TAIWANESE JOURNAL OF MATHEMATICS
Vol. 20, No. 3, pp. 577–587, June 2016
DOI: 10.11650/tjm.20.2016.6291
This paper is available online at http://journal.tms.org.tw

Global Stability of a Nonlocal Epidemic Model with Delay

Liang Zhang and Jian-Wen Sun*

Abstract. In this work, we investigate the nonlocal time-delayed and reaction-diffusion epidemic model studied by Guo et al. [Z. Guo, F. Wang and X. Zou, Threshold dynamics of an infective disease model with a fixed latent period and non-local infections, J. Math. Biol. 65 (2012) 1387–410]. In the case that the coefficients are independent of the spatial variable, we obtain the global stability of the disease-free equilibrium and the unique endemic equilibrium, which partially answers the open problem proposed by Guo et al.

1. Introduction

Incorporating a fixed latency and spatial mobility into a disease, Guo et al. [2] derived a nonlocal and time-delayed reaction-diffusion SIR model

$$
\begin{cases}
\dfrac{\partial u_1(t,x)}{\partial t} = \nabla \cdot [D_1(x)\nabla u_1(t,x)] + \mu(x) \\
\qquad\qquad -d(x)u_1(t,x) - r(x)u_1(t,x)u_2(t,x), & t > 0,\ x \in \Omega, \\
\dfrac{\partial u_2(t,x)}{\partial t} = \nabla \cdot [D_2(x)\nabla u_2(t,x)] - \beta(x)u_2(t,x) \\
\qquad\qquad + \displaystyle\int_\Omega \Gamma(\tau,x,y)r(y)u_1(t-\tau,y)u_2(t-\tau,y)\,dy, & t > 0,\ x \in \Omega, \\
[D_i(x)\nabla u_i(x,t)] \cdot \nu = 0, & t > 0,\ x \in \partial\Omega.
\end{cases}
\tag{1.1}
$$

In (1.1), Ω is a bounded smooth domain, $u_1(t,x)$ and $u_2(t,x)$ represent the density of susceptible and infectious individuals at time t and location x. The functions $\mu(x), d(x)$ and $r(x)$ represent the recruiting rate, the natural death rate and transmission rate of the disease, respectively. $\beta(x) = \sigma_I(x) + \gamma_I(x) + d(x)$, where $\sigma_I(x)$ and $\gamma_I(x)$ represent the disease-induced mortality rate and the recovery rate in the infective period at location x. The discrete delay τ means the fixed latent period of the disease, Γ is the Green function of the operator $\nabla\cdot[D_L(\cdot)\nabla]-\beta_L(\cdot)$ associated with the zero flux boundary condition (see [1]),

Received May 17, 2015, accepted November 2, 2015.
Communicated by Cheng-Hsiung Hsu.
2010 *Mathematics Subject Classification.* 35K57, 35B35, 35B40.
Key words and phrases. Steady state, Disease-free equilibrium, Global stability, Lyapunov functional.
This work was supported in part by NSF of China (11401277) and (lzujbky-2013-210).
*Corresponding author.

and $\beta_L(x) = \sigma_L(x) + \gamma_L(x) + d(x)$, where $\sigma_L(x)$ and $\gamma_L(x)$ represent the disease-induced mortality rate and the recovery rate in the latent period at location x, respectively.

Let $\mathbb{X} := C(\overline{\Omega}, \mathbb{R}^2)$ be the Banach space with the supremum norm $\|\cdot\|_{\mathbb{X}}$. Let $\tau \geq 0$ and $C_\tau := C([-\tau,0],\mathbb{X})$ with the norm $\|\phi\| := \max_{\theta\in[-\tau,0]} \|\phi(\theta)\|_{\mathbb{X}}$, $\forall \phi \in C_\tau$. Define $\mathbb{X}^+ := C(\overline{\Omega}, \mathbb{R}^2_+)$ and $C_\tau^+ := C([-\tau,0],\mathbb{X}^+)$, then $(\mathbb{X},\mathbb{X}^+)$ and (C_τ, C_τ^+) are strongly ordered spaces. For $\sigma > 0$ and a given function $u(t)\colon [-\tau,\sigma) \to \mathbb{X}$, we define $u_t \in C_\tau$ by $u_t(\theta) = u(t+\theta)$, $\forall\, \theta \in [-\tau,0]$. We know that for any initial function $\phi \in C_\tau^+$, system (1.1) has a unique solution $\boldsymbol{u}(t,\cdot,\phi) \in C_\tau^+$ on $[0,\infty)$ with $\boldsymbol{u}_0 = \phi$ [2, 12]. Moreover, the semiflow $\Phi(t) = \boldsymbol{u}_t(\cdot)\colon C_\tau^+ \to C_\tau^+$ generated by (1.1), i.e.,

$$(\Phi(t)\phi)(\theta,x) = \boldsymbol{u}(t+\theta,x,\phi), \quad \forall\, t \geq 0,\ \theta \in [-\tau,0],\ x \in \overline{\Omega}$$

has a global compact attractor in C_τ^+, $\forall\, t \geq 0$.

