Eq. (3.1) is stable.

Surprisingly, when $K(t)$ is time-dependent, we can find the conditions imposed on $K(t)$ such that there exits a local unstable manifold of $\mathcal{E}$-class around the solution $u_0(t,\cdot)$ leading to the instability of that solution. To do this, Remark 2.8 (see also [12, Corollary 6.6]) is actually sufficient for our purpose since the nonlinear function F is locally φ-Lipschitz on any ball $B_\rho := \{v \in X : \|v\| \le \rho\}$ with $\varphi(t) = \frac{2\rho r}{K(t)}$ for any fixed $\rho > 0$. However, we would like to perform here the cut-off technique to obtain a "modified" equation of (3.2) (see Eq. (3.3) below) so that we can apply Theorems (2.7) and (2.9) to the modified equation, and when restricted to the ball B_ρ, the two equations are identical. Concretely, we fix any $\rho > 0$ and define the cut-off mapping on X as follows.

$$G(t,\phi)(x) = \begin{cases} -\dfrac{r}{K(t)}\phi^2(x) & \text{if } \phi \in B_\rho, \\ -\dfrac{r}{K(t)}\dfrac{\phi^2(x)}{\|\phi\|^2} & \text{if } \phi \notin B_\rho. \end{cases}$$

Obviously, $G(t,0) = 0$. We now check that G is φ-Lipschitz with $\varphi(t) = \frac{4\rho r}{K(t)}$ for all $t \in \mathbb{R}$. Indeed, let $v, w \in X$. Then,

- if $v, w \in B_\rho$, we have $\|G(t,v) - G(t,w)\| \le \varphi(t)\,\|v - w\|$.
- if $v, w \notin B_\rho$, assuming without loss of generality that $\|v\| \le \|w\|$, then

$$\begin{aligned} |G(t,v)(x) - G(t,w)(x)| &= \frac{r}{K(t)}\left|\frac{\|v\|^2 (w^2 - v^2) + v^2(\|v\|^2 - \|w\|^2)}{\|v\|^2\,\|w\|^2}\right| \\ &\le \frac{r}{K(t)}\left(2\,\|w - v\| + \frac{(\|v\| + \|w\|)\,\|w - v\|}{\|w\|^2}\right) \\ &\le \varphi(t)\,\|v - w\|\,. \end{aligned}$$

- if $v \in B_\rho$, $w \notin B_\rho$, we have

$$\begin{aligned} |G(t,v)(x) - G(t,w)(x)| &= \frac{r}{K(t)}\left|\frac{v^2 - w^2 + v^2(\|w\|^2 - 1)}{\|w\|^2}\right| \\ &\le \frac{r}{K(t)}\left(2\,\|v - w\| + \frac{(\|w\| + 1)(\|w\| - \|v\|)}{\|w\|^2}\right) \\ &\le \varphi(t)\,\|v - w\|\,. \end{aligned}$$

Thus, G is φ-Lipschitz with $\varphi(t) = \frac{4\rho r}{K(t)}$.

Next, we consider the following abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}v(t,\cdot) = \widetilde{A}(t)v(t,\cdot) + G(t, v(t,\cdot)) & \text{for } t \ge s, \\ v(s,\cdot) = \widetilde{\phi}(\cdot) \in X. \end{cases} \tag{3.3}$$

This equation can be considered as a "modified" equation of Eq. (3.2). Clearly, the solutions of (3.3), which are staying on B_ρ as $t \to -\infty$, are also the solutions of (3.2) and vice versa. That is to say, the intersections of B_ρ with the surfaces $(\boldsymbol{U}_t)_{t\in\mathbb{R}}$ of invariant-unstable manifold of $\mathcal{E}$-class for (3.3) form the surfaces of a local unstable manifold of $\mathcal{E}$-class for (3.2) near the trivial solution.

As seen above, A generates an evolution family having exponential dichotomy with the corresponding dichotomy constants $N', \nu_1 > 0$ and dichotomy projection operator P. Then, using [7, Corollary 5.3] we obtain that if $K(t)$ is continuous and satisfy

$$\sup_{t\in\mathbb{R}} \frac{2r\,\|u_0(t,\cdot)\|}{K(t)} < \frac{\nu_1}{2N'\,\|P\|\,(1+N'+N'\,\|P\|)}, \tag{3.4}$$

then $\widetilde{A}(t)$ generates an evolution family having exponential dichotomy with the corresponding dichotomy constants, say, $N, \nu > 0$ and dichotomy projections $P(t)$.

