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# Convergence of the Relative Pareto Efficient Sets

Nguyen Van Tuyen

Abstract. The aim of this paper is to present new results on the convergence of relative Pareto efficient sets and the lower semicontinuity of relative Pareto efficient point multifunctions under perturbations. Our results extend some results of Luc et al. [16, Theorem 2.1], Bednarczuk [4, Theorem 4] and [5, Proposition 3.1], Lucchetti and Miglierina [17, Proposition 3.1]. Some remarks and examples are provided for analysing the results obtained and for comparing them with the preceding results.

## 1. Introduction

Stability analysis is one of the most important and interesting topics and its role has been widely recognized in optimization theory. In the literature, two classical approaches to studying stability of vector optimization problems can be found. One is to investigate the set-convergence of efficient sets of perturbed sets converging to a given set. Another is to study continuity properties of the optimal multifunctions. For instance, the lower (upper) semicontinuity of the optimal multifunctions have been examined by Penot [20]. Luc, Lucchetti and Malivert [16] investigated the stability of vector optimization in terms of the convergence of the efficient sets. Miglierina and Molho [18, 19] obtained some results on stability of convex vector optimization problems by considering the convergence of efficient sets. For more results concerning the use of convexity in stability analysis, we refer readers to [15, 17]. Various stability results on the optimal multifunctions were presented in the monographs [15, 21] and papers (see, e.g., [4–8, 12, 14, 20]). Using the so-called *domination property*, *containment property*, and *dual containment property*, Bednarczuk [4–8] studied the Hausdorff upper semicontinuity, the $C$-Hausdorff upper semicontinuity and the lower (upper) semicontinuity of the efficient solution map and the efficient point multifunctions. Recently, by using the approach of Bednarczuk [4, 6] and introducing the new concepts of *local containment property*, $K$-*local domination property* and *uniformly local closedness* of a multifunction around a given point, Chuong, Yao, and Yen [12] have

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obtained further results on the lower semicontinuity of efficient point multifunctions taking values in Hausdorff topological vector spaces.

In the present paper we study the stability of a vector optimization problem using the notion of relative Pareto efficiency. By using the approach of [16] we obtain results on the set-convergence of the relative Pareto efficient sets. Our results can be seen as an extension of [16, Theorem 2.1] and [17, Proposition 3.1] from the weak Pareto efficient sets to the relative Pareto efficient sets. To obtain a result on the lower semicontinuity of relative Pareto efficient point multifunctions under perturbations for cones with possibly empty interior, we propose new concepts called *relative containment property*, *relative lower semicontinuity* and *relative upper Hausdorff semicontinuity* of a multifunction around a given point. Our results generalize and strengthen the corresponding ones in [4, 5].

The rest of paper is organized as follows. In Section 2 we establish the upper convergence in the sense of Kuratowski-Painlevé of the relative Pareto efficient sets. Section 3 concerns the lower convergence in the sense of Kuratowski-Painlevé of the relative Pareto efficient sets. Section 4 introduces new concepts called *relative containment property*, *relative lower semicontinuity*, and *relative upper Hausdorff semicontinuity* of a multifunction around a given point, and derives some sufficient conditions for the lower semicontinuity of relative Pareto efficient point multifunctions under perturbations for cones with *possibly empty interior*. Some topological properties of the relative Pareto efficient set are examined, and examples for analyzing the obtained results are provided.

## 2. Upper convergence of relative Pareto efficient sets

Let $Z$ be a Banach space. For a set $A \subset Z$, the following notations will be used throughout: $\operatorname{int} A$, $\operatorname{cl} A$ (or $\overline{A}$), $\operatorname{bd} A$, $\operatorname{ri} A$ and $\operatorname{aff}(A)$ mean the interior, closure, boundary, relative interior and affine hull of $A$ in $Z$. By $\mathcal{N}(z)$ we denote the set of all neighborhoods of $z \in Z$. The origin in $Z$ is denoted by $0_Z$. The closed unit ball in $Z$ is abbreviated to $\mathbb{B}$. The closed ball with center $z$ and radius $\rho$ is denoted by $\mathbb{B}(x, \rho)$.

Recall that the relative interior $\operatorname{ri} A$ of a convex set $A$ is defined as the interior of $A$ with respect to the closed affine hull of $A$. It is well known that $\operatorname{ri} A$ *is nonempty for every nonempty convex set* $A$ *in finite dimensions*. Further properties of relative interiors of convex sets in Banach spaces can be found in [10, 11].

Suppose that $C$ is a convex cone in $Z$. Then, $C$ induces a partial order in $Z$ as follows: $z_1, z_2 \in Z$, $z_1 \leq z_2$ if $z_2 - z_1 \in C$.

**Definition 2.1.** Let $A$ be a nonempty subset in $Z$. We say that

(i) $\overline{z} \in A$ is a *Pareto efficient* of $A$ with respect to $C$ if

$$A \cap (\overline{z} - C) = \{\overline{z}\};$$

(ii) $\overline{z} \in A$ is a *relative Pareto efficient* (or *Slater efficient*) of $A$ with respect to $C$ if

$$A \cap (\overline{z} - \operatorname{ri} C) = \emptyset$$

provided that $\operatorname{ri} C \neq \emptyset$;

(iii) $\overline{z} \in A$ is a *weakly Pareto efficient* of $A$ with respect to $C$ if

$$A \cap (\overline{z} - \operatorname{int} C) = \emptyset$$

provided that $\operatorname{int} C \neq \emptyset$.

The set of efficient points defined in Definition 2.1 are denoted by $\operatorname{Min}(A \mid C)$, $\operatorname{ReMin}(A \mid C)$ and $\operatorname{WMin}(A \mid C))$, respectively.

*Remark* 2.2. (a) Observe that if $\operatorname{int} C \neq \emptyset$, then $\operatorname{ri} C = \operatorname{int} C$ and

$$\operatorname{WMin}(A \mid C) = \operatorname{ReMin}(A \mid C).$$

(b) If $\operatorname{ri} C \neq \emptyset$ and $C \setminus (-C) \neq \emptyset$, then $0 \notin \operatorname{ri} C$. Thus

$$\operatorname{Min}(A \mid C) \subset \operatorname{ReMin}(A \mid C).$$

The opposite inclusion does not hold in general. However, if $A$ is a rotund set, i.e., $A$ is convex and the boundary of $A$ does not contain line segments, then we have $\operatorname{Min}(A \mid C) = \operatorname{ReMin}(A \mid C)$.

**Proposition 2.3.** *Suppose that* $\operatorname{ri} C \neq \emptyset$ *and* $0 \notin \operatorname{ri} C$. *If $A$ is a rotund set, then*

$$\operatorname{ReMin}(A \mid C) = \operatorname{Min}(A \mid C).$$

*Proof.* By Remark 2.2(b), it suffices to prove that $\operatorname{ReMin}(A \mid C) \subset \operatorname{Min}(A \mid C)$. Suppose to the contrary that there is an element $\overline{z} \in \operatorname{ReMin}(A \mid C) \setminus \operatorname{Min}(A \mid C)$. Then there exists $z \in A$ such that $z - \overline{z} \in (-C \setminus (-\operatorname{ri} C \cup \{0\}))$. From $z, \overline{z} \in A$ and the rotundity of $A$ it follows that

$$(\overline{z}, z] := \{x \in Z \mid x = \overline{z} + \alpha(z - \overline{z}), \alpha \in (0, 1]\}$$

does not lie entirely in the boundary of $A$. Hence there exist $\overline{\alpha} \in (0, 1]$ such that $\overline{x} := \overline{z} + \overline{\alpha}(z - \overline{z}) \in \operatorname{int} A$. From $\overline{\alpha}(z - \overline{z}) \in (-C \setminus (-\operatorname{ri} C \cup \{0\}))$ we have $\overline{x} - \overline{z} \in (-C \setminus (-\operatorname{ri} C \cup \{0\}))$. Since $\overline{x} \in \operatorname{int} A$, we can find $e \in -\operatorname{ri} C$ such that $\overline{x} + e \in A$. Then

$$\begin{aligned} \overline{x} + e - \overline{z} &= (\overline{x} - \overline{z}) + e \subset (-C \setminus (-\operatorname{ri} C \cup \{0\})) - \operatorname{ri} C \\ &\subset -C - \operatorname{ri} C \subset -\operatorname{ri} C \end{aligned}$$

or, equivalently, $\overline{x} + e \in (\overline{z} - \operatorname{ri} C)$. Thus $(\overline{x} + e) \in A \cap (\overline{z} - \operatorname{ri} C)$, contrary to $\overline{z} \in \operatorname{ReMin}(A \mid C)$. The proof is complete. □

**Corollary 2.4.** (see [19, Proposition 4.3]) *Let $C$ be a convex pointed cone with* $\operatorname{int} C \neq \emptyset$*. If $A$ is a rotund set, then*

$$\mathrm{WMin}(A \mid C) = \mathrm{Min}(A \mid C).$$

We now recall some concepts of convergence of a sequence of sets. Let $(A_n)$ be a sequence of subsets in $Z$ and $A \subset Z$ be a nonempty subset.