As stated in [2], for spatially heterogeneous system (1.1), the stability of steady state $\widehat{u}(x)$ is an important but very difficult problem. And we mention that it is also harder to obtain the uniqueness of $\widehat{u}(x)$. Based on this, the authors further considered the model that the coefficients are independent of spatial variable:

$$\text{(1.2)} \quad \begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = D_1 \Delta u_1(t,x) + \mu - d u_1(t,x) - r u_1(t,x) u_2(t,x), & t > 0,\ x \in \Omega, \\ \dfrac{\partial u_2(t,x)}{\partial t} = D_2 \Delta u_2(t,x) - \beta u_2(t,x) & \\ \qquad + \displaystyle\int_\Omega \Gamma(\tau,x,y) r u_1(t-\tau,y) u_2(t-\tau,y)\,dy, & t > 0,\ x \in \Omega, \\ \dfrac{\partial u_1(t,x)}{\partial \nu} = \dfrac{\partial u_2(t,x)}{\partial \nu} = 0, & t > 0,\ x \in \partial\Omega. \end{cases}$$

To establish threshold dynamics, the authors in [2] defined the spectral radius of the next generation operator as the basic reproduction number $\mathcal{R}_0$ for the model (1.2), which played a threshold role. It follows from [2] that the basic reproduction number $\mathcal{R}_0$ is given by

$$\text{(1.3)} \quad \mathcal{R}_0 = e^{-\beta_L \tau} \frac{r\mu}{\beta d}.$$

Then one of the main results in [2] is the disease persistence and extinction of (1.2).

Theorem 1.1. *Suppose $\boldsymbol{u}(t,x,\phi)$ is the solution of system* (1.2) *with $\boldsymbol{u}_0 = \phi \in C_\tau^+$. Then the following statements hold.*

(i) *If $\mathcal{R}_0 < 1$, then the disease free equilibrium $(\mu/d, 0)$ is globally attractive.*

(ii) *If $\mathcal{R}_0 > 1$, the system* (1.2) *admits at least one positive steady state (endemic steady state), and there exists an $\eta > 0$ such that for any $\phi \in C_\tau^+$ with $\phi_2(0,\cdot) \not\equiv 0$, we have*

$$\liminf_{t\to\infty} u_i(t,x) \geq \eta, \quad \forall\, i = 1,2$$

uniformly for all $x \in \overline{\Omega}$.

Note that the system (1.2) admits a unique positive constant steady state $\widehat{E} = (\widehat{u}_1, \widehat{u}_2)$. When the demographic function in (1.2) is replaced by the Logistic function $u_1(x,t)\,[c_0 - c_1 u_1(x,t)]$, the mass action infection incidence function is replaced by $\frac{ru_1u_2}{b_0+b_2u_2}$ with $b_0 > 0$ and $b_2 > 0$, the authors [2] stated that the stability of $\widehat{u}$ could be obtained by employing [11, Theorem 4.1(i)] under some conditions on the constant parameters. When $b_2 = 0$, that is, the mass action infection incidence function is considered, however, these conditions will not hold and thus [11, Theorem 4.1(i)] does not apply to (1.2).

The objective of this work is to attempt to derive the global stability of the steady state of (1.2). Our results partially answer the open problem in [2, Section 5] by constructing the Lyapunov functional for the system (1.2). And when we show the stability of the steady state, restrictions on the coefficients in (1.2) are never imposed. The rest of this work is organized as follows. Section 2 is devoted to the proof the stability of the steady state of (1.2). As a discussion, Section 3 completes this work.

2. Global stability

In this section, motivated by the arguments of [4,6], we discuss the global stability of the positive constant steady state $\widehat{E} = (\widehat{u}_1, \widehat{u}_2)$ and disease-free equilibrium $E_0 = (u_1^*, 0) = (\mu/d, 0)$.

Consider the following spatially homogeneous system associated with (1.2). Letting $\boldsymbol{u}(t,x) = \boldsymbol{u}(t) = (u_1(t), u_2(t))$, we have

$$(2.1)\qquad \begin{cases} \dfrac{du_1}{dt} = \mu - du_1(t) - ru_1(t)u_2(t), \\ \dfrac{du_2}{dt} = e^{-\beta_L\tau} ru_1(t-\tau)u_2(t-\tau) - \beta u_2(t) \end{cases}$$

with initial condition

$$(u_1(s), u_2(s)) = (\varphi_1(s), \varphi_2(s)) \geq 0,\ s \in [-\tau, 0],\ \varphi = (\varphi_1, \varphi_2) \in C([-\tau, 0], \mathbb{R}^2),\ \varphi_2(0) > 0.$$

It is not difficult to see that the basic reproduction number for the system (2.1) is given by

$$\mathcal{R}_0^* = e^{-\beta_L\tau}\frac{r\mu}{\beta d} = \mathcal{R}_0,$$

and $\widehat{E} = (\widehat{u}_1, \widehat{u}_2) = \left(\frac{\beta e^{\beta_L\tau}}{r}, \frac{\mu r - d\beta e^{\beta_L\tau}}{\beta r e^{\beta_L\tau}}\right)$ is the unique interior equilibrium of (2.1). At what follows, motivated by the arguments of [6], we construct a Lyapunov functional of (2.1) to obtain the global stability of $\widehat{E}$ for (2.1). Consider the ordinary differential equation

$$(2.2)\qquad \begin{cases} \dfrac{du_1}{dt} = \mu - du_1 - ru_1u_2, \\ \dfrac{du_2}{dt} = e^{-\beta_L\tau} ru_1u_2 - \beta u_2. \end{cases}$$