We now choose $K(t) := be^{\beta|t|}$ for $t \in \mathbb{R}$ with constants $\beta > \nu$, and $b > 0$. Putting $\varphi(t) = \frac{4\rho r}{K(t)}$ for $t \in \mathbb{R}$ we can see that $\varphi \in L_p(\mathbb{R})$ for $1 \le p < +\infty$ and

$$\|\varphi\|_{L_p} = 4b\rho r \left(\int_{-\infty}^{\infty} e^{-p\beta|t|}\,dt\right)^{\frac{1}{p}} = 4b\rho r\left(\frac{2}{\beta p}\right)^{\frac{1}{p}}.$$

Then, the function

$$h_\nu(t) = \left(\int_{-\infty}^{\infty} e^{-p\nu|t-\tau|}\varphi^p(\tau)\,d\tau\right)^{\frac{1}{p}} \quad \text{for } t\in\mathbb{R}$$

is an even function since the function $\varphi(\cdot)$ is even. For $t \ge 0$ we compute

$$\begin{aligned}
h_\nu(t) &= \left(\int_{-\infty}^{\infty} e^{-p\nu|t-\tau|}\varphi^p(\tau)\,d\tau\right)^{\frac{1}{p}} = \left(\int_{-\infty}^{\infty} e^{-p\nu|t-\tau|}(4b\rho r e^{-\beta|t|})^p\,d\tau\right)^{\frac{1}{p}}\\
&= 4b\rho r\left(\int_{-\infty}^{0} e^{-p\nu|t-\tau|}e^{-\beta p|t|}\,d\tau + \int_0^t e^{-p\nu|t-\tau|}e^{-\beta p|t|}\,d\tau\right.\\
&\qquad\left. + \int_t^{+\infty} e^{-p\nu|t-\tau|}e^{-\beta p|t|}\,d\tau\right)^{\frac{1}{p}}\\
&= 4b\rho r\left(\frac{e^{-p\nu t}+e^{-p\beta t}}{p(\beta+\nu)} + \frac{e^{-p\nu t}-e^{-p\beta t}}{p(\beta-\nu)}\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence, using the fact that the function h_ν is even we have

$$h_\nu(t) = 4b\rho r\left(\frac{e^{-p\nu|t|}+e^{-p\beta|t|}}{p(\beta+\nu)} + \frac{e^{-p\nu|t|}-e^{-p\beta|t|}}{p(\beta-\nu)}\right)^{\frac{1}{p}} \quad \text{for } t\in\mathbb{R}.$$

Therefore, $h_\nu \in L_q(\mathbb{R})$ for $\frac{1}{q}+\frac{1}{p} = 1$ and

$$\begin{aligned}
\|h_\nu\|_{L_q} &= \left(\int_{-\infty}^{\infty} (h_\nu(t))^q\,dt\right)^{\frac{1}{q}} \le \left(2\int_0^{\infty} (h_\nu(t))^q\,dt\right)^{\frac{1}{q}}\\
&\le 4r\rho b\left(\frac{2\beta}{p(\beta+\nu)(\beta-\nu)}\right)^{\frac{1}{p}}\left(\frac{4}{\nu q}\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.5}$$

By Theorems 2.7 and 2.9 we see that if

$$N(1+H)4b\rho r\left(\frac{4}{\nu q}\right)^{\frac{1}{q}}\left[NN_1\left(\frac{2}{\beta p}\right)^{\frac{1}{p}}+\left(\frac{2\beta}{p(\beta+\nu)(\beta-\nu)}\right)^{\frac{1}{p}}\right]<1$$

then there is an invariant unstable manifold $\boldsymbol{U}=\{(t,\boldsymbol{U}_t)\}_{t\in\mathbb{R}}$ of L_p-class for mild solutions of Eq. (3.3) and this manifold attracts all mild solutions to Eq. (3.3).

As argued above, the intersection $\{(t,\boldsymbol{U}_t\cap B_\rho)\}_{t\in\mathbb{R}}$ is the local unstable manifold of L_p-class for mild solutions of Eq. (3.1) around solution $u_0(t,\cdot)$ leading to the instability of this solution.

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