- The convergence in the sense of *Kuratowski-Painlevé*:

  The *Kuratowski-Painlevé lower* and *upper limits* of $(A_n)$ are defined as

$$\operatorname{Lim\,inf} A_n := \left\{ z \in Z \mid z = \lim_{n\to\infty} z_n, z_n \in A_n \text{ for all large } n \right\},$$
$$\operatorname{Lim\,sup} A_n := \left\{ z \in Z \mid z = \lim_{n\to\infty} z_k, z_k \in A_{n_k} \text{ for some } (A_{n_k}) \subset (A_n) \right\}.$$

  Clearly, $\operatorname{Lim\,inf} A_n \subset \operatorname{Lim\,sup} A_n$. If $\operatorname{Lim\,sup} A_n \subset A \subset \operatorname{Lim\,inf} A_n$, then we say that $(A_n)$ converges to $A$ in the sense of *Kuratowski-Painlevé* and we denote $A_n \xrightarrow{K} A$. The condition $\operatorname{Lim\,sup} A_n \subset A$ will be called *upper Kuratowski-Painlevé convergence*, whereas the condition $A \subset \operatorname{Lim\,inf} A_n$ will be called *lower Kuratowski-Painlevé convergence*. The closedness of $\operatorname{Lim\,sup} A_n$ and $\operatorname{Lim\,inf} A_n$ implies that if $A_n \xrightarrow{K} A$, then $A$ is a closed subset. When the limits are considered in the weak topology on $Z$, we denote the lower and the upper limits above by $w - \operatorname{Lim\,inf} A_n$ and $w - \operatorname{Lim\,sup} A_n$. If $w - \operatorname{Lim\,sup} A_n \subset A \subset \operatorname{Lim\,inf} A_n$, we say that $(A_n)$ converges to $A$ in the sense of *Mosco* and we write $A_n \xrightarrow{M} A$.

  Let $x \in Z$ and let $A$, $B$ be nonempty subsets in $Z$. Define

$$\begin{aligned}
d(x, A) &= \inf_{a\in A} d(x, a) \quad (d(x, \emptyset) = \infty);\\
e(A, B) &= \sup_{a\in A} d(a, B) \quad (e(\emptyset, B) = 0,\ e(\emptyset, \emptyset) = 0,\ e(A, \emptyset) = \infty);\\
h(A, B) &= \max\{e(A, B), e(B, A)\};\\
e_\rho(A, B) &= e(A \cap \mathbb{B}_\rho, B), \quad \mathbb{B}_\rho = \mathbb{B}(0, \rho);\\
h_\rho(A, B) &= \max\{e_\rho(A, B), e_\rho(B, A)\}.
\end{aligned}$$

- The convergence in the sense of *Wijsman*:

  We say that $(A_n)$ converges to $A$ in the sense of *Wijsman* if

$$\lim_{n\to\infty} d(A_n, x) = d(A, x), \quad \forall\, x \in Z,$$

  where $d(A, x) = \inf_{a\in A} d(a, x)$. Clearly, $d(x, A) = d(A, x)$.

- The convergence in the sense of *Hausdorff*:

  We say that the sequence $(A_n) \subset Z$ converges to $A$ in the sense *Hausdorff* if

$$\lim_{n\to\infty} h(A_n, A) = 0.$$

  The condition $\lim_{n\to\infty} e(A_n, A) = 0$ will be called *upper Hausdorff convergence*, whereas the condition $\lim_{n\to\infty} e(A, A_n) = 0$ will be called *lower Hausdorff convergence.*

- The convergence in the sense of *Attouch-Wets*:

  We say that the sequence $(A_n) \subset Z$ converges to $A$ in the sense *Attouch-Wets* if

$$\lim_{n\to\infty} h_\rho(A_n, A) = 0$$

  for all $\rho > 0$. We can split this notion of convergence into an upper part and a lower part as follows

$$\lim_{n\to\infty} e_\rho(A_n, A) = 0,$$

  and

$$\lim_{n\to\infty} e_\rho(A, A_n) = 0.$$

  Sonntag and Zălinescu [22] have shown that the upper convergence in the sense of Attouch-Wets is equivalent to

$$\liminf_{n\to\infty} d(A_n, B) \geq d(A, B),$$

  for each nonempty bounded set $B$. Here $d(A, B)$ stands for $\inf_{a\in A} \inf_{b\in B} d(a, b)$, where we agree that $d(A, B) := +\infty$ if and only if at least one of the two sets is empty.

For the relationships between the various notions of set convergence recalled here, see, e.g., [1, 22]. It is well known that, if $Z$ is a finite dimensional space, the above quoted notions of set-convergence coincide whenever we consider a sequence $(A_n)$ of closed sets.

**Lemma 2.5.** *Suppose that $C \subset Z$ is a convex set with nonempty relative interior. Then $z \in \operatorname{ri} C$ if and only if for every $y \in C$, there exists $\mu > 1$ such that*

$$(1-\mu)y + \mu z \in C.$$

*Proof.* ($\Rightarrow$) Let $z \in \operatorname{ri} C$ and $y$ be an arbitrary element belong to $C$. For each $\mu > 1$, put $x_\mu = (1-\mu)y + \mu z$. We have $x_\mu \in \operatorname{aff}(C)$ for all $\mu > 1$. Since $z \in \operatorname{ri} C$, there is a $\delta > 0$

such that $\mathbb{B}(z,\delta) \cap \overline{\mathrm{aff}}(C) \subset C$. As $x_\mu = z - (\mu - 1)(y - z)$, $x_\mu$ belongs to $\mathbb{B}(z,\delta)$ with $\mu > 1$ and $\mu - 1$ sufficiently small. Thus

$$\begin{aligned} x_\mu &\in \mathbb{B}(z,\delta) \cap \mathrm{aff}(C) \\ &\subset \mathbb{B}(z,\delta) \cap \overline{\mathrm{aff}}(C) \\ &\subset C, \end{aligned}$$

with $\mu - 1$ sufficiently small, as required.

($\Leftarrow$) Suppose that $z$ satisfies the condition: for every $y \in C$, there exists $\mu > 1$ such that

$$(1 - \mu)y + \mu z \in C. \tag{2.1}$$

Select a point $x \in \mathrm{ri}\, C$. By (2.1), $(1 - \mu)x + \mu z \in C$. Thus there is $y \in C$ satisfying $y = (1 - \mu)x + \mu z$. Consequently, $z = (1 - \lambda)x + \lambda y$, where $0 < \lambda = \frac{1}{\mu} < 1$. By [11, Lemma 3.1], we have $z \in \mathrm{ri}\, C$. The proof is complete. □

Let $(C_n)$ be a sequence of convex cones in $Z$, and $C \subset Z$ be a convex cone. For brevity, in the sequel we write $\mathrm{ReMin}\, A$, $\mathrm{ReMin}\, A_n$, $\mathrm{Min}\, A$, $\mathrm{Min}\, A_n$, $\mathrm{WMin}\, A$, and $\mathrm{WMin}\, A_n$ instead of $\mathrm{ReMin}(A \mid C)$, $\mathrm{ReMin}(A_n \mid C_n)$, $\mathrm{Min}(A \mid C)$, $\mathrm{Min}(A_n \mid C_n)$, $\mathrm{WMin}(A \mid C)$, and $\mathrm{WMin}(A_n \mid C_n)$, respectively.

**Theorem 2.6.** *Let $(C_n)$ and $C$ be convex cones in $Z$ with nonempty relative interior. If*

(i) $A_n \xrightarrow{K} A$,

(ii) $\mathrm{Lim\,sup}\, C_n^c \subset (\mathrm{ri}\, C)^c$, *where* $C^c := Z \setminus C$,

*then*

$$\mathrm{Lim\,sup}\, \mathrm{ReMin}\, A_n \subset \mathrm{ReMin}\, A.$$

*Proof.* Arguing by contradiction, assume that there is $x \in \mathrm{Lim\,sup}\, \mathrm{ReMin}\, A_n \setminus \mathrm{ReMin}\, A$. By $x \in \mathrm{Lim\,sup}\, \mathrm{ReMin}\, A_n$ and (i), we see that for each $k \in \mathbb{N}$ there exist $x_k \in \mathrm{ReMin}\, A_{n_k}$ such that $\lim_{k\to\infty} x_k = x$ and $x \in A$. Since $x \notin \mathrm{ReMin}\, A$, there exists $a \in A$ satisfying $x - a \in \mathrm{ri}\, C$ or, equivalently,

$$x - a \notin (\mathrm{ri}\, C)^c. \tag{2.2}$$

We claim that $x - a \notin \mathrm{Lim\,sup}(\mathrm{ri}\, C_n)^c$. Indeed, if this is false, then there exists

$$z_k \in (\mathrm{ri}\, C_k)^c \tag{2.3}$$

satisfying $\lim_{k\to\infty} z_k = x - a$, where $(\operatorname{ri} C_k)$ is a subsequence of $(\operatorname{ri} C_n)$. By (2.3) and Lemma 2.5, for each $k \in \mathbb{N}$ there exists $y_k \in C_k$ such that

$$(1-\mu)y_k + \mu z_k \notin C_k \tag{2.4}$$

for all $\mu > 1$. Substituting $\mu = 1 + \frac{1}{m}$ into the left side of (2.4) we obtain

$$-\frac{1}{m}y_k + \left(1 + \frac{1}{m}\right) z_k \notin C_k, \tag{2.5}$$

for all $m \in \mathbb{N}$. Letting $m \to \infty$ in (2.5), we have

$$z_k \in \operatorname{cl}(C_k^c), \quad \forall\, k \in \mathbb{N}. \tag{2.6}$$