Notice that $(\widehat{u}_1, \widehat{u}_2)$ is also the interior equilibrium of (2.2). Let $\boldsymbol{u} = (u_1, u_2)$, denote by $f(\boldsymbol{u})$ the vector field given by (2.2). Define

$$V_0(\boldsymbol{u}) = e^{-\beta_L \tau}(u_1 - \widehat{u}_1 \ln u_1) + (u_2 - \widehat{u}_2 \ln u_2),$$

an easy calculation from [7] gives that

$$\nabla V_0(\boldsymbol{u}) \cdot f(\boldsymbol{u}) = e^{-\beta_L \tau} d\widehat{u}_1 \left(2 - \frac{u_1}{\widehat{u}_1} - \frac{\widehat{u}_1}{u_1}\right) + e^{-\beta_L \tau} r\widehat{u}_1\widehat{u}_2 \left(2 - \frac{u_1}{\widehat{u}_1} - \frac{\widehat{u}_1}{u_1}\right).$$

Denote

$$V(\boldsymbol{u}, \boldsymbol{u_t}) = V_0(\boldsymbol{u}) + V_1(\boldsymbol{u_t}),$$

where

$$V_1(\boldsymbol{u_t}) = e^{-\beta_L \tau} r\widehat{u}_1\widehat{u}_2 \int_0^\tau H\left(\frac{u_1(t-\eta)u_2(t-\eta)}{\widehat{u}_1\widehat{u}_2}\right) d\eta$$

and $H(s) = s - 1 - \ln s$. Note that $V_1(\boldsymbol{u}_t) \geq 0$ and $V_1(\boldsymbol{u}_t) = 0$ if and only if $u_1(t)u_2(t) = \widehat{u}_1\widehat{u}_2$ identically. By virtue of [2, Theorems 2.1 and 3.1], the solutions of (2.1) is bounded above and bounded away from zero for large time provided $\mathcal{R}_0^* > 1$ (see also Theorem 1.1(ii)). Thus V is defined for all $t \geq 0$. In order to obtain $\frac{dV}{dt}$ along the positive solution to the system (2.1), we calculate the derivatives of V_0 and V_1 separately.

$$\begin{aligned}
\frac{dV_0(\boldsymbol{u}(t))}{dt} &= e^{-\beta_L \tau}\left(1 - \frac{\widehat{u}_1}{u_1}\right)(\mu - du_1 - ru_1u_2) \\
&\quad + \left(1 - \frac{\widehat{u}_2}{u_2}\right)\left(e^{-\beta_L \tau} ru_1(t-\tau)u_2(t-\tau) - \beta u_2\right) \\
&= e^{-\beta_L \tau}\left(1 - \frac{\widehat{u}_1}{u_1}\right)(\mu - du_1 - ru_1u_2) + \left(1 - \frac{\widehat{u}_2}{u_2}\right)\left(e^{-\beta_L \tau} ru_1u_2 - \beta u_2\right) \\
&\quad + \left(1 - \frac{\widehat{u}_2}{u_2}\right)\left(e^{-\beta_L \tau} ru_1(t-\tau)u_2(t-\tau) - e^{-\beta_L \tau} ru_1u_2\right) \\
&= \nabla V_0(\boldsymbol{u}) \cdot f(\boldsymbol{u}) + \Big[e^{-\beta_L \tau} ru_1(t-\tau)u_2(t-\tau) - \frac{u_2}{\widehat{u}_2} e^{-\beta_L \tau} ru_1(t-\tau)u_2(t-\tau) \\
&\qquad\qquad - e^{-\beta_L \tau} ru_1u_2 + \frac{u_2}{\widehat{u}_2} e^{-\beta_L \tau} ru_1u_2\Big].
\end{aligned}$$

In view of the expression of $\nabla V_0(\boldsymbol{u}) \cdot f(\boldsymbol{u})$, we have

$$\begin{aligned}
\frac{dV_0(\boldsymbol{u}(t))}{dt} &= e^{-\beta_L \tau} d\widehat{u}_1 \left(2 - \frac{u_1}{\widehat{u}_1} - \frac{\widehat{u}_1}{u_1}\right) + e^{-\beta_L \tau} r\widehat{u}_1\widehat{u}_2 \left(2 - \frac{u_1}{\widehat{u}_1} - \frac{\widehat{u}_1}{u_1}\right) \\
&\quad + e^{-\beta_L \tau} r\widehat{u}_1\widehat{u}_2 \left(\frac{u_1(t-\tau)u_2(t-\tau)}{\widehat{u}_1\widehat{u}_2} - \frac{u_1(t-\tau)u_2(t-\tau)}{\widehat{u}_1 u_2} - \frac{u_1u_2}{\widehat{u}_1\widehat{u}_2} + \frac{u_1}{\widehat{u}_1}\right) \\
&= e^{-\beta_L \tau} d\widehat{u}_1 \left(2 - \frac{u_1}{\widehat{u}_1} - \frac{\widehat{u}_1}{u_1}\right) + e^{-\beta_L \tau} r\widehat{u}_1\widehat{u}_2 \left(2 - \frac{\widehat{u}_1}{u_1} - \frac{u_1(t-\tau)u_2(t-\tau)}{\widehat{u}_1 u_2}\right) \\
&\quad + e^{-\beta_L \tau} r\widehat{u}_1\widehat{u}_2 \left(\frac{u_1(t-\tau)u_2(t-\tau)}{\widehat{u}_1\widehat{u}_2} - \frac{u_1u_2}{\widehat{u}_1\widehat{u}_2}\right).
\end{aligned}$$