Passing (2.6) to the limit as $k \to \infty$ yields $x - a \in \operatorname{Lim\,sup}[\operatorname{cl}(C_n^c)]$. Furthermore,

$$\operatorname{Lim\,sup}[\operatorname{cl}(C_n^c)] = \operatorname{Lim\,sup} C_n^c \subset (\operatorname{ri} C)^c.$$

Thus $x - a \in (\operatorname{ri} C)^c$, contrary to (2.2). Since $\operatorname{Lim\,sup}(\operatorname{ri} C_{n_k})^c \subset \operatorname{Lim\,sup}(\operatorname{ri} C_n)^c$, $x - a \notin \operatorname{Lim\,sup}(\operatorname{ri} C_{n_k})^c$. The inclusion $a \in A$ and the assumption (i) imply that there exist $a_n \in A_n$ satisfying $\lim_{n\to\infty} a_n = a$. Thus $\lim_{k\to\infty} a_{n_k} = a$, where $a_{n_k} \in A_{n_k}$ for all $k \in \mathbb{N}$. From $\lim_{k\to\infty}(x_k - a_{n_k}) = x - a \notin \operatorname{Lim\,sup}(\operatorname{ri} C_{n_k})^c$ we see that there exists $k_0 \in \mathbb{N}$ such that $x_{k_0} - a_{n_{k_0}} \notin (\operatorname{ri} C_{n_{k_0}})^c$ or, equivalently, $x_{k_0} - a_{n_{k_0}} \in \operatorname{ri} C_{n_{k_0}}$, contradicting the fact that $x_{k_0} \in \operatorname{ReMin} A_{n_{k_0}}$. The proof is complete. □

If $\operatorname{ri} C \neq \emptyset$ and $\operatorname{int} C = \emptyset$, then the condition (ii) in Theorem 2.6 cannot be replaced the condition "$\operatorname{Lim\,sup} C_n^c \subset \operatorname{cl}(C^c)$." To see this, we consider the following example.

**Example 2.7.** Let $Z = \mathbb{R}^2$, $C = \mathbb{R}_+ \times \{0\}$, $A = [-1, 1] \times \{0\}$,

$$A_n = \left\{ z = (z_1, z_2) \in \mathbb{R}^2 \;\middle|\; z_2 = -\frac{1}{n} z_1, -1 \le z_1 \le 1 \right\},$$

and $C_n = C$, $\forall\, n \in \mathbb{N}$. It is easy to see that $A_n \xrightarrow{K} A$ and $\operatorname{Lim\,sup} C_n^c = \operatorname{cl}(C^c) = \mathbb{R}^2$. We have $\operatorname{ReMin} A_n = A_n$ for all $n \in \mathbb{N}$ and $\operatorname{ReMin} A = \{(-1, 0)\}$. Clearly,

$$\operatorname{Lim\,sup} \operatorname{ReMin} A_n = A \supsetneq \operatorname{ReMin} A.$$

However, when $(C_n)$ and $C$ are cones with nonempty interior, then we have the following result.

**Corollary 2.8.** *Let $(C_n)$ and $C$ be convex cones in $Z$ with nonempty interior. If*

(i) $A_n \xrightarrow{K} A$,

(ii) $\operatorname{Lim\,sup} C_n^c \subset \operatorname{cl}(C^c)$, *then*

$$\operatorname{Lim\,sup} \operatorname{WMin} A_n \subset \operatorname{WMin} A.$$

*Proof.* It is easy to see that $\operatorname{cl}(C^c) \subset (\operatorname{int} C)^c$. This and (ii) imply that $\operatorname{Lim\,sup} C_n^c \subset (\operatorname{int} C)^c$. An analysis similar to that in the proof of Theorem 2.6 shows that

$$\operatorname{Lim\,sup} \operatorname{WMin} A_n \subset \operatorname{WMin} A.$$

The proof is complete. □

*Remark* 2.9. In [17, Proposition 3.1], the authors derived sufficient conditions for the upper convergence of weakly Pareto efficient sets under perturbations of the feasible region only. If $C_n = C$ for all $n \in \mathbb{N}$ and $\operatorname{int} C \neq \emptyset$, then the condition (ii) in Corollary 2.8 is fulfilled. Thus our result is an extension of [17, Proposition 3.1].

**Theorem 2.10.** *Let $C_n$ and $C$ be convex cones in $Z$ with nonempty relative interior. If*

(i) $A_n \xrightarrow{K} A$,

(ii) $\operatorname{Lim\,sup} C_n^c \subset (\operatorname{ri} C)^c$,

(iii) *for every bounded subset of $\bigcup_{n=1}^{\infty} \operatorname{ReMin} A_n$ is relatively compact,*

*then*

$$\liminf_{n\to\infty} d(\operatorname{ReMin} A_n, B) \geq d(\operatorname{ReMin} A, B),$$

*for each bounded subset $B$.*

*Proof.* The conclusion of the theorem is trivial if $\liminf_{n\to\infty} d(\operatorname{ReMin} A_n, B) = +\infty$. Thus it suffices to consider the case $\liminf_{n\to\infty} d(\operatorname{ReMin} A_n, B)$ is finite. Suppose on the contrary that there is some bounded subset $B \subset Z$ and some positive number $\gamma > 0$ such that

$$\liminf_{n\to\infty} d(\operatorname{ReMin} A_n, B) < \gamma < \alpha,$$

where $\alpha := d(\operatorname{ReMin} A, B)$. By taking a subsequence of $(A_n)$ if necessary we may assume that $d(\operatorname{ReMin} A_n, B) < \gamma$ for all $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, there exists $y_n \in \operatorname{ReMin} A_n$ satisfying $d(y_n, B) < \gamma$. Hence the sequence $(y_n)$ is bounded. By the inclusion $(y_n) \subset \bigcup_{n=1}^{\infty} \operatorname{ReMin} A_n$ and the assumption (iii), without loss of generality, we can assume that the sequence $(y_n)$ converges to $y_0 \in Z$. By the assumption (i), we have $y_0 \in A$. From

$$d(y_0, B) \leq \gamma < \alpha = d(\operatorname{ReMin} A, B)$$

it follows that $y_0 \notin \operatorname{ReMin} A$. Thus there exists $a \in A$ satisfying $y_0 - a \in \operatorname{ri} C$ or, equivalently,

$$y_0 - a \notin (\operatorname{ri} C)^c. \tag{2.7}$$

Similar as the proof of Theorem 2.6, inclusion (2.7) gives $y_0 - a \notin \operatorname{Lim\,sup}(\operatorname{ri} C_n)^c$. The inclusion $a \in A$ and the assumption (i) imply that there is a sequence $(a_n)$, $a_n \in A_n$ such that $\lim_{n\to\infty} a_n = a$. Consequently,

$$\lim_{n\to\infty}(y_n - a_n) = y_0 - a \notin \operatorname{Lim\,sup}(\operatorname{ri} C_n)^c.$$

Thus there exists $n_0 \in \mathbb{N}$ satisfying $y_{n_0} - a_{n_0} \notin (\operatorname{ri} C_{n_0})^c$ or, equivalently,

$$y_{n_0} - a_{n_0} \in \operatorname{ri} C_{n_0},$$

contradicting the fact that $y_{n_0} \in \operatorname{ReMin} A_{n_0}$. The proof is complete. □

**Corollary 2.11.** *Let $C_n$ and $C$ be convex cones in $Z$ with nonempty interior. If*

(i) $A_n \xrightarrow{K} A$,

(ii) $\operatorname{Lim\,sup} C_n^c \subset (\operatorname{int} C)^c$,

(iii) *for every bounded subset of $\bigcup_{n=1}^{\infty} \operatorname{ReMin} A_n$ is relatively compact,*

*then*

$$\liminf_{n\to\infty} d(\operatorname{WMin} A_n, B) \geq d(\operatorname{WMin} A, B),$$

*for each bounded subset $B$.*

*Remark* 2.12. Theorem 2.10 can be seen as an extension of [16, Theorem 2.1] from the weak Pareto efficient sets to the relative Pareto efficient sets. Moreover, if $\operatorname{int} C \neq \emptyset$, then $\operatorname{int} C = \operatorname{ri} C$ and $\operatorname{cl}(C^c) \subset (\operatorname{int} C)^c$. Thus the condition "$\operatorname{Lim\,sup} C_n^c \subset (\operatorname{int} C)^c$" in Corollary 2.11 is weaker than the condition "$\operatorname{Lim\,sup} C_n^c \subset \operatorname{cl}(C^c)$" in [16, Theorem 2.1].