According to [6, inequality (7)], we substitute $a_1 = u_1$, $a_2 = \widehat{u}_1 u_2$, $b_1 = \widehat{u}_1$, $b_2 = u_1 u_2$, $b_2' = u_1(t-\tau)u_2(t-\tau)$ for $n = 2$, then there holds

$$2 - \frac{\widehat{u}_1}{u_1} - \frac{u_1(t-\tau,x)u_2(t-\tau,x)}{\widehat{u}_1 u_2} + \ln\frac{u_1(t-\tau,x)u_2(t-\tau,x)}{u_1 u_2} \le 0. \tag{2.3}$$

We continue calculation for the derivative of $V_0(\boldsymbol{u})$:

$$\begin{aligned} \frac{dV_0(\boldsymbol{u}(t))}{dt} &= e^{-\beta_L\tau} d\widehat{u}_1\left(2 - \frac{u_1}{\widehat{u}_1} - \frac{\widehat{u}_1}{u_1}\right) \\ &\quad + e^{-\beta_L\tau} r\widehat{u}_1\widehat{u}_2\left(2 - \frac{\widehat{u}_1}{u_1} - \frac{u_1(t-\tau)u_2(t-\tau)}{\widehat{u}_1 u_2} + \ln\frac{u_1(t-\tau)u_2(t-\tau)}{u_1u_2}\right) \\ &\quad + e^{-\beta_L\tau} r\widehat{u}_1\widehat{u}_2\left(\frac{u_1(t-\tau)u_2(t-\tau)}{\widehat{u}_1\widehat{u}_2} - \frac{u_1u_2}{\widehat{u}_1\widehat{u}_2} - \ln\frac{u_1(t-\tau)u_2(t-\tau)}{u_1u_2}\right). \end{aligned} \tag{2.4}$$

Now we calculate the derivative of $V_1(\boldsymbol{u}_t)$.

$$\frac{dV_1(\boldsymbol{u}_t)}{dt} = e^{-\beta_L\tau} r\widehat{u}_1\widehat{u}_2\left(\frac{u_1u_2}{\widehat{u}_1\widehat{u}_2} - \frac{u_1(t-\tau,x)u_2(t-\tau,x)}{\widehat{u}_1\widehat{u}_2} + \ln\frac{u_1(t-\tau)u_2(t-\tau)}{u_1u_2}\right). \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\begin{aligned} \frac{dV}{dt} &= e^{-\beta_L\tau} d\widehat{u}_1\left(2 - \frac{u_1}{\widehat{u}_1} - \frac{\widehat{u}_1}{u_1}\right) \\ &\quad + e^{-\beta_L\tau} r\widehat{u}_1\widehat{u}_2\left(2 - \frac{\widehat{u}_1}{u_1} - \frac{u_1(t-\tau)u_2(t-\tau)}{\widehat{u}_1 u_2} + \ln\frac{u_1(t-\tau)u_2(t-\tau)}{u_1u_2}\right). \end{aligned}$$

Thus, the inequality (2.3) implies that

$$\frac{dV(\boldsymbol{u},\boldsymbol{u}_t)}{dt} \le 0,$$

which gives that V is a Lyapunov functional. Moreover, $\frac{dV}{dt} = 0$ if and only if $u_1 = \widehat{u}_1$ and $u_2 = \widehat{u}_2$. So the largest compact invariant set in $\widehat{\Gamma} = \left\{(u_1,u_2) \in C(\mathbb{R}_+) \mid \frac{dV}{dt} = 0\right\}$ is the singleton $\widehat{E}$. By [3, Theorem 5.3.1], $\boldsymbol{u} \to \widehat{E}$ as $t \to \infty$. Then we have the following result on the global stability of $\widehat{E}$ for system (2.1).

Proposition 2.1. *For any $\varphi = (\varphi_1,\varphi_2) \in C([-\tau,0],\mathbb{R}_+^2)$ with $\varphi_2(0) > 0$, let $\boldsymbol{u}(t,\varphi) = (u_1(t,\varphi),u_2(t,\varphi))$ be the solution of system (2.1). If $\mathcal{R}_0^* > 1$, then $\lim_{t\to\infty}(u_1(t),u_2(t)) = (\widehat{u}_1,\widehat{u}_2)$.*

Now we are in a position to state the main result.