**Theorem 2.13.** *Let $Z$ be a reflexive Banach space, let $(C_n)$ and $C$ be convex cones with nonempty relative interior. If*

(i) $w - \operatorname{Lim\,sup} A_n \subset A \subset w - \operatorname{Lim\,inf} A_n$,

(ii) $w - \operatorname{Lim\,sup}(\operatorname{ri} C_n)^c \subset (\operatorname{ri} C)^c$,

*then*

$$\liminf_{n\to\infty} d(\operatorname{ReMin} A_n, x) \geq d(\operatorname{ReMin} A, x), \quad \forall\, x \in Z.$$

*Proof.* Let $x$ be an arbitrary element in $Z$. As in the proof of Theorem 2.10, we need only to consider the case where $\liminf_{n\to\infty} d(\operatorname{ReMin} A_n, x)$ is finite. Arguing by contradiction, assume that

$$\liminf_{n\to\infty} d(\operatorname{ReMin} A_n, x) < d(\operatorname{ReMin} A, x)$$

for some $x \in Z$. By taking a subsequence of $(A_n)$ if necessary we can find $y_n \in \operatorname{ReMin} A_n$ and a positive number $\gamma$ satisfying

$$d(y_n, x) < \gamma < d(\operatorname{ReMin} A, x), \quad \forall n \in \mathbb{N}.$$

By the reflexivity of $Z$ and the boundedness of $(y_n)$, there exists a subsequence of $(y_n)$ weakly converges to $y_0 \in Z$. From (i) we obtain $y_0 \in A$. Hence

$$d(y_0, x) \le \gamma < d(\operatorname{ReMin} A, x).$$

This implies that $y_0 \notin \operatorname{ReMin} A$. Thus there exists $a \in A$ such that $y_0 - a \in \operatorname{ri} C$. By the assumption (i), there is a sequence $(a_n)$, $a_n \in A_n$ for $n$ large enough satisfying

$$w - \lim_{n\to\infty} a_n = a.$$

We claim that there exists $n_0$ such that $y_n - a_n \in \operatorname{ri} C_n$ for all $n \ge n_0$. Indeed, otherwise there exists $(y_{n_k} - a_{n_k}) \subset (y_n - a_n)$ satisfying $y_{n_k} - a_{n_k} \in (\operatorname{ri} C_{n_k})^c$ for all $k \in \mathbb{N}$. From $w - \lim_{k\to\infty}(y_{n_k} - a_{n_k}) = y_0 - a$ we have

$$y_0 - a \in w - \operatorname{Lim\,sup}(\operatorname{ri} C_n)^c \subset (\operatorname{ri} C)^c.$$

Thus $y_0 - a \in (\operatorname{ri} C)^c$, which contradicts the fact that $y_0 - a \in \operatorname{ri} C$. This completes the proof. □

## 3. Lower convergence of relative Pareto efficient sets

Let $Z$ be a Banach space, $\emptyset \ne A \subset Z$ and let $C \subset Z$ be a closed convex cone with $0 \notin \operatorname{ri} C$. Put $\Theta = \operatorname{ri} C \cup \{0\}$.

**Definition 3.1.** We say that the *relative domination property*, denoted by (RDP), holds for a set $A \subset Z$ if

$$A \subset \operatorname{ReMin} A + \Theta.$$

*Remark* 3.2. (a) The relative domination property is an extended version of the weak domination property (see [15, Definition 4.9]). If $\operatorname{int} C \ne \emptyset$, then $\Theta = \operatorname{int} C \cup \{0\}$ and the relative domination property coincides with the weak domination property.

(b) It is easy to check that $\Theta = \operatorname{ri} C \cup \{0\}$ is a correct cone.

(c) From $\operatorname{ReMin} A = \operatorname{Min}(A \mid \Theta)$ and the correctness of $\Theta$ we see that (RDP) holds for every nonempty compact set $A$.

In this section we assume that $C_n = C$ for all $n \in \mathbb{N}$, where $C$ is a closed convex cone with nonempty relative interior and $0 \notin \operatorname{ri} C$.

**Theorem 3.3.** *Suppose that the following conditions hold:*

(i) $A_n \xrightarrow{K} A$,

(ii) (RDP) *holds for* $(A_n)$ *for all* $n$ *large enough,*

(iii) $\bigcup_{n=1}^{\infty} \operatorname{ReMin} A_n$ *is relative compact.*

*Then* $\operatorname{ReMin} A$ *is nonempty and* $\operatorname{ReMin} A \subset \operatorname{Lim\,inf} \operatorname{ReMin} A_n$.

*Proof.* We first show that $\operatorname{ReMin} A$ is nonempty. Put $A_0 = \operatorname{Lim\,sup} \operatorname{ReMin} A_n$. Then $A_0$ is a closed subset in $Z$. By the nonemptyness of $A$ and the inclusion $A \subset \operatorname{Lim\,inf} A_n$, $A_n$ is nonempty for all $n$ large enough. By the assumptions (ii) and (iii), it follows that $A_0$ is a nonempty compact set. By the correctness of $\Theta$ and the compactness of $A_0$, $\operatorname{ReMin} A_0 = \operatorname{Min}(A_0 \mid \Theta)$ is nonempty. We claim that $\operatorname{ReMin} A_0 \subset \operatorname{ReMin} A$. On the contrary, suppose that there exists $e \in \operatorname{ReMin} A_0 \setminus \operatorname{ReMin} A$. Then there is $a \in A$ such that

$$e \in a + \operatorname{ri} C. \tag{3.1}$$

By the assumption (i), there is a sequence $(a_n)$ such that $a_n \in A_n$ for all $n$ and $\lim_{n\to\infty} a_n = a$. The assumption (ii) implies that there exists $e_n \in \operatorname{ReMin} A_n$ satisfying $a_n \in e_n + \Theta$ for all $n$ large enough. Consequently, $a_n \in e_n + C$ for all $n$ large enough. In view of (iii) we may assume that the sequence $(e_n)$ converges to some $e_0 \in Z$. It is easy to see that $e_0 \in A_0$ and $a \in e_0 + C$. From this and (3.1) it follows that

$$e \in e_0 + C + \operatorname{ri} C \subset e_0 + \operatorname{ri} C.$$

Consequently, $e \in e_0 + \operatorname{ri} C$, which contradicts the minimality of $e$. Thus $\operatorname{ReMin} A_0 \subset \operatorname{ReMin} A$ and $\operatorname{ReMin} A$ is nonempty.

We now claim that $\operatorname{ReMin} A \subset \operatorname{Lim\,inf} \operatorname{ReMin} A_n$. Taking any $a \in \operatorname{ReMin} A$. The assumption (i) implies that there is a sequence $(a_n)$ such that $a_n \in A_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$. Since (RDP) holds for $A_n$, there exists $e_n \in \operatorname{ReMin} A_n$ such that

$$a_n \in e_n + \Theta. \tag{3.2}$$

By the assumption (iii) we may assume that $(e_n)$ converges to some $e \in A$. We shall show that $e = a$. Indeed, assume that $e \neq a$. Letting $n \to \infty$ in (3.2) we obtain $a - e \in C$. From $\lim_{n\to\infty}(a_n - e_n) = a - e \neq 0$ we have

$$0 \neq a_n - e_n \in \Theta,$$

for all large enough $n$. Consequently,

$$0 \neq a_n - e_n \in \operatorname{ri} C,$$

for all large enough $n$. Let $x$ be an arbitrary element in $C$. Then there exists $\mu > 1$ satisfying

$$(1-\mu)x + \mu(a_n - e_n) \in C, \quad \forall\, n \in \mathbb{N}. \tag{3.3}$$

Letting $n \to \infty$ in (3.3), by the closedness of $C$ we obtain $(1-\mu)x + \mu(a-e) \in C$. Thus $a - e \in \operatorname{ri} C$, which contradicts the fact that $a \in \operatorname{ReMin} A$. Thus $a = e$. From this and $e \in \operatorname{Lim\,inf} \operatorname{ReMin} A_n$ we have $a \in \operatorname{Lim\,inf} \operatorname{ReMin} A_n$. Thus

$$\operatorname{ReMin} A \subset \operatorname{Lim\,inf} \operatorname{ReMin} A_n.$$

The proof is complete. □

*Remark* 3.4. We see that $\operatorname{ReMin} A = \operatorname{Min}(A \mid \Theta)$, where $\Theta = \{0\} \cup \operatorname{ri} C$. However, Theorem 3.3 cannot be deduced from [16, Theorem 3.4]. To see this, we consider the following example.

**Example 3.5.** Let $Z = \mathbb{R}^3$ and $C = \mathbb{R}^2_+ \times \{0\}$. We have

$$\operatorname{ri} C = \big\{z = (z_1, z_2) \in \mathbb{R}^2 \mid z_1 > 0, z_2 > 0\big\}$$

and $\Theta = \{0\} \cup \big\{z = (z_1, z_2) \in \mathbb{R}^2 \mid z_1 > 0, z_2 > 0\big\}$. It is easy to see that

$$\operatorname{Lim\,sup} \Theta = \operatorname{cl} \Theta = C \subsetneq \Theta.$$

Consequently, the condition (iv) in [16, Theorem 3.4] does not hold. Thus Theorem 3.4 in [16] is not applicable to this case. Meanwhile, Theorem 3.3 works well.

**Theorem 3.6.** *Suppose that the following conditions hold:*

(i) $A_n \xrightarrow{K} A$,

(ii) (RDP) *holds for* $(A_n)$ *for all* $n$ *large enough,*

(iii) *if* $a_n \in A_n$ *is such that* $\lim_{n\to\infty} a_n$ *exists and*

$$e_n \in \operatorname{ReMin} A_n \cap (a_n - \Theta),$$

*then* $(e_n)$ *admits a convergent subsequence.*

(iv) *for any* $\rho > 0$, $\operatorname{ReMin} A \cap \mathbb{B}(0, \rho)$ *is relatively compact.*

*Then for each* $\rho > 0$ *we have*

$$\lim_{n\to\infty} e_\rho(\operatorname{ReMin} A, \operatorname{ReMin} A_n) = 0.$$

*Proof.* We will follow the proof scheme of [16, Theorem 3.3]. If the conclusion of our theorem does not hold, then there exist $\rho > 0$, $\epsilon > 0$ and a subsequence $(A_{n_k})$ of $(A_n)$ such that

$$e_\rho(\mathrm{ReMin}\, A, \mathrm{ReMin}\, A_{n_k}) > \epsilon, \quad \forall\, k \in \mathbb{N}.$$

Thus for each $k \in \mathbb{N}$ there is $e_k \in \mathrm{ReMin}\, A \cap \mathbb{B}_\rho$ satisfying

$$d(e_k, \mathrm{ReMin}\, A_{n_k}) > \epsilon. \tag{3.4}$$

By the assumption (iv), $(e_k)$ admits a subsequence converging to some $e \in Z$ and then by (3.4) there exists $k_0$ such that

$$d(e, \mathrm{ReMin}\, A_{n_k}) > \frac{\epsilon}{2}, \quad \forall\, k > k_0. \tag{3.5}$$

By Theorem 3.3, $\mathrm{ReMin}\, A \subset \mathrm{Lim\,inf}\, \mathrm{ReMin}\, A_n$. Thus, for each $k$, there exists a sequence $(e_i^k)$ such that $e_i^k \in \mathrm{ReMin}\, A_{n_i}$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} e_i^k = e_k$. Consequently, for each $k \in \mathbb{N}$, there is a sequence $(e_{i(k)}^k)$ such that $e_{i(k)}^k \in \mathrm{ReMin}\, A_{n_{i(k)}}$ and

$$d(e_k, e_{i(k)}^k) < \frac{1}{k}.$$

Clearly, $(e_{i(k)}^k)$ converges to $e$, contrary to (3.5). □

*Remark* 3.7. The results in Sections 2 and 3 can be generalized to the *intrinsic relative Pareto efficient sets* and *quasi relative Pareto efficient sets*. For details about these concepts we refer the reader to [2, 3].

## 4. The relative containment property and the lower semicontinuity of relative Pareto efficient point multifunctions

Let $A$ be a nonempty subset in a Banach space $Z$ and $B \subset A$. Suppose that $C \subset Z$ is a convex cone.

**Definition 4.1.** We say that the *domination property* (DP) holds for $(A, B) \subset Z \times Z$, if

$$A \subset B + C.$$

**Definition 4.2.** We say that the containment property (CP) holds for $(A, B) \subset Z \times Z$, if for each $W \in \mathcal{N}(0_Z)$ there exists $V \in \mathcal{N}(0_Z)$ such that

$$[A \setminus (B + W)] + V \subset B + C.$$

The domination property has been used by many authors (see, e.g., [4–9, 15, 21]). The containment property was first introduced by Bednarczuk in [4]. On the relationships between (CP) and (DP) were established in [8].

The following definition gives a weaker form of the notion containment property.

**Definition 4.3.** We say that the *relative containment property* ((RCP) for brevity) holds for $(A, B) \subset Z \times Z$ if for each $W \in \mathcal{N}(0_Z)$ there exists $V \in \mathcal{N}(0_Z)$ such that

$$[A \setminus (B + W)] + [V \cap \overline{\mathrm{aff}}(C)] \subset B + C.$$

If $\operatorname{int} C \neq \emptyset$, then the relative containment property coincides with the containment property. But it does not hold in general.

**Example 4.4.** Let $A = \{(z_1, z_2) \in \mathbb{R}^2 \mid 0 \le z_1 \le 1, 0 \le z_2 \le 1\}$, $C = \mathbb{R}_+ \times \{0\}$ and $B = \operatorname{Min} A$. We have

$$\operatorname{Min} A = \{0\} \times [0, 1] \quad \text{and} \quad \operatorname{Min} A + C = \{(z_1, z_2) \in \mathbb{R}^2 \mid 0 \le z_1, 0 \le z_2 \le 1\}.$$

It is easy to see that $A \subset \operatorname{Min} A + C$. Thus (DP) holds for $(A, \operatorname{Min} A)$. But (CP) does not hold for this pair. Indeed, take $W = \mathbb{B}(0_{\mathbb{R}^2}, \frac{1}{2})$. We have

$$A \setminus (\operatorname{Min} A + W) = \left\{(z_1, z_2) \in \mathbb{R}^2 \mid \frac{1}{2} \le z_1 \le 1, 0 \le z_2 \le 1\right\}.$$

Hence for any $V \in \mathcal{N}(0_{\mathbb{R}^2})$ then $[A \setminus (\operatorname{Min} A + W)] + V \subsetneq \operatorname{Min} A + C$. We now claim that (RCP) holds for $(A, \operatorname{Min} A)$. Let $W = \mathbb{B}(0_{\mathbb{R}^2}, \epsilon)$ be an arbitrary neighborhood of the origin. An easy computation shows that

$$A \setminus (\operatorname{Min} A + W) = \{(z_1, z_2) \in \mathbb{R}^2 \mid \epsilon \le z_1 \le 1, 0 \le z_2 \le 1\} \quad \text{and} \quad \overline{\mathrm{aff}} C = \mathbb{R} \times \{0\}.$$

Choose $V = \mathbb{B}(0_{\mathbb{R}^2}, \frac{\epsilon}{2}) \in \mathcal{N}(0_{\mathbb{R}^2})$. Then $[A \setminus (\operatorname{Min} A + W)] + [V \cap \overline{\mathrm{aff}} C] \subset \operatorname{Min} A + C$. Thus (RCP) holds for $(A, \operatorname{Min} A)$.

The next result shows that if the relative containment property holds for $(A, \operatorname{Min} A)$, then $\operatorname{Min} A$ is dense in $\operatorname{ReMin} A$.

**Proposition 4.5.** *Let $A$ be a nonempty subset in $Z$ and $C \subset Z$ be a pointed convex cone with $\operatorname{ri} C \neq \emptyset$. If* (RCP) *holds for $(A, \operatorname{Min} A)$, then*

$$\operatorname{Min} A \subset \operatorname{ReMin} A \subset \operatorname{cl} \operatorname{Min} A.$$

*Proof.* Due to Remark 2.2(b), to finish the proof it suffices to show that

$$\operatorname{ReMin} A \subset \operatorname{cl} \operatorname{Min} A.$$

Arguing by contradiction, assume that there is $\overline{z} \in \operatorname{ReMin} A \setminus \operatorname{cl} \operatorname{Min} A$. Since $\overline{z} \in \operatorname{ReMin} A$, we have

$$(\overline{z} - \operatorname{ri} C) \cap A = \emptyset. \tag{4.1}$$

We claim that

$$(\overline{z} - \operatorname{ri} C) \cap (\operatorname{Min} A + C) = \emptyset. \tag{4.2}$$

Otherwise, there exist $\theta \in \operatorname{ri} C$, $a \in \operatorname{Min} A$ and $c \in C$ such that $\overline{z} - \theta = a + c$. This implies that

$$a = \overline{z} - (\theta + c) \subset \overline{z} - (\operatorname{ri} C + C) \subset \overline{z} - \operatorname{ri} C.$$

Thus $a \in (\overline{z} - \operatorname{ri} C) \cap A$, contrary to (4.1). Since $\overline{z} \notin \operatorname{cl} \operatorname{Min} A$, there exists $W \in \mathcal{N}(0_Z)$ such that $\overline{z} \notin (\operatorname{Min} A + W)$. Since (RCP) holds for $(A, \operatorname{Min} A)$ there exists $V \in \mathcal{N}(0_Z)$ such that

$$[A \setminus (\operatorname{Min} A + W)] + [V \cap \overline{\operatorname{aff}}(C)] \subset \operatorname{Min} A + C.$$

Thus $\overline{z} + [V \cap \overline{\operatorname{aff}}(C)] \subset \operatorname{Min} A + C$. Clearly, $(\overline{z} - \operatorname{ri} C) \cap \big(\overline{z} + [V \cap \overline{\operatorname{aff}}(C)]\big) \neq \emptyset$. Hence $(\overline{z} - \operatorname{ri} C) \cap (\operatorname{Min} A + C) \neq \emptyset$, contrary to (4.2). □

*Remark* 4.6. (a) Proposition 4.5 generalizes Proposition 3.1 of [5] to the relative Pareto efficient case.

(b) Note that the set ReMin $A$ may not be closed even in case that $A$ is closed. Indeed, let

$$A = \left\{z = (z_1, z_2) \in \mathbb{R}^2 \mid z_1^2 + z_2^2 \leq 1\right\} \cup \{(-2, 0)\} \quad \text{and} \quad C = \mathbb{R}_+ \times \{0\}.$$

Then

$$\operatorname{ReMin} A = \left\{z = (z_1, z_2) \in \mathbb{R}^2 \mid z_1^2 + z_2^2 = 1, z_1 \leq 0, z_2 \neq 0\right\} \cup \{(-2, 0)\}$$

and it is not closed.

However, if $A \subset \mathbb{R}^m$ is polyhedral, then ReMin $A$ is closed. To prove this we need the following lemma.