Theorem 2.2. *Suppose $\boldsymbol{u}(t,x,\phi)$ is the solution of system (1.2) with $u_0 = \phi \in C_\tau^+$. If $\mathcal{R}_0 > 1$, for any $\phi = (\phi_1,\phi_2) \in C_\tau^+$ with $\phi_2(0,\cdot) \not\equiv 0$, then $\boldsymbol{u}(t,x)$ tends to $\widehat{E}$ as $t \to \infty$ uniformly for $x \in \overline{\Omega}$.*

Proof. With the aid of [2, Lemma 2.3], it follows that for any $\phi = (\phi_1, \phi_2) \in C_\tau^+$ with $\phi_2(0,\cdot) \not\equiv 0$,

$$u_i(t,x) > 0, \quad \forall\, t > 0,\ x \in \Omega.$$

Then we can define

$$W = \int_\Omega V(\boldsymbol{u}(t,x), \boldsymbol{u}_t(\cdot,x))\, dx,$$

where

$$\begin{aligned} V(\boldsymbol{u}(t,x), \boldsymbol{u}_t(\cdot,x)) &= V_0(\boldsymbol{u}(t,x)) + V_1(\boldsymbol{u}_t(\cdot,x)) \\ &= e^{-\beta_L\tau}\left(u_1(t,x) - \widehat{u}_1 \ln u_2(t,x)\right) + \left(u_2(t,x) - \widehat{u}_2 \ln u_2(t,x)\right) \\ &\quad + e^{-\beta_L\tau} r\widehat{u}_1\widehat{u}_2 \int_0^\tau H\left(\frac{u_1(t-\eta,x)u_2(t-\eta,x)}{\widehat{u}_1\widehat{u}_2}\right) d\eta. \end{aligned}$$

The derivative of W with respect to time t along the positive solution of (1.2) is

$$\frac{dW(\boldsymbol{u},\boldsymbol{u}_t)}{dt} = \int_\Omega \frac{\partial V(\boldsymbol{u},\boldsymbol{u}_t)}{\partial t}\, dx,$$

$$\begin{aligned} \frac{\partial V(\boldsymbol{u},\boldsymbol{u}_t)}{\partial t} &= e^{-\beta_L\tau}\left(1 - \frac{\widehat{u}_1}{u_1}\right)\cdot\frac{\partial u_1}{\partial t} + \left(1 - \frac{\widehat{u}_2}{u_2}\right)\cdot\frac{\partial u_2}{\partial t} \\ &\quad + e^{-\beta_L\tau} r\widehat{u}_1\widehat{u}_2\left(\frac{u_1u_2}{\widehat{u}_1\widehat{u}_2} - \frac{u_1(t-\tau,x)u_2(t-\tau,x)}{\widehat{u}_1\widehat{u}_2} + \ln\frac{u_1(t-\tau,x)u_2(t-\tau,x)}{u_1u_2}\right) \\ &= e^{-\beta_L\tau}\left(1 - \frac{\widehat{u}_1}{u_1}\right)(D_1\Delta u_1 + \mu - du_1 - ru_1u_2) \\ &\quad + \left(1 - \frac{\widehat{u}_2}{u_2}\right)\left(D_2\Delta u_2 - \beta u_2 + \int_\Omega \Gamma(\tau,x,y) r u_1(t-\tau,y)u_2(t-\tau,y)\, dy\right) \\ &\quad + e^{-\beta_L\tau} r\widehat{u}_1\widehat{u}_2\left(\frac{u_1u_2}{\widehat{u}_1\widehat{u}_2} - \frac{u_1(t-\tau,x)u_2(t-\tau,x)}{\widehat{u}_1\widehat{u}_2} + \ln\frac{u_1(t-\tau,x)u_2(t-\tau,x)}{u_1u_2}\right) \\ &= e^{-\beta_L\tau}\left(1 - \frac{\widehat{u}_1}{u_1}\right)\cdot D_1\Delta u_1 + \left(1 - \frac{\widehat{u}_2}{u_2}\right)\cdot D_2\Delta u_2 \\ &\quad + e^{-\beta_L\tau}\left(1 - \frac{\widehat{u}_1}{u_1}\right)(\mu - du_1 - ru_1u_2) \\ &\quad + \left(1 - \frac{\widehat{u}_2}{u_2}\right)\left(-\beta u_2 + \int_\Omega \Gamma(\tau,x,y) r u_1(t-\tau,y)u_2(t-\tau,y)\, dy\right) \\ &\quad + e^{-\beta_L\tau} r\widehat{u}_1\widehat{u}_2\left(\frac{u_1u_2}{\widehat{u}_1\widehat{u}_2} - \frac{u_1(t-\tau,x)u_2(t-\tau,x)}{\widehat{u}_1\widehat{u}_2} + \ln\frac{u_1(t-\tau,x)u_2(t-\tau,x)}{u_1u_2}\right). \end{aligned}$$

By using the Neumann boundary conditions and Green's formula, we have

$$\int_\Omega e^{-\beta_L\tau}\left(1 - \frac{\widehat{u}_1}{u_1}\right)\cdot D_1\Delta u_1\, dx = -D_1 e^{-\beta_L\tau}\widehat{u}_1 \int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\, dx \tag{2.6}$$

and

$$\int_\Omega \left(1 - \frac{\widehat{u}_2}{u_2}\right)\cdot D_2\Delta u_2\, dx = -D_2\widehat{u}_2 \int_\Omega \frac{|\nabla u_2|^2}{u_2^2}\, dx. \tag{2.7}$$

Additionally, note that $\int_\Omega \Gamma(\tau,x,y)\,dx = \int_\Omega \Gamma(\tau,x,y)\,dy = e^{-\beta_L\tau}$ for $x,y\in\Omega$. Then

$$\begin{aligned}&\int_\Omega \left(1-\frac{\widehat{u}_2}{u_2}\right)\left(-\beta u_2+\int_\Omega \Gamma(\tau,x,y) r u_1(t-\tau,y)u_2(t-\tau,y)\,dy\right)dx\\ &=\int_\Omega \left[\left(1-\frac{\widehat{u}_2}{u_2}\right)\left(-\beta u_2+e^{-\beta_L\tau} r u_1(t-\tau,x)u_2(t-\tau,x)\right)\right]dx.\end{aligned}\tag{2.8}$$

Combing equations (2.6)-(2.8), we obtain

$$\begin{aligned}\frac{dW}{dt} &= -\left\{D_1 e^{-\beta_L\tau}\widehat{u}_1\int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\,dx + D_2\widehat{u}_2\int_\Omega \frac{|\nabla u_2|^2}{u_2^2}\,dx\right\}\\ &\quad+\left\{\int_\Omega \left[e^{-\beta_L\tau}\left(1-\frac{\widehat{u}_1}{u_1}\right)(\mu-du_1-ru_1u_2)\right.\right.\\ &\quad+\left(1-\frac{\widehat{u}_2}{u_2}\right)\left(-\beta u_2+e^{-\beta_L\tau}ru_1(t-\tau,x)u_2(t-\tau,x)\right)\\ &\quad+\left.\left. e^{-\beta_L\tau} r\widehat{u}_1\widehat{u}_2\left(\frac{u_1u_2}{\widehat{u}_1\widehat{u}_2}-\frac{u_1(t-\tau,x)u_2(t-\tau,x)}{\widehat{u}_1\widehat{u}_2}+\ln\frac{u_1(t-\tau,x)u_2(t-\tau,x)}{u_1u_2}\right)\right]dx\right\}\\ &:= M_1+M_2.\end{aligned}$$

Clearly, $M_1\le 0$. By similar arguments to the proof of Proposition 2.1, we have $M_2\le 0$ and $M_2=0$ if and only if $\boldsymbol{u}=\widehat{E}$. Thus, $\frac{dW}{dt}\le 0$ and $\frac{dW}{dt}<0$, if $\boldsymbol{u}\ne\widehat{E}$. Note that $\boldsymbol{u}$ is bounded. Then by a similar proof to [5, Lemma 2.1], it follows that u_i and u_{it}, $i=1,2$ are bounded in C^1 (see also [9]). Since $\overline{\Gamma} := \left\{\boldsymbol{u}\in C^1([0,\infty)\times\Omega)\mid \frac{dW(\boldsymbol{u},\boldsymbol{u}_t)}{dt}=0\right\}$ is just $\left\{\widehat{E}\right\}$, applying a similar argument to the proof of [10, Theorem 2], we obtain $\boldsymbol{u}_t\to 0$ uniformly. This complete the proof. □

In the rest of this section, we discuss the global stability of the disease-free equilibrium $E_0=(u_1^*,0)$. Note that the global stability of E_0 for system (1.2) is just a special case in [2, Theorem 3.1(i)] when $\mathcal{R}_0<1$. However, we provide a distinct method to prove the global stability of E_0, which is valid for $\mathcal{R}_0\le 1$.

Theorem 2.3. *Suppose $\boldsymbol{u}(t,x,\phi)$ is the solution of system (1.2) with $\boldsymbol{u}_0=\phi\in C_\tau^+$. If $\mathcal{R}_0\le 1$, then the disease free equilibrium $E_0=(u_1^*,0)$ is globally stable in C_τ^+.*

Proof. Note that for any $\phi\in C_\tau^+$, the solutions of (1.2) satisfy

$$u_1(t,x)>0,\quad u_2(t,x)\ge 0,\quad \forall\, t>0,\ x\in\overline{\Omega}.$$

We construct the following Lyapunov functional

$$W'=\int_\Omega\left\{e^{-\beta_L\tau}\Psi\left(\frac{u_1(t,x)}{u_1^*}\right)+u_2(t,x)+e^{-\beta_L\tau}r\int_{t-\tau}^t u_1(s,x)u_2(s,x)\,ds\right\}dx,$$

where $\Psi(z) = z - 1 - \ln z$, $z \in \mathbb{R}^+$. Calculating the time derivative of W' along the solution of system (1.2), we get