**Lemma 4.7.** *Suppose that $A$ is a polyhedral subset of $\mathbb{R}^m$ defined by*

$$A = \{z \in \mathbb{R}^m \mid \langle a_i, z\rangle \leq b_i, i = 1, 2, \ldots, N\}, \tag{4.3}$$

*where $a_i \in \mathbb{R}^m$ and $b_i \in \mathbb{R}$ for all $i \in I := \{1, 2, \ldots, N\}$. Let $C \subset Z$ be a pointed closed convex cone. Then* ReMin $A$ *is nonempty if and only if*

$$\operatorname{Rec}(A) \cap (-\operatorname{ri} C) = \emptyset,$$

*where* $\operatorname{Rec}(A)$ *is the recession cone of $A$ which is given by*

$$\operatorname{Rec}(A) = \{z \in \mathbb{R}^m \mid \langle a_i, z\rangle \leq 0, i \in I\}.$$

*Proof.* Put $\Theta = \{0\} \cup \operatorname{ri} C$. We have $\operatorname{ReMin} A = \operatorname{Min}(A \mid \Theta)$. By Theorem 3.18 of [15, Chapter 2], $\operatorname{ReMin} A$ is nonempty if and only if

$$\operatorname{Rec}(A) \cap (-\Theta) = \{0\}\,. \tag{4.4}$$

From the pointedness of $C$ we see that $0 \notin \operatorname{ri} C$. Thus the condition (4.4) is equivalent to $\operatorname{Rec}(A) \cap (-\operatorname{ri} C) = \emptyset$, completing the proof. □

**Theorem 4.8.** *If $A$ is a polyhedral subset in $\mathbb{R}^m$ given by (4.3) and $C \subset \mathbb{R}^m$ is a pointed closed convex cone, then* $\operatorname{ReMin} A$ *is closed.*

*Proof.* We will follow the proof scheme of [8, Proposition 4.1]. Suppose on the contrary that $\operatorname{ReMin} A$ is not closed. Then there exists a sequence $(z_n) \subset \operatorname{ReMin} A$ which converges to $\overline{z}$ and $\overline{z} \notin \operatorname{ReMin} A$. Thus there is $z \in A$ satisfying $\overline{z} - z \in \operatorname{ri} C$. For each $n \in \mathbb{N}$, put $I_n = \{i \in I \mid \langle a_i, z_n\rangle = b_i\}$. As $I$ is finite and $I_n \subset I$ for all $n \in \mathbb{N}$, there exist infinitely many integers $n$ such that $I_n = I_1$. Without loss of generality, we can assume that $I_n = I_1$ for all $n \in \mathbb{N}$. This means that

$$\langle a_i, z_n\rangle = b_i,\ i \in I_1 \quad \text{and} \quad \langle a_i, z_n\rangle < b_i,\ i \in I \setminus I_1$$

for all $n \in \mathbb{N}$. Thus $\langle a_i, \overline{z}\rangle = b_i$ and $\langle a_i, \overline{z}\rangle \geq \langle a_i, z\rangle$ for $i \in I_1$. Furthermore, there exists $i \in I \setminus I_1$ such that $\langle a_i, \overline{z}\rangle < \langle a_i, z\rangle$. Otherwise, $\langle a_i, \overline{z}\rangle \geq \langle a_i, z\rangle$ for all $i \in I$ or, equivalently,

$$\langle a_i, z - \overline{z}\rangle \leq 0$$

for all $i \in I$. This implies that $z - \overline{z} \in \operatorname{Rec}(A)$. Hence $z - \overline{z} \in [\operatorname{Rec}(A) \cap (-\operatorname{ri} C)]$, which is impossible. Thus there are two index subsets $J_1, J_2 \subset I$ with $J_2 \neq \emptyset$ satisfying

$$\langle a_i, z - \overline{z}\rangle \leq 0,\ i \in J_1 \supset I_1, \quad \text{and} \quad \langle a_i, z - \overline{z}\rangle > 0,\ i \in J_2.$$

For each $n \in \mathbb{N}$ put

$$\lambda_n = \min_{i \in J_2} \frac{b_i - \langle a_i, z_n\rangle}{\langle a_i, z - \overline{z}\rangle} > 0,$$

and $y_n = z_n + \lambda_n (z - \overline{z})$. We have

$$\begin{aligned}\langle a_i, y_n\rangle &= \langle a_i, z_n\rangle + \lambda_n \langle a_i, z - \overline{z}\rangle \\ &\leq \langle a_i, z_n\rangle + (b_i - \langle a_i, z_n\rangle) \\ &\leq b_i\end{aligned}$$

for all $i \in J_2$. Clearly, $\langle a_i, y_n\rangle \leq b_i$ for all $i \in I \setminus J_2$. Thus $y_n \in A$ for all $n \in \mathbb{N}$. Moreover, we have $y_n - z_n \in -\operatorname{ri} C$, contradicting the fact that $z_n \in \operatorname{ReMin} A$. The proof is complete. □

**Corollary 4.9.** *Let $A$ be a polyhedral subset in $\mathbb{R}^m$ and $C$ be a pointed closed convex cone. If* (RCP) *holds for* $(A, \operatorname{Min} A)$*, then* $\operatorname{ReMin} A = \operatorname{cl} \operatorname{Min} A$.

The following proposition gives a characterization of (RCP) whenever $\operatorname{ri} C \neq \emptyset$.

**Proposition 4.10.** *If $\operatorname{ri} C \neq \emptyset$, then the following two properties are equivalent:*

(i) (RCP) *holds for* $(A, B)$*;*

(ii) *For each $W \in \mathcal{N}(0_Z)$ there is $W_0 \in \mathcal{N}(0_Z)$ such that for all*

$$y \in A \setminus (B + W)$$

*there exist $\eta_y \in B$ and $c_y \in C$ satisfying*

$$y = \eta_y + c_y, \quad (c_y + W_0) \cap \overline{\operatorname{aff}}(C) \subset C.$$

*Proof.* Sine the implication (ii) $\Rightarrow$ (i) is obvious, it suffices to show that (i) $\Rightarrow$ (ii). For each $W \in \mathcal{N}(0_Z)$, put $C_W = \{c \in C \mid (c + W) \cap \overline{\operatorname{aff}}(C) \subset C\}$. Clearly, $\operatorname{ri} C = \bigcup_{W \in \mathcal{N}(0_Z)} C_W$. We claim that for any $V \in \mathcal{N}(0_Z)$ there exists $W_V \in \mathcal{N}(0_Z)$ such that

$$\{z \in Z \mid z + [V \cap \overline{\operatorname{aff}}(C)] \in B + C\} \subset B + C_{W_V}. \tag{4.5}$$

Indeed, since $0_Z \in (-C) = \operatorname{cl}(-\operatorname{ri} C)$, there exists $W_V \in \mathcal{N}(0_Z)$ satisfying $V \cap (-C_{W_V}) \neq \emptyset$. Obviously, $-C_{W_V} \subset \overline{\operatorname{aff}}(C)$. Thus

$$V \cap (-C_{W_V}) = V \cap [(-C_{W_V}) \cap \overline{\operatorname{aff}}(C)] = [V \cap \overline{\operatorname{aff}}(C)] \cap (-C_{W_V}) \neq \emptyset.$$

Choose $z_V \in [V \cap \overline{\operatorname{aff}}(C)] \cap (-C_{W_V})$. Take any $z \in \{c \in Z \mid c + [V \cap \overline{\operatorname{aff}}(C)] \subset B + C\}$, i.e., $z + [V \cap \overline{\operatorname{aff}}(C)] \subset B + C$. We have $z + z_V \in B + C$. On the other hand, we have $C + C_{W_V} \subset C_{W_V}$. Indeed, take any $c_1 \in C$ and $c_2 \in C_{W_V}$. We claim that $(c_1 + c_2) \in C_{W_V}$ or, equivalently, $[(c_1 + c_2) + W_V] \cap \overline{\operatorname{aff}}(C) \subset C$. Let $u$ be an arbitrary point in $[(c_1 + c_2) + W_V] \cap \overline{\operatorname{aff}}(C)$. Then there is $w \in W_V$ such that $u = c_1 + c_2 + w$ and $c_1 + c_2 + w \in \overline{\operatorname{aff}}(C)$. Since $C$ is a convex cone, we have $\overline{\operatorname{aff}}(C) + C = \overline{\operatorname{aff}}(C)$. Thus $c_2 + w = u - c_1 \in \overline{\operatorname{aff}}(C) - C = \overline{\operatorname{aff}}(C)$. From this and $c_2 \in C_{W_V}$ imply that $c_2 + w \in C$. Thus $u = c_1 + (c_2 + w) \in C + C = C$. This means that $[(c_1 + c_2) + W_V] \cap \overline{\operatorname{aff}}(C) \subset C$ for any $c_1 \in C$ and $c_2 \in C_{W_V}$. Hence $C + C_{W_V} \subset C_{W_V}$. This implies that

$$z \in B + C - z_V \subset B + C + C_{W_V} \subset B + C_{W_V}.$$

Next, take any $W \in \mathcal{N}(0_Z)$. Since (RCP) holds for $(A, B)$ there exists $V \in \mathcal{N}(0_Z)$ such that

$$[A \setminus (B + W)] + [V \cap \overline{\operatorname{aff}}(C)] \subset B + C. \tag{4.6}$$

By virtue of (4.5) we can find $W_V \in \mathcal{N}(0_Z)$ satisfying

$$\{z \in Z \mid z + [V \cap \overline{\text{aff}}(C)] \in B + C\} \subset B + C_{W_V}. \tag{4.7}$$

For each $y \in A \setminus (B + W)$, it follows from (4.6) and (4.7) that

$$y + [V \cap \overline{\text{aff}}(C)] \subset B + C \subset B + C_{W_V}.$$

Thus there exist $\eta_y \in B$ and $c_y \in C$ satisfying

$$y = \eta_y + c_y, \quad (c_y + W_0) \cap \overline{\text{aff}}(C) \subset C,$$

where $W_0 := W_V$. □

*Remark* 4.11. If $\operatorname{int} C \neq \emptyset$, then Proposition 2.2 in [7] is deduced from the above.

**Definition 4.12.** Given a multifunction $F\colon P \rightrightarrows Z$, where $P$ is a topological space, put $\mathcal{F}(p) = \text{Min}(F(p))$, $\mathcal{R}(p) = \text{ReMin}(F(p))$ and call $\mathcal{F}\colon P \rightrightarrows Z$ and $\mathcal{R}\colon P \rightrightarrows Z$ are the *Pareto efficient point multifunction* and the *relative Pareto efficient point multifunction* corresponding to the quadruplet $\{F, P, Z, C\}$, respectively.

We recall some concepts of upper and lower continuities of a multifunction.

**Definition 4.13.** Let $F\colon P \rightrightarrows Z$ be a multifunction and $p_0 \in P$.

(i) $F$ is *upper semicontinuous* (usc for brevity) at $p_0$ if for every open set V containing $F(p_0)$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $F(p) \subset V$ for all $p \in U_0$.

(ii) $F$ is *lower semicontinuous* (lsc) at $p_0 \in \operatorname{dom} F$ if for any open set $V \subset Z$ satisfying $V \cap F(p_0) \neq \emptyset$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $V \cap F(p) \neq \emptyset$ for all $p \in U_0$.

(iii) $F$ is *Hausdorff upper semicontinuous* (H-usc) at $p_0$ if for every $W \in \mathcal{N}(0_Z)$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $F(p) \subset F(p_0) + W$ for all $p \in U_0$.

(iv) $F$ is *Hausdorff lower semicontinuous* (H-lsc) at $p_0$ if for every $W \in \mathcal{N}(0_Z)$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $F(p_0) \subset F(p) + W$ for all $p \in U_0$.

We now introduce some stronger forms of the properties lsc and H-usc of a multifunction, which will be used in this section.

**Definition 4.14.** (i) $F$ is said to be *relatively lower semicontinuous* (r-lsc for brevity) at $p_0 \in \operatorname{dom} F$ if for any $\overline{z} \in F(p_0)$ and $W \in \mathcal{N}(0_Z)$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $[\overline{z} + (W \cap \overline{\text{aff}}(C))] \cap F(p) \neq \emptyset$ for all $p \in U_0$.

(ii) $F$ is said to be *relatively Hausdorff upper semicontinuous* (r-H-usc) at $p_0$ if for every $W \in \mathcal{N}(0_Z)$ there exists $U_0 \in \mathcal{N}(p_0)$ such that $F(p) \subset F(p_0) + [W \cap \overline{\text{aff}}(C)]$ for all $p \in U_0$.

*Remark* 4.15. The following observations are simple:

(a) If $F$ is upper semicontinuous (Hausdorff lower semicontinuous), then $F$ is Hausdorff upper semicontinuous (lower semicontinuous).

(b) If $\text{int}\, C \neq \emptyset$ then $\overline{\text{aff}}(C) = Z$. Thus the properties r-usc (r-H-usc) and usc (H-usc) are coincide.

(c) If $F$ is relatively lower semicontinuous, then $F$ is lower semicontinuous. The converse does not hold in general. For instance, let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined by

$$F(p) = \left\{(z_1, z_2) \in \mathbb{R}^2 \mid 0 < z_2 \leq 1\right\}, \quad \forall\, p \in \mathbb{R} \setminus \{0\},$$

$F(0) = \{0_{\mathbb{R}^2}\}$ and $C = \mathbb{R}_+ \times \{0\}$. Then $F$ is lsc at every points but not r-lsc at $p = 0$.

(d) If $F$ is relatively Hausdorff upper semicontinuous, then $F$ is Hausdorff upper semicontinuous, but not vice versa. For example, let $P = \mathbb{R}$, $Z = \mathbb{R}^2$, $C = \mathbb{R}_+ \times \{0\}$. Let $F\colon \mathbb{R} \rightrightarrows \mathbb{R}^2$ be defined by $F(0) = \{(z_1, z_2) \mid z_1 = 0\} \setminus \{0_{\mathbb{R}^2}\}$ and

$$F(p) = \{(z_1, z_2) \mid -|p| \leq z_1 \leq |p|\}, \quad \forall\, p \in \mathbb{R} \setminus \{0\}.$$

Then for any $\delta > 0$ we have

$$F(p) \subset F(0) + \mathbb{B}(0_{\mathbb{R}^2}, \delta), \quad \forall\, p \in \mathbb{R},\ |p| < \frac{\delta}{2}.$$

Thus $F$ is H-usc at $p_0 = 0$. However, for each $\delta > 0$ we have

$$F(0) + [\mathbb{B}(0_{\mathbb{R}^2}, \delta) \cap \overline{\text{aff}}(C)] = \{(z_1, z_2) \mid |z_1| < \delta \text{ and } z_1 \neq 0\}.$$

Hence $F(p) \subsetneq F(0) + [\mathbb{B}(0_{\mathbb{R}^2}, \delta) \cap \overline{\text{aff}}(C)]$, $\forall\, p \in \mathbb{R} \setminus \{0\}$. Thus $F$ is not r-H-usc at 0.

Let $F$ and $\mathcal{G}$ be two multifunctions from $P$ to $Z$. The following definition gives a weaker form of notion uniform containment property of a multifunction around a point in [4].

**Definition 4.16.** We say that (RCP) holds for pair $(F, \mathcal{G})$ uniformly around a certain $p_0$ if for any neighborhood $W \in \mathcal{N}(0_Z)$ there exist $V \in \mathcal{N}(0_Z)$ and $U_0 \in \mathcal{N}(p_0)$ such that

$$[F(p) \setminus (\mathcal{G}(p) + W)] + [V \cap \overline{\text{aff}}(C)] \subset \mathcal{G}(p) + C, \quad \forall\, p \in U_0.$$

It is clear that if (CP) holds for pair $(F, \mathcal{G})$ uniformly around $p_0$ then (RCP) holds for pair $(F, \mathcal{G})$ uniformly around $p_0$. The converse is not true in general (see Example 4.20 below).

**Theorem 4.17.** *Suppose that* $\operatorname{ri} C \neq \emptyset$ *and* (RCP) *holds for pair* $(F, \mathcal{R})$ *uniformly around* $p_0$*. If* $F$ *is r-H-usc and r-lsc at* $p_0$*, then* $\mathcal{R}$ *is lsc at* $p_0$*.*

*Proof.* Let $\overline{z} \in \mathcal{R}(p_0)$ and $W \in \mathcal{N}(0_Z)$. The proof will be completed if we can show that there exists $U_W \in \mathcal{N}(p_0)$ such that

$$(\overline{z} + W) \cap \mathcal{R}(p) \neq \emptyset, \quad \forall\, p \in U_W. \tag{4.8}$$

Take any $W_1 \in \mathcal{N}_B(0_Z)$ satisfying $W_1 + W_1 \subset W$. Since (RCP) holds for pair $(F, \mathcal{R})$ uniformly around $p_0$, there exist $W_0 \in \mathcal{N}(0_Z)$ and $U_0 \in \mathcal{N}(p_0)$ such that for all $p \in U_0$ and $y \in [F(p) \setminus (\mathcal{R}(p) + W_1)]$ we can find $\eta_y \in \mathcal{R}(p)$ and $c_y \in C$ satisfying

$$y = \eta_y + c_y, \quad (c_y + W_0) \cap \overline{\operatorname{aff}}(C) \subset C. \tag{4.9}$$

Choose $W_2 \in \mathcal{N}_B(0_Z)$ satisfying $W_2 + W_2 \subset W_0$. By the relative lower semicontinuity of $F$ at $p_0$, there exists a neighborhood $U_1$ of $p_0$, $U_1 \subset U_0$ such that

$$[\overline{z} + (W_1 \cap W_2) \cap \overline{\operatorname{aff}}(C)] \cap F(p) \neq \emptyset, \quad \forall\, p \in U_1.$$

For each $p \in U_1$ let

$$y_p \in [\overline{z} + (W_1 \cap W_2) \cap \overline{\operatorname{aff}}(C)] \cap F(p). \tag{4.10}$$

By the relative Hausdorff upper semicontinuity of $F$ at $p_0$, there exists $U_2 \in \mathcal{N}(p_0)$, $U_2 \subset U_1$, such that

$$F(p) \subset F(p_0) + [(W_1 \cap W_2) \cap \overline{\operatorname{aff}}(C)], \quad \forall\, p \in U_2. \tag{4.11}$$

Suppose first that there exists $\overline{U} \in \mathcal{N}(p_0)$, $\overline{U} \subset U_0$, such that

$$y_p \in \mathcal{R}(p) + W_1, \quad \forall\, p \in \overline{U}. \tag{4.12}$$

For each $p \in U_1 \cap \overline{U}$, (4.10) and (4.12) imply that there exist $w_p \in W_1 \cap W_2$, $\eta_p \in \mathcal{R}(p)$ and $\overline{w}_p \in W_1$ satisfying $y_p = \overline{z} + w_p = \eta_p + \overline{w}_p$. Thus

$$\eta_p = \overline{z} + w_p - \overline{w}_p \in \overline{z} + W_1 + W_1 \subset \overline{z} + W.$$

This means that

$$(\overline{z} + W) \cap \mathcal{R}(p) \neq \emptyset, \quad \forall\, p \in U_1 \cap \overline{U},$$

which proves assertion (4.8) with $U_W = U_1 \cap \overline{U}$.