$$\begin{aligned}
\frac{dW'}{dt} &= \int_\Omega \left[e^{-\beta_L \tau} \left(1 - \frac{u_1^*}{u_1}\right)(\mu - du_1 - ru_1u_2) \right] dx - e^{-\beta_L\tau} u_1^* \int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\, dx \\
&\quad + \int_\Omega \left[\int_\Omega \Gamma(\tau, x, y) ru_1(t-\tau, y) u_2(t-\tau, y)\, dy - \beta u_2 \right] dx \\
&\quad + e^{-\beta_L\tau} r \int_\Omega u_1 u_2\, dx - e^{-\beta_L\tau} r \int_\Omega u_1(t-\tau, x) u_2(t-\tau, x)\, dx \\
&= -e^{-\beta_L\tau} \int_\Omega \left[d\frac{(u_1 - u_1^*)^2}{u_1} + ru_1u_2 - ru_1^*u_2 \right] dx - e^{-\beta_L\tau} u_1^* \int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\, dx \\
&\quad + e^{-\beta_L\tau} \int_\Omega ru_1(t-\tau, x) u_2(t-\tau, x)\, dx - e^{-\beta_L\tau} \int_\Omega \beta u_2\, dx \\
&\quad + e^{-\beta_L\tau} \int_\Omega ru_1u_2\, dx - e^{-\beta_L\tau} \int_\Omega ru_1(t-\tau, x) u_2(t-\tau, x)\, dx \\
&= -e^{-\beta_L\tau} d \int_\Omega \frac{(u_1 - u_1^*)^2}{u_1}\, dx + \int_\Omega \beta u_2 \left[e^{-\beta_L\tau} \frac{ru_1^*}{\beta} - 1 \right] dx - e^{-\beta_L\tau} u_1^* \int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\, dx.
\end{aligned}$$

It then follows from (1.3) that

$$\frac{dW'}{dt} = -e^{-\beta_L\tau} d \int_\Omega \frac{(u_1 - u_1^*)^2}{u_1}\, dx + \int_\Omega \beta u_2(\mathcal{R}_0 - 1)\, dx - e^{-\beta_L\tau} u_1^* \int_\Omega \frac{|\nabla u_1|^2}{u_1^2}\, dx.$$

Since $\mathcal{R}_0 \le 1$, we have $\frac{dW'}{dt} \le 0$. It is easy to see that $\frac{dW'}{dt} = 0$ if $u_1 = u_1^*$ and $u_2(\mathcal{R}_0 - 1) = 0$. If $\mathcal{R}_0 = 1$, then the first equation in (1.2) implies that $u_2 = 0$. Thus the set $\widehat{\Gamma} = \left\{ \boldsymbol{u} \in C^1([0,\infty) \times \Omega) \mid \frac{dW'(\boldsymbol{u}, \boldsymbol{u}_t)}{dt} = 0 \right\}$ is just the singleton E_0. As mentioned in Theorem 2.2, u_i and u_{it}, $i = 1, 2$ are bounded in C^1. Hence, by applying a similar argument to the proof of [10, Theorem 2], we obtain that $\boldsymbol{u}_t \to E_0$ uniformly. This completes the proof. □

3. Discussion

Generally, the study of the global stability of steady states for differential equation models is an important and difficult problem for non-local and time-delayed reaction-diffusion models. In [11], the fluctuation method is used to establish the global attractivity of the positive constant steady state for a spatially homogeneous reaction-diffusion predator-prey model with time-delayed and nonlocal effect. However, it holds under some restriction on the constant parameters of the model system, which leads to unavailablity of [11, Theorem 4.1(i)] for system (1.2) if $\mathcal{R}_0 > 1$. In this paper, by constructing the modified Lyapunov functional, we get the global stability of the positive constant steady state of system (1.2) if $\mathcal{R}_0 > 1$. Furthermore, the global stability of the disease-free equilibrium is obtained by constructing suitable Lyapunov functional if $\mathcal{R}_0 \le 1$, which seems to be a weaker condition than it in Guo et al. [2].

For the epidemic model with nonlinear incidence function

(3.1)
$$\begin{cases} \dfrac{\partial u_1(t,x)}{\partial t} = D_1 \Delta u_1(t,x) + \mu - d u_1(t,x) - r u_1(t,x) \dfrac{u_2(t,x)}{b_0 + b_2 u_2(t,x)}, & t > 0,\ x \in \Omega, \\ \dfrac{\partial u_2(t,x)}{\partial t} = D_2 \Delta u_2(t,x) - \beta u_2(t,x) & \\ \qquad + \displaystyle\int_\Omega \Gamma(\tau, x, y) r u_1(t-\tau, y) \dfrac{u_2(t-\tau, y)}{b_0 + b_2 u_2(t-\tau, y)}\, dy, & t > 0,\ x \in \Omega, \\ \dfrac{\partial u_1(t,x)}{\partial n} = \dfrac{\partial u_2(t,x)}{\partial n} = 0, & t > 0,\ x \in \partial\Omega, \end{cases}$$

if $b_2 = 0$, system (3.1) degenerate into system (1.2). If $b_2 > 0$, it is easy to obtain a unique positive constant steady state $\widehat{E} = (\widehat{u}_1, \widehat{u}_2)$ when $\mathcal{R}_0 > 1$. To establish the global stability of $\widehat{u}$, we construct suitable Lyapunov functional through the following ordinary differential equation

$$\begin{cases} \dfrac{du_1}{dt} = \mu - d u_1 - r u_1 g(u_2), \\ \dfrac{du_2}{dt} = e^{-\beta_L \tau} r u_1 g(u_2) - \beta u_2, \end{cases} \tag{3.2}$$

where $g(u_2) = \frac{u_2}{b_0 + b_2 u_2}$. Note that $\widehat{E}$ is also the unique positive steady state of system (3.2). Similar to Section 2, we define V_0 by

$$V_0(\boldsymbol{u}) = e^{-\beta_L \tau}(u_1 - \widehat{u}_1 \ln u_1) + (u_2 - \widehat{u}_2 \ln u_2).$$