Next, suppose that for all $U \in \mathcal{N}(p_0)$, $U \subset U_2$, there exists $p \in U$ such that

$$y_p \notin \mathcal{R}(p) + W_1. \tag{4.13}$$

Combining (4.13) with (4.9) yields $y_p \in [F(p) \setminus (\mathcal{R}(p) + W_1)]$. By (4.9), there exist $\eta_p \in \mathcal{R}(p)$ and $c_p \in C$ satisfying

$$y_p = \eta_p + c_p, \quad (c_p + W_0) \cap \overline{\mathrm{aff}}(C) \subset C. \tag{4.14}$$

By (4.11) and the relation $\eta_p \in \mathcal{R}(p) \subset F(p)$, there exist $z_0 \in F(p_0)$ and $w_0 \in (W_1 \cap W_2) \cap \overline{\mathrm{aff}}(C)$ such that

$$\eta_p = z_0 + w_0. \tag{4.15}$$

By (4.10), there is $w_p \in (W_1 \cap W_2) \cap \overline{\mathrm{aff}}(C)$ such that

$$y_p = \overline{z} + w_p. \tag{4.16}$$

Using (4.16), (4.14) and (4.15) we get $\overline{z} + w_p = \eta_p + c_p = z_0 + w_0 + c_p$. This implies that $\overline{z} = z_0 + c_p + w_0 - w_p$. Furthermore,

$$\begin{aligned} c_p + w_0 - w_p &\in c_p + [(W_1 \cap W_2) \cap \overline{\mathrm{aff}}(C)] - [(W_1 \cap W_2) \cap \overline{\mathrm{aff}}(C)] \\ &\subset c_p + [W_2 \cap \overline{\mathrm{aff}}(C)] + [W_2 \cap \overline{\mathrm{aff}}(C)] \\ &\subset (c_p + W_0) \cap \overline{\mathrm{aff}}(C) \subset C. \end{aligned}$$

Put $k_0 = c_p + w_0 - w_p$. Then $k_0 \in \mathrm{ri}\, C$ and $z_0 = \overline{z} - k_0$. Thus $F(p_0) \cap (\overline{z} - \mathrm{ri}\, C) \neq \emptyset$, which contradicts the fact that $\overline{z} \in \mathcal{R}(p_0)$. The proof is complete. □

Note that if $0 \notin \mathrm{ri}\, C$ and $k_0 \in \mathrm{ri}\, C$, then $k_0 \neq 0$. From $\overline{z} - k_0 \in F(p_0)$ and $k_0 \neq 0$ it follows that $F(p_0) \cap (\overline{z} - \mathrm{ri}\, C) \neq \{\overline{z}\}$. Consequently, $F(p_0) \cap (\overline{z} - C) \neq \{\overline{z}\}$. Thus, replacing $\mathcal{R}$ by $\mathcal{F}$ in Theorem 4.17, we have the following assertion.

**Theorem 4.18.** *Suppose that $C$ is a convex cone with $\mathrm{ri}\, C \neq \emptyset$ and $0 \notin \mathrm{ri}\, C$, and* (RCP) *holds for pair $(F, \mathcal{F})$ uniformly around $p_0$. If $F$ is r-H-usc and r-lsc at $p_0$, then $\mathcal{F}$ is lsc at $p_0$.*

*Remark* 4.19. (a) Theorem 4.17 can be seen as an extension of [4, Theorem 4] from the Pareto efficient point multifunction to the relative Pareto efficient point multifunction.

(b) We stress that the assumption made about the ordering cone $C$ in Theorem 4.18 is weaker than in [4, Theorem 4]. Moreover, when $\mathrm{int}\, C \neq \emptyset$, then Theorem 4 in [4] is deduced from Theorem 4.18.

**Example 4.20.** Let $P = [0,1]$, $Z = \mathbb{R}^2$, $C = \mathbb{R}_+ \times \{0\}$. Let $F\colon P \rightrightarrows \mathbb{R}^2$ be defined by setting $F(p) = \{(z_1, z_2) \mid f(z_1) \leq z_2 \leq -z_1 + 1\}$ for all $p \in [0,1]$, where

$$f(t) = \begin{cases} -t + p & \text{if } t \leq p, \\ 0 & \text{if } p < t \leq 1, \\ -t + 1 & \text{if } t > 1. \end{cases}$$

Clearly, $\operatorname{ri} C = (0, +\infty) \times \{0\}$ and $\operatorname{int} C = \emptyset$. For each $p \in [0,1]$ we have

$$\mathcal{F}(p) = \{(z_1, z_2) \mid z_2 = -z_1 + p, z_1 \leq p\} \cup \{(z_1, z_2) \mid z_2 = -z_1 + 1, z_1 > 1\}.$$

It is a simple matter to check that $F$ is r-H-usc and r-lsc at $p_0$, and (RCP) holds for $(F, \mathcal{F})$ uniformly around $p_0$. Thus $\mathcal{F}$ is lsc at $p_0$. Furthermore, we can see that (CP) does not hold for $(F, \mathcal{F})$ uniformly around $p_0$. Thus, [4, Theorem 4] is not applicable to $F$ at $p_0$. Meanwhile, Theorem 4.18 works well for the multifunction at $p_0$.

It is well known that the lower convergence in the sense of Kuratowski-Painlevé of efficient sets is obtained by the lower semicontinuity of efficient point multifunctions. For instance, the following result follows directly from [4, Theorem 4] and the same results can be deduced from Theorem 4.17 and 4.18.

**Corollary 4.21.** (see [13, Theorem 3.1]) *Let $C$ be a convex cone with nonempty interior. Suppose that the following condition hold:*

(i) $A \subset \operatorname{Lim\,inf} A_n$,

(ii) $(A_n)$ *upper Hausdorff converges to* $A$,

(iii) (CP) *holds for* $(A_n)$ *for all large* $n$.

*Then* $\operatorname{Min} A \subset \operatorname{Lim\,inf} \operatorname{Min} A_n$.

*Remark* 4.22. In Corollary 4.21, the condition "(CP) *holds for* $(A_n)$ *for all large* $n$" cannot be replaced by "(DP) *holds for* $(A_n)$ *for all large* $n$" (see [13, Example 3.1]). Thus the domination property together with the assumptions (i) and (ii) of Corollary 4.21 do not suffice for the lower convergence of $(\operatorname{Min} A_n)$ to $\operatorname{Min} A$.

Finally, we refer to the results by Chuong et al. [12, Theorem 3.2]. By using the approach of Bednarczuk [4,6] and introducing the new concepts of *local containment property*, denoted by (locCP), the authors obtained further results on the lower semicontinuity of Pareto efficient point multifunctions taking values in Hausdorff topological vector spaces. In [12], the authors showed that "*if* (CP) *holds for pair* $(F, \mathcal{F})$ *uniformly around* $p_0$ *then* (locCP) *holds for pair* $(F, \mathcal{F})$ *uniformly around* $p_0$." However, the property (locCP) and (RCP) are independent of each other. To see this, we consider the following examples.

**Example 4.23.** Let $(F, P, Z, C)$ be as in Example 4.20. It is easy to check that (locCP) (see [12, Definition 3.1]) does not hold for $(F, \mathcal{F})$ uniformly around $p_0 = 0$. Meanwhile, (RCP) holds for $(F, \mathcal{F})$ uniformly around $p_0$.

**Example 4.24.** (see [12, Example 3.5]) Let $P = [0, 1]$, $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$. Let $F\colon P \rightrightarrows \mathbb{R}^2$ be defined by setting $F(0) = \{(z_1, z_2) \mid -z_1 \le z_2 \le -z_1 + 2\}$ and

$$F(p) = \{(z_1, z_2) \mid f(z_1) \le z_2 \le -z_1 + 2\}$$

for every $p \in P \setminus \{0\}$, where

$$f(t) = \begin{cases} -t + p & \text{if } t \le \frac{1}{p}, \\ p - \frac{1}{p} & \text{if } \frac{1}{p} < t \le \frac{1}{p} + 2 - p, \\ -t + 2 & \text{if } t > \frac{1}{p} + 2 - p \end{cases}$$

for all $t \in \mathbb{R}$. We have $\mathcal{F}(0) = \{(z_1, z_2) \mid z_2 = -z_1\}$ and

$$\mathcal{F}(p) = \left\{(z_1, z_2) \mid z_2 = -z_1 + p, z_1 \le \frac{1}{p}\right\} \cup \left\{(z_1, z_2) \mid z_2 = -z_1 + 2, z_1 > \frac{1}{p} + 2 - p\right\}.$$

It is easy to show that (RCP) holds for $(F, \mathcal{F})$ at $p_0 = 0$. But (RCP) does not hold for $(F, \mathcal{F})$ with any $p \in P \setminus \{0\}$. Thus (RCP) does not hold for $(F, \mathcal{F})$ uniformly around $p_0$. Meanwhile, it is a simple matter to check that (locCP) holds for $(F, \mathcal{F})$ uniformly around $p_0$.

Note that although the property (locCP) and (RCP) are independent of each other, the assumption made about the ordering cone $C$ in Theorem 4.18 is weaker than in [12, Theorem 3.2].

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Nguyen Van Tuyen
Department of Mathematics, Hanoi Pedagogical University No. 2, Xuan Hoa, Phuc Yen, Vinh Phuc, Vietnam
*E-mail address*: tuyensp2@yahoo.com