By Korobeinikov [8], $\nabla V_0(\boldsymbol{u}) \cdot f(\boldsymbol{u})$ can be calculated as follows

$$\begin{aligned} \nabla V_0(\boldsymbol{u}) \cdot f(\boldsymbol{u}) = {} & e^{-\beta_L \tau} d \widehat{u}_1 \left(2 - \frac{u_1}{\widehat{u}_1} - \frac{\widehat{u}_1}{u_1} \right) \\ & + e^{-\beta_L \tau} r \widehat{u}_1 g(\widehat{u}_2) \left(3 - \frac{u_1}{\widehat{u}_1} - \frac{u_1 \widehat{u}_2 g(u_2)}{\widehat{u}_1 u_2 g(\widehat{u}_2)} - \frac{u_2 g(\widehat{u}_2)}{\widehat{u}_2 g(u_2)} \right) \\ & + e^{-\beta_L \tau} r \widehat{u}_1 g(\widehat{u}_2) \left(\frac{u_2 g(\widehat{u}_2)}{\widehat{u}_2 g(u_2)} - 1 - \frac{u_2}{\widehat{u}_2} + \frac{g(u_2)}{g(\widehat{u}_2)} \right). \end{aligned}$$

To deal with the delay term, we introduce $V_1(\boldsymbol{u_t})$ by

$$V_1(\boldsymbol{u_t}) = \int_0^\tau H\left(\frac{u_1(t-\eta) u_2(t-\eta)}{\widehat{u}_1 \widehat{u}_2} \right) d\eta,$$

where $H(s) = s - 1 - \ln s$. Let

$$V(\boldsymbol{u}, \boldsymbol{u_t}) = V_0(\boldsymbol{u}) + e^{-\beta_L \tau} r \widehat{u}_1 g(\widehat{u}_2) V_1(\boldsymbol{u_t}).$$

So we have the following Lyapunov functional for the model (3.1)

$$W = \int_\Omega V(\boldsymbol{u}(t,x), \boldsymbol{u}_t(\cdot, x))\, dx.$$

Applying similar method as in Section 2 for dealing with calculating the time derivative of W along the solution of system (3.1), we can obtain the global stability of $\widehat{E}$ for system (3.1) if $\mathcal{R}_0 > 1$. Although we have never extended the method in [11, Theorem 4.1(i)] to obtain the global stability of the steady state of (3.1), from the above, our method is not dependent of the restriction on the constant parameters.

References

[1] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[2] Z. Guo, F.-B. Wang and X. Zou, *Threshold dynamics of an infective disease model with a fixed latent period and non-local infections*, J. Math. Biol. **65** (2012), no. 6-7, 1387–1410. http://dx.doi.org/10.1007/s00285-011-0500-y

[3] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences **99**, Springer-Verlag, New York, 1993. http://dx.doi.org/10.1007/978-1-4612-4342-7

[4] K. Hattaf and N. Yousfi, *Global stability for reaction-diffusion equations in biology*, Comput. Math. Appl. **66** (2013), no. 8, 1488–1497. http://dx.doi.org/10.1016/j.camwa.2013.08.023

[5] V. Hutson and W. Moran, *Repellers in reaction-diffusion systems*, Rocky Mountain J. Math. **17** (1987), no. 2, 301–314. http://dx.doi.org/10.1216/rmj-1987-17-2-301

[6] T. Kajiwara, T. Sasaki and Y. Takeuchi, *Construction of Lyapunov functionals for delay differential equations in virology and epidemiology*, Nonlinear Anal. Real World Appl. **13** (2012), no. 4, 1802–1826. http://dx.doi.org/10.1016/j.nonrwa.2011.12.011

[7] A. Korobeinikov, *Global properties of basic virus dynamics models*, Bull. Math. Biol. **66** (2004), no. 4, 879–883. http://dx.doi.org/10.1016/j.bulm.2004.02.001

[8] ———, *Global properties of infectious disease models with nonlinear incidence*, Bull. Math. Biol. **69** (2007), no. 6, 1871–1886. http://dx.doi.org/10.1007/s11538-007-9196-y

[9] Y. Kuang and H. L. Smith, *Global stability in diffusive delay Lotka-Volterra systems*, Differential Integral Equations **4** (1991), no. 1, 117–128.

[10] R. Redheffer, R. Redlinger and W. Walter, *A theorem of La Salle-Lyapunov type for parabolic systems*, SIAM J. Math. Anal. **19** (1988), no. 1, 121–132. http://dx.doi.org/10.1137/0519009

[11] H. R. Thieme and X.-Q. Zhao, *A non-local delayed and diffusive predator-prey model*, Nonlinear Anal. Real World Appl. **2** (2001), no. 2, 145–160. http://dx.doi.org/10.1016/s0362-546x(00)00112-7

[12] S.-L. Wu, C.-H. Hsu and Y. Xiao, *Global attractivity, spreading speeds and traveling waves of delayed nonlocal reaction-diffusion systems*, J. Differential Equations **258** (2015), no. 4, 1058–1105. http://dx.doi.org/10.1016/j.jde.2014.10.009

Liang Zhang and Jian-Wen Sun
School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu, 730000, P. R. China
E-mail address: zhangll-2009@lzu.edu.cn, jianwensun@lzu.edu.